Numerical Solution of Parametrically Excited System by Perturbation Method

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Abstract

This paper discusses the perturbation method for analysing parametrically excited systems of the higher order from the practical and computational point of view. Furthermore, we propose a multistep perturbation method. This method is powerful when the system has large parameters as perturbed terms. As an example of the perturbation method, we deal with an equation derived from the resonant transfer circuit. In addition, we also treat the Mathieu equation in order to assure the multistep perturbation method.

1. Introduction

Conventionally, there have been many studies on parametrically excited systems of the low order, aimed at their stability. Most of these studies are based on the method of Hill's infinite determinant. However, the Hill's method is not effective when we try to analyse the systems of the higher order. In practical problems, we need to know both the stability and the wave forms of the solutions. For that purpose, we often use the perturbation method. In this paper, we discuss the perturbation method from the practical and computational points of view.

As is well known, the solutions of the system obtained by the perturbation method do not converge for large parameters. We propose a multistep perturbation method for those systems with large parameters.

2. Perturbation method[1)]

We consider the periodic system

$$\left.\begin{aligned} &\frac{d\boldsymbol{x}}{d\tau} = \boldsymbol{P}(\tau)\boldsymbol{x} \\ &\boldsymbol{P}(\tau+2\pi) = \boldsymbol{P}(\tau) \end{aligned}\right\} \qquad (1)$$

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where $\boldsymbol{x}$ is a real n vector and $\boldsymbol{P}(\tau)$ is a real $n\times n$ matrix with the period 2π. By the Floquet theorem, the normalized fundamental matrix of Eq.(1) is given by

$$\left.\begin{aligned}&\boldsymbol{\Phi}(\tau)=\boldsymbol{Z}(\tau)\exp(\boldsymbol{W}\tau)\\&\boldsymbol{Z}(0)=\boldsymbol{1}\end{aligned}\right\}\tag{2}$$

where $\boldsymbol{Z}(\tau)$ is a real $n\times n$ matrix with the period 2π, and $\boldsymbol{W}$ is an $n\times n$ constant matrix.

Here, we introduce a small parameter ε as follows:

$$\left.\begin{aligned}&\boldsymbol{P}_0=\frac{1}{2\pi}\int_0^{2\pi}\boldsymbol{P}(\tau)d\tau\\&\varepsilon\boldsymbol{P}_1(\tau)\triangleq\boldsymbol{P}(\tau)-\boldsymbol{P}_0\end{aligned}\right\}\tag{3}$$

We have from Eq. (1)

$$\frac{d\boldsymbol{x}}{d\tau}=(\boldsymbol{P}_0+\varepsilon\boldsymbol{P}_1(\tau))\boldsymbol{x}\,.\tag{4}$$

The matrix differential equation associated with Eq. (4) is given by

$$\frac{d\boldsymbol{X}}{d\tau}=(\boldsymbol{P}_0+\varepsilon\boldsymbol{P}_1(\tau))\boldsymbol{X}\tag{5}$$

where $\boldsymbol{X}$ is a real $n\times n$ matrix.

In the next section, we discuss how to obtain the fundamental matrix of Eq.(5).

2.1 Construction of the fundamental matrix

We introduce the parameter ε for Eq. (1). Then, the fundamental matrix of Eq. (5) can be represented by

$$\boldsymbol{\Phi}(\tau,\,\varepsilon)=\boldsymbol{Z}(\tau,\,\varepsilon)\exp(\boldsymbol{W}(\varepsilon)\tau)\tag{6}$$

where $\boldsymbol{Z}(\tau,\,\varepsilon)$ is an $n\times n$ matrix with the period 2π. The matrices $\boldsymbol{Z}(\tau,\,\varepsilon)$ and $\boldsymbol{W}(\varepsilon)$ are assumed to be expanded by the power series of ε,

$$\left.\begin{aligned}&\boldsymbol{Z}(\tau,\,\varepsilon)=\boldsymbol{Z}_0(\tau)+\varepsilon\boldsymbol{Z}_1(\tau)+\varepsilon^2\boldsymbol{Z}_2(\tau)+\cdots\\&\boldsymbol{W}(\varepsilon)=\boldsymbol{W}_0+\varepsilon\boldsymbol{W}_1+\varepsilon^2\boldsymbol{W}_2+\cdots\end{aligned}\right\}\tag{7}$$

Here, $\boldsymbol{Z}_k(\tau)$ $(k=0,\,1,\,\cdots)$ is a periodic matrix with the period 2π. The fundamental matrix of the unperturbed system of Eq. (5) is given by

$$\boldsymbol{\Phi}(\tau)=\exp(\boldsymbol{P}_0\tau)\,.\tag{8}$$

From Eqs. (6) and (7), we have

$$\boldsymbol{\Phi}(\tau, 0) = \boldsymbol{Z}_0(\tau) \exp(\boldsymbol{W}_0\tau) . \tag{9}$$

Therefore, if we take $\boldsymbol{W}_0$ as the real matrix, we then have

$$\boldsymbol{Z}_0(\tau) = \boldsymbol{1}, \quad \boldsymbol{W}_0 = \boldsymbol{P}_0 . \tag{10}$$

Accordingly, substituting Eqs. (6) and (7) into Eq. (5) and equating the like power of ε, we obtain the series of the matrix equations

$$\frac{d\boldsymbol{Z}_0(\tau)}{d\tau} = \boldsymbol{P}_0\boldsymbol{Z}_0 - \boldsymbol{Z}_0\boldsymbol{W}_0 \tag{11}_0$$

$$\frac{d\boldsymbol{Z}_1(\tau)}{d\tau} = \boldsymbol{P}_0\boldsymbol{Z}_1(\tau) - \boldsymbol{Z}_1(\tau)\boldsymbol{W}_0 - \boldsymbol{W}_1 - \boldsymbol{P}_1(\tau)\boldsymbol{Z}_0(\tau) \tag{11}_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d\boldsymbol{Z}_k(\tau)}{d\tau} = \boldsymbol{P}_0\boldsymbol{Z}_k(\tau) - \boldsymbol{Z}_k(\tau)\boldsymbol{W}_0 - \boldsymbol{W}_k - \boldsymbol{U}_k(\tau) \tag{11}_k$$

where

$$\boldsymbol{U}_k(\tau) = \sum_{i=1}^{k-1} \boldsymbol{Z}_i(\tau)\boldsymbol{W}_{k-i} - \boldsymbol{U}_0(\tau)\boldsymbol{Z}_{k-1}(\tau), \quad \boldsymbol{U}_0(\tau) = \boldsymbol{P}_1(\tau) . \tag{12}$$

As is easily seen, $\boldsymbol{U}_k(\tau)$ is the periodic matrix with the period 2π.

We obtain the solution of Eq. (11) by the method of variation of constants. The complementary solution of Eq. $(11)_k$ can be written by

$$\boldsymbol{Z}_k(\tau) = \exp(\boldsymbol{P}_0\tau)\boldsymbol{C}_k \exp(-\boldsymbol{P}_0\tau) . \tag{13}$$

Substituting Eq. (13) into Eq. $(11)_k$, we obtain the differential equation on $\boldsymbol{C}_k$, and the solution is given by

$$\boldsymbol{C}_k = -\int_0^\tau \exp(-\boldsymbol{U}_0\tau)\{\boldsymbol{W}_k + \boldsymbol{P}_k(\tau)\} \exp(\boldsymbol{P}_0\tau)d\tau \tag{14}$$

where $\boldsymbol{Z}_k(0) = \boldsymbol{0}$ $(k=1, 2, \cdots)$. Therefore, from Eqs. (13) and (14), we obtain

$$\boldsymbol{Z}_k(\tau) = -\int_0^\tau \exp\{\boldsymbol{P}_0(\tau-\xi)\}\{\boldsymbol{W}_k + \boldsymbol{U}_k(\xi)\} \exp\{-\boldsymbol{P}_0(\tau-\xi)\}d\xi . \tag{15}$$

Because $\boldsymbol{Z}_k(\tau)$ is periodic, we have

$$\int_0^{2\pi} \exp(\boldsymbol{P}_0\xi)\boldsymbol{W}_k \exp(-\boldsymbol{P}_0\xi)d\xi = -\int_0^{2\pi} \exp(\boldsymbol{P}_0\xi)\boldsymbol{U}_k(2\pi-\xi) \exp(-\boldsymbol{P}_0\xi)d\xi . \tag{16}$$

From Eq. (16), we can find $\boldsymbol{W}_k$. Therefore, substituting $\boldsymbol{W}_k$ into Eq. (15), we can determine $\boldsymbol{Z}_k(\tau)$.

2.2 Computation of $\boldsymbol{W}_k$

We can not easily determine the value of $\boldsymbol{W}_k$ directly from Eq. (16). There-

fore, we transform Eq. (16) by use of the modal matrix $\boldsymbol{S}_0$, formulated by the eigenvectors of $\boldsymbol{P}_0$. The classical canonical form of the matrix $\boldsymbol{P}_0$ is represented by

$$\boldsymbol{J}=\boldsymbol{S}_0^{-1}\boldsymbol{P}_0\boldsymbol{S}_0\,,\quad \boldsymbol{J}=\bigoplus_{i=1}^{s}\boldsymbol{J}_i\,,\quad \boldsymbol{J}_i=\lambda_i\boldsymbol{1}_{m_i}+\boldsymbol{H}_{m_i} \tag{17}$$

where $\boldsymbol{1}_{m_i}$ and $\boldsymbol{H}_{m_i}$ are the $m_i\times m_i$ unit and the well-known nilpotent matrices, respectively, and m_i is an order of the Jordan block $\boldsymbol{J}_i$. Therefore, we have

$$\exp(\boldsymbol{J}\tau)=\bigoplus_{i=1}^{s}\exp(\boldsymbol{J}_i\tau)$$

$$\exp(\boldsymbol{J}_i\tau)\triangleq\!\!\equiv e^{\lambda_i\tau}\begin{pmatrix} 1 & \tau & \cdots\cdots & \dfrac{1}{(m_i-1)!}\tau^{m_i-1}\\ \cdot & 1 & \tau & \cdots & \dfrac{1}{(m_i-2)!}\tau^{m_i-2}\\ & & \ddots & \ddots & \vdots\\ & & & \ddots & \tau\\ \cdot & \cdot & \cdot & \cdots\cdots & 1 \end{pmatrix}. \tag{18}$$

Eq. (16) becomes

$$\int_0^{2\pi}e^{J_i\xi}\tilde{\boldsymbol{W}}_{ij}^{(k)}e^{-J_j\xi}d\xi=-\int_0^{2\pi}e^{J_i\xi}\tilde{\boldsymbol{U}}_{ij}^{(k)}(2\pi-\xi)e^{-J_j\xi}d\xi \tag{19}$$

where $\tilde{\boldsymbol{W}}_{ij}^{(k)}$ and $\tilde{\boldsymbol{U}}_{ij}^{(k)}$ are the i,j block matrices given by

$$\left.\begin{aligned}\tilde{\boldsymbol{W}}_k&\triangleq(\tilde{\boldsymbol{W}}_{ij}^{(k)})=\boldsymbol{S}_0^{-1}\boldsymbol{W}_k\boldsymbol{S}_0\\ \tilde{\boldsymbol{U}}_k&\triangleq\!\!\equiv(\tilde{\boldsymbol{U}}_{ij}^{(k)})=\boldsymbol{S}_0^{-1}\boldsymbol{U}_k\boldsymbol{S}_0\end{aligned}\right\}. \tag{20}$$

If $\boldsymbol{P}_0$ has a simple structure, then $\exp(\boldsymbol{J}\tau)$ is a diagonal matrix. Therefore, the elements of $\boldsymbol{W}_k$ can be obtained analytically by

$$\tilde{W}_{ii}^{(k)}=\frac{-1}{2\pi}\int_0^{2\pi}\tilde{U}_{ii}^{(k)}(\xi)d\xi \tag{21}_1$$

$$\tilde{W}_{ij}^{(k)}=\begin{cases}-\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\tilde{U}_{ij}^{(k)}(\xi)d\xi & (\lambda_i=\lambda_j) & (21)_2\\ -\dfrac{\lambda_i-\lambda_j}{e^{2\pi(\lambda_i-\lambda_j)}-1}\displaystyle\int_0^{2\pi}e^{(\lambda_i-\lambda_j)\xi}\tilde{U}_{ij}^{(k)}(2\pi-\xi)d\xi & (\lambda_i\neq\lambda_j) & (21)_3\end{cases}$$

where $\tilde{W}_{ij}^{(k)}$ and $\tilde{U}_{ij}^{(k)}$ are the i,j elements of the matrices $\tilde{\boldsymbol{W}}^{(k)}$ and $\tilde{\boldsymbol{U}}^{(k)}$, respectively.

2.3 Modification algorithm of the solution

When the eigenvalues of $\boldsymbol{W}$ are complex or purely imaginary numbers,

$\exp(\boldsymbol{W}\tau)$ has oscillatory terms. Then, we include the periodically oscillatory terms into $\boldsymbol{Z}(\tau)$ so that $\log \boldsymbol{W}$ has the principal value. We modify $\boldsymbol{W}$ in such a way that the absolute values of the imaginary parts for each eigenvalue take the minimum values. The algorithm is as follows:

S0: $\boldsymbol{J} \leftarrow \boldsymbol{S}_0^{-1}\boldsymbol{W}\boldsymbol{S}_0$

S1: $\hat{\boldsymbol{J}} \leftarrow R_k(I_m(\boldsymbol{J}))$ where the function $R_k(\ )$ is defined by

$$R_k(x) \triangleq \frac{1}{2^k}\operatorname{sgn}(x)\left[2^k|x|+\frac{1}{2}\right]$$

where $I_m(\boldsymbol{J})$ denotes the imaginary part of $\boldsymbol{J}$.

S2: $\boldsymbol{W} \leftarrow \boldsymbol{W} - j\boldsymbol{S}_0\hat{\boldsymbol{J}}\boldsymbol{S}_0^{-1}$, $\boldsymbol{Z}(\tau) \leftarrow \boldsymbol{Z}(\tau)\boldsymbol{S}_0 \exp(j\hat{\boldsymbol{J}}\tau)\boldsymbol{S}_0^{-1}$ where $j=\sqrt{-1}$.

The precise discussion is given in the appendix.

3. Multistep perturbation method

When the parameter ε in Eq. (5) is large, the perturbation method in section 2 is not applicable because the solution does not converge. Here, we develop a technique to obtain the solution for large parameters. We divide $\varepsilon\boldsymbol{P}_1(\tau)$ in Eq. (5) into M small terms,

$$\left.\begin{aligned}\boldsymbol{P}(\tau) &= \boldsymbol{P}_0+\hat{\varepsilon}_1\hat{\boldsymbol{P}}_1(\tau)+\hat{\varepsilon}_2\hat{\boldsymbol{P}}_2(\tau)+\cdots+\hat{\varepsilon}_M\hat{\boldsymbol{P}}_M(\tau)\\ \varepsilon\boldsymbol{P}_1(\tau) &\triangleq \sum_{l=1}^{M}\hat{\varepsilon}_l\hat{\boldsymbol{P}}_l(\tau)\end{aligned}\right\} \tag{22}$$

and consider the following series of matrix differential equations:

$$\frac{d\boldsymbol{X}_0}{d\tau} = \boldsymbol{P}_0\boldsymbol{X}_0 \tag{23}_0$$

$$\frac{d\boldsymbol{X}_1}{d\tau} = \{\boldsymbol{P}_0+\hat{\varepsilon}_1\hat{\boldsymbol{P}}_1(\tau)\}\boldsymbol{X}_1 \tag{23}_1$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{d\boldsymbol{X}_l}{d\tau} = \{\boldsymbol{P}_0+\hat{\varepsilon}_1\hat{\boldsymbol{P}}_1(\tau)+\cdots+\hat{\varepsilon}_l\hat{\boldsymbol{P}}_l(\tau)\}\boldsymbol{X}_l\,. \tag{23}_l$$

The Floquet solution of Eq. $(23)_l$ is given by

$$\boldsymbol{X}_l(\tau) = \boldsymbol{Z}^{[l]}(\tau)\exp(\boldsymbol{W}^{[l]}\tau) \tag{24}$$

$$l = 0, 1, \cdots, M\,.$$

If we can obtain the solutions $\boldsymbol{X}_0, \boldsymbol{X}_1, \cdots, \boldsymbol{X}_l$, then we can find $\boldsymbol{X}_{l+1}$ by the perturbation method, because Eq. $(23)_{l+1}$ is led by adding the smaller term $\hat{\varepsilon}_{l+1}\boldsymbol{P}_{l+1}(\tau)$ to the matrix of Eq. $(23)_l$. The matrices $\boldsymbol{Z}^{[l+1]}(\tau)$ and $\boldsymbol{W}^{[l+1]}$ are represented by

$$\left.\begin{aligned} \boldsymbol{Z}^{[l+1]}(\tau) &= \boldsymbol{Z}^{[l]}(\tau)+\sum_{k_{l+1}=1}^{\infty}(\hat{\varepsilon}_{l+1})^{k_{l+1}}\boldsymbol{Z}_{k_{l+1}}^{[l]}(\tau) \\ \boldsymbol{W}^{[l+1]} &= \boldsymbol{W}^{[l]}+\sum_{k_{l+1}=1}^{\infty}(\hat{\varepsilon}_{l+1})^{k_{l+1}}\boldsymbol{W}_{k_{l+1}}^{[l]}. \end{aligned}\right\} \tag{25}$$

Therefore, the matrices $\boldsymbol{Z}^{[M]}(\tau)$ and $\boldsymbol{W}^{[M]}$ are given by

$$\left.\begin{aligned} \boldsymbol{Z}^{[M]}(\tau) &= \boldsymbol{Z}^{[0]}(\tau)+\sum_{l=1}^{M}\Big\{\sum_{k_l=1}^{\infty}(\hat{\varepsilon}_l)^{k_l}\boldsymbol{Z}_{k_l}^{[l-1]}(\tau)\Big\} \\ \boldsymbol{W}^{[M]} &= \boldsymbol{W}^{[0]}+\sum_{l=1}^{M}\Big\{\sum_{k_l=1}^{\infty}(\hat{\varepsilon}_l)^{k_l}\boldsymbol{W}_{k_l}^{[l-1]}\Big\}. \end{aligned}\right\} \tag{26}$$

From Eq. (26), we can obtain $\boldsymbol{X}^{[M]}(\tau)=\boldsymbol{Z}^{[M]}\exp(\boldsymbol{W}^{[M]}\tau)$.

3.1 Correction of the solution

When we try to apply the multistep perturbation method, we repeatedly use the perturbation method. Therefore, the solution comes to include accumulated errors. Accordingly, the algorithm must be such an accurate one that the errors become as small as possible.

Let us compute the coefficient matrix $\varepsilon\boldsymbol{P}(\tau)$ from the numerical results of step l. Substituting $\boldsymbol{X}^{[l]}=\boldsymbol{Z}^{[l]}\exp(\boldsymbol{W}^{[l]}\tau)$ into both sides of Eq. $(23)_l$, we have

$$\begin{aligned} \frac{d\boldsymbol{X}^{[l]}}{d\tau} &= \left\{\frac{d}{d\tau}\boldsymbol{Z}^{[l]}(\tau)+\boldsymbol{Z}^{[l]}(\tau)\,\boldsymbol{W}^{[l]}\right\}\exp(\boldsymbol{W}^{[l]}\tau) \\ &= \{\boldsymbol{P}_0+\sum_{k=1}^{l}\hat{\varepsilon}_k\boldsymbol{P}_k(\tau)\}\boldsymbol{Z}^{[l]}(\tau)\exp(\boldsymbol{W}^{[l]}\tau). \end{aligned} \tag{27}$$

From Eq. (27), we obtain the new coefficient matrix $\boldsymbol{P}_l(\tau)$ given by

$$\boldsymbol{P}_l'(\tau)\triangleq\left\{\frac{d}{d\tau}\boldsymbol{Z}^{[l]}(\tau)+\boldsymbol{Z}^{[l]}(\tau)\,\boldsymbol{W}^{[l]}\right\}(\boldsymbol{Z}^{[l]}(\tau)^{-1}). \tag{28}$$

Let us put

$$\varepsilon''\boldsymbol{P}_l(\tau) = \{\boldsymbol{P}_0+\sum_{k=1}^{l}\hat{\varepsilon}_k\hat{\boldsymbol{P}}_k(\tau)\}-\boldsymbol{P}_l'(\tau) \tag{29}$$

Then, $\varepsilon''\boldsymbol{P}_l(\tau)$ must become very small. Therefore, we seek the solution of the unperturbed system of the equation

$$\frac{d\boldsymbol{X}}{d\tau} = \{\boldsymbol{P}_l'(\tau)+\varepsilon''\boldsymbol{P}_l(\tau)\}\boldsymbol{X}. \tag{30}$$

The solution of Eq. (30) is a more accurate solution than $\boldsymbol{X}^{[l]}$.

4. Some techniques of numerical computation

In this section, we describe the numerical method by which we need to carry out the multistep perturbation method.

4.1 Computation of $\tilde{W}_k$

We obtain Eq. (21) by the actual calculation of the left-hand side of Eq. (19). However, if Eq. (21) is computed by a numerical integration, then the error of the values $\tilde{W}_{ij}^{(k)}$ becomes large. This is because the method of integration is different for each side of Eq. (19). Therefore, we must compute both sides numerically. Here, we evaluate the values of the elements $\tilde{W}_{ij}^{(k)}$ by the trapezoidal rule as follows:

$$\tilde{W}_{ij}^{(k)} = \begin{cases} \dfrac{1}{2\pi}\tilde{U}_{ij}^{(k)} & (\lambda_i = \lambda_j) \\ \dfrac{2}{\Delta\tau}\dfrac{1-e^{(\lambda_j-\lambda_i)\Delta\tau}}{1+e^{(\lambda_j-\lambda_i)\Delta\tau}}\dfrac{\tilde{U}_{ij}^{(k)}}{1-e^{2(\lambda_j-\lambda_i)\pi}} & (\lambda_i \neq \lambda_j) \end{cases} \tag{31}$$

where $\Delta\tau$ is the step size.

4.2 Computation of the derivative of $\boldsymbol{Z}^{[l]}(\tau)$

When we correct the solution $\boldsymbol{X}^{[l]}(\tau)$, we compute its derivative by the DFT. Because the matrix $\boldsymbol{Z}^{[l]}(\tau)$ is periodic with the period 2π, it is expanded by the Fourier series

$$\boldsymbol{Z}^{[l]} = \boldsymbol{F}_0 + \sum_{k=1}^{N} \{\boldsymbol{F}_{ck}\cos k\tau + \boldsymbol{F}_{sk}\sin k\tau\} \tag{32}$$

where $\boldsymbol{F}_0$, $\boldsymbol{F}_{ck}$ and $\boldsymbol{F}_{sk}$ are real $n\times n$ matrices. Therefore, we differentiate both sides of Eq. (32) with respect to τ in order to obtain the derivative of $\boldsymbol{Z}^{[l]}(\tau)$. The algorithm is as follows:

S0 : Compute $\boldsymbol{F}_0$, $\boldsymbol{F}_{ck}$ and $\boldsymbol{F}_{sk}$ by the DFT of the sequence $\{\boldsymbol{Z}^{[l]}(\tau_p)\}$ $p=0, 1, \cdots, N$.

S1 : Set $\boldsymbol{F}_0 \leftarrow \boldsymbol{0}$, $\boldsymbol{F}_{ck} \leftarrow \boldsymbol{F}_{sk}$, $\boldsymbol{F}_{sk} \leftarrow -k\boldsymbol{F}_{ck}$ and obtain the sequences $\{\boldsymbol{F}_{sk}\}$ and $\{\boldsymbol{F}_{ck}\}$ $k=0, 1, \cdots, N$.

S2 : Compute the sequence $\left\{\dfrac{d\boldsymbol{Z}^{[l]}}{d\tau}\right\}$ by the IDFT of the sequences $\{\boldsymbol{F}_{ck}\}$ and $\{\boldsymbol{F}_{sk}\}$.

4.3 Convergence of the series

If the condition

$$\|\varepsilon^{K_{\max}}\boldsymbol{W}_{K_{\max}}\| < \varepsilon_p \| \sum_{k=0}^{K_{\max}-1} \varepsilon^k \boldsymbol{W}_k \| \tag{33}$$

is satisfied, the series is considered to be numerically converged. Here, ε_p is a small positive number and $K_{\max}$ is the order of approximation. The norm of the matrix is given by $\|\boldsymbol{A}\| \triangleq \max_i \sum_j |a_{ij}|$.

5. Algorithm

The above consideration leads us to the following algorithm of the perturbation method:

S0 : Give the sequence $\{\boldsymbol{P}(\tau_p)\}$ $p=0, 1, \cdots, N$ and ε_p, $K_{\max}$.

S1 : Compute $\boldsymbol{P}_0$ by Eq. (3) and set $\boldsymbol{W}_0 \leftarrow \boldsymbol{P}_0$, $\boldsymbol{Z}_0 \leftarrow \boldsymbol{I}$, $\boldsymbol{P}_1(\tau_p) \leftarrow \{\boldsymbol{P}(\tau_p) - \boldsymbol{P}_0\}$, $p=0, 1, \cdots, N$.

S2 : Compute the eigen values of $\boldsymbol{W}_0$ and determine $\boldsymbol{S}_0$, $\boldsymbol{J}_0$ and $\boldsymbol{S}_0^{-1}$.

S3 : Compute $\exp(\boldsymbol{W}_0\tau_p)$ and $\exp(-\boldsymbol{W}_0\tau_p)$ for $p=0, 1, \cdots, N$.

S4 : Set $\boldsymbol{U}_0(\tau_p) \leftarrow \boldsymbol{P}_1(\tau_p)$, $p=0, 1, \cdots, N$.

S5 : Set $k \leftarrow 1$.

S6 : Compute $\boldsymbol{U}_k(\tau_p)$ by Eq. (12) for $p=0, 1, \cdots, N$.

S7 : Compute $\boldsymbol{W}_k$ by Eqs. (20) and (31).

S8 : Compute $\boldsymbol{Z}_k(\tau_p)$ by Eq. (15) for $p=0, 1, \cdots, N$.

S9 : Set $k \leftarrow k+1$. If k is larger than $K_{\max}$, then go to S10. Otherwise, return to S6.

S10 : Test the convergence of $\boldsymbol{W}$ by Eq. (33).
Comment; At this step, we can examine the stalibity of the solution by the eigenvalues of $\boldsymbol{W}$.

S11 : Transform Eq. (6) into the principal value.

S12 : Compute the harmonic component of $\{\boldsymbol{Z}(\tau_p)\}$ by the DFT.

S13 : Stop.

6. Some applications

6.1 Mathieu equation

To assure the multistep perturbation method, we are going to obtain the Floquet solution of the Mathieu equation written by

$$\frac{d}{d\tau}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{a}{4} - 4q\cos\tau & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{34}$$

We choose the parameters $a=1.0$, $q=0.75$. For these parameters, we can not obtain the solution by the conventional perturbation method, because the solution does not converge. However, by the multistep perturbation method we can obtain the solution as indicated in Table 1. We take the period of $\boldsymbol{Z}(\tau)$ as 4π to make $\boldsymbol{W}$ real and set $N=256$. To compare the result, we demonstrate the solution by the method of the matrix function[2]. Both results are in good agreement.

In Fig. 1, we show the step size of q on the Mathieu chart, for which the solu-

Table 1. Computational results for Mathieu equation

	Multistep perturbation method	Method of matrix function
W	$\begin{bmatrix} 0 & -0.10113\times10^{-1} \\ -0.12026\times10^{2} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -0.10115\times10^{-1} \\ -0.12025\times10^{2} & 0 \end{bmatrix}$
eigenvalues of W	±0.34873	±0.34876
$Z_c^{(1)}$	$\begin{bmatrix} 0.64104\times10^{1} & 0 \\ 0 & 0.15877 \end{bmatrix}$	$\begin{bmatrix} 0.64098\times10^{1} & 0 \\ 0 & 0.15877 \end{bmatrix}$
$Z_s^{(1)}$	$\begin{bmatrix} 0 & 0.44720 \\ -0.85832\times10^{1} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.44720 \\ -0.85825\times10^{1} & 0 \end{bmatrix}$
$Z_c^{(3)}$	$\begin{bmatrix} -0.33447\times10^{1} & 0 \\ 0 & 0.58541 \end{bmatrix}$	$\begin{bmatrix} -0.33444\times10^{1} & 0 \\ 0 & 0.58542 \end{bmatrix}$
$Z_s^{(3)}$	$\begin{bmatrix} 0 & 0.36773 \\ 0.59478 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.36772 \\ 0.59470 & 0 \end{bmatrix}$
$Z_c^{(5)}$	$\begin{bmatrix} -0.17542\times10^{1} & 0 \\ 0 & 0.22008 \end{bmatrix}$	$\begin{bmatrix} -0.17540\times10^{1} & 0 \\ 0 & 0.22008 \end{bmatrix}$
$Z_s^{(5)}$	$\begin{bmatrix} 0 & 0.08094 \\ 0.34122\times10^{1} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.08093 \\ 0.34118\times10^{1} & 0 \end{bmatrix}$
$Z_c^{(7)}$	$\begin{bmatrix} -0.28548 & 0 \\ 0 & 0.03294 \end{bmatrix}$	$\begin{bmatrix} -0.28544 & 0 \\ 0 & 0.03294 \end{bmatrix}$
$Z_s^{(7)}$	$\begin{bmatrix} 0 & 0.00859 \\ 0.89591 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.00859 \\ 0.89580 & 0 \end{bmatrix}$
$Z_c^{(9)}$	$\begin{bmatrix} -0.02457 & 0 \\ 0 & 0.00267 \end{bmatrix}$	$\begin{bmatrix} -0.02456 & 0 \\ 0 & 0.00267 \end{bmatrix}$
$Z_s^{(9)}$	$\begin{bmatrix} 0 & 0.00054 \\ 0.10409 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.00054 \\ 0.10407 & 0 \end{bmatrix}$

Notes 1. We take 6 step sizes of $q(=4\varepsilon)$. These are 0.085, 0.206, 0.376, 0.629, 0.734, 0.85. We start at $q=0$.

2. $Z(\tau)=Z^{(0)}+\sum_{k=1}^{K}(Z_c^{(k)}\cos k\tau+Z_s^{(k)}\sin k\tau)$.

3. There exist the odd harmonic components $Z_c^{(k)}$, $Z_s^{(k)}$ for $k\geq11$. Both methods are in good agreement for these components.

4. $N=256$ for the multistep perturbation method. $N=128$ for the method of matrix function.

tion converges. The dots on the straight lines indicate the steps of the multistep perturbation method. For example, in the first unstable region, the solution does not change its form. Therefore, we can take the large step size. However, when q takes its value near the boundaries, the small step size must be taken, because the solution does not converge for the large step size. In Fig. 2, we show the

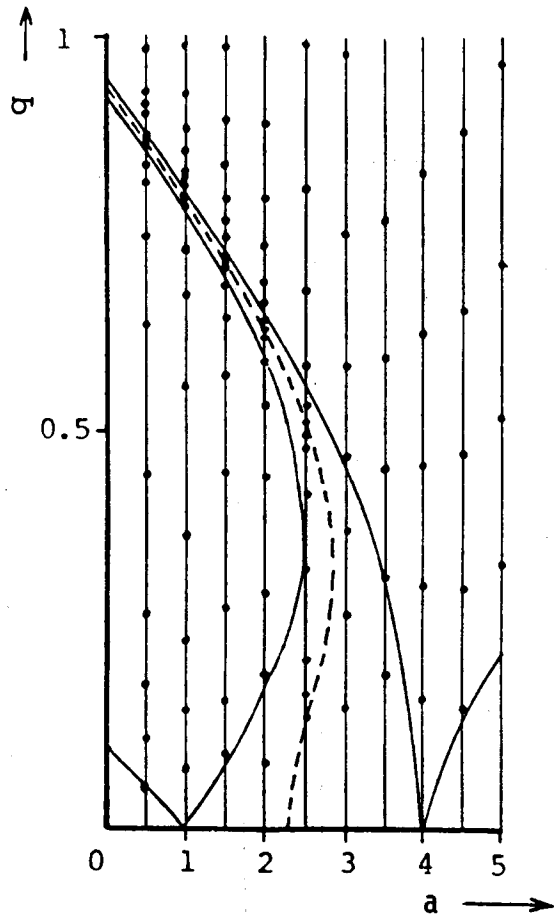

Fig. 1. Mathieu chart and the step size of q.
(The dashed line is the boundary of two regions where the harmonic components of $Z(\tau)$ are all odd and all even.)

enlarged charts of Fig. 1 near the boundaries. The perturbed solution has a very small convergence radius near the boundaries, illustrated by the dashed lines. When the step size q takes so large a value as to be across the boundary, the solution takes another form. On the real lines, the solution is periodic and characteristic exponents are purely imaginary numbers.

In this example, we correct the solution by the method stated in section 3.1 at every four steps.

6.2 Equation derived from resonant transfer circuit[3)]

As an example of the system of the higher order, we deal with the resonant transfer circuit, as shown in Fig. 3. The inductor and capacitor are parametrically excited to compensate the circuit losses. The inductance and capacitance

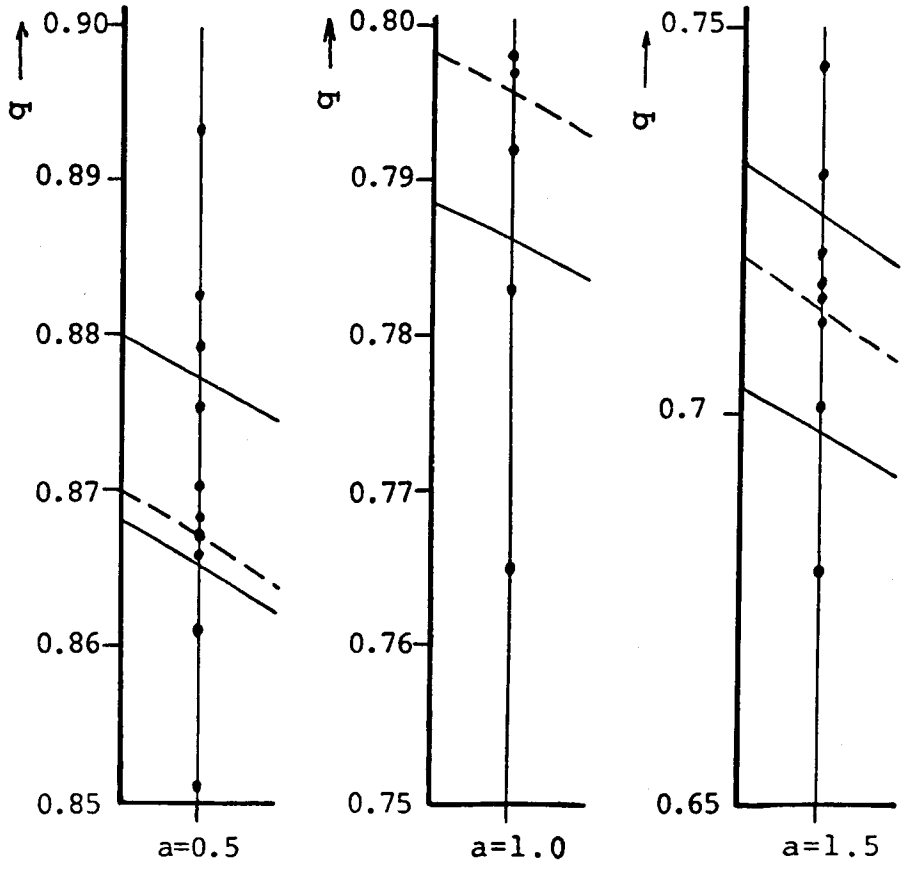

Fig. 2. Enlarged charts near the boundary.

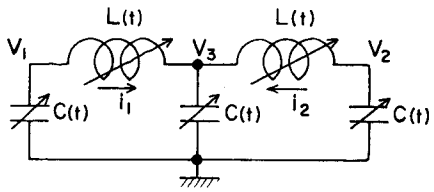

Fig. 3. The resonant transfer circuit.

are varied by

$$\left.\begin{aligned} L &= L_0\{1-\gamma_L \cos(2\omega t+\theta)\} \\ C' &= \frac{1}{k} C\{1-\gamma_c \cos(4\omega t+\phi)\} \end{aligned}\right\} \tag{35}$$

respectively, where $\omega^2 = 1/L_0C$.

Putting $\tau = \omega t, \quad i_1' = \omega L_0 i_1, \quad i_2' = \omega L_0 i_2$

and $\boldsymbol{x} \triangleq {}^t(v_1, v_2, v_3, i_1', i_2')$,

we have Eq. (1), where the 5×5 periodic matrix is given by

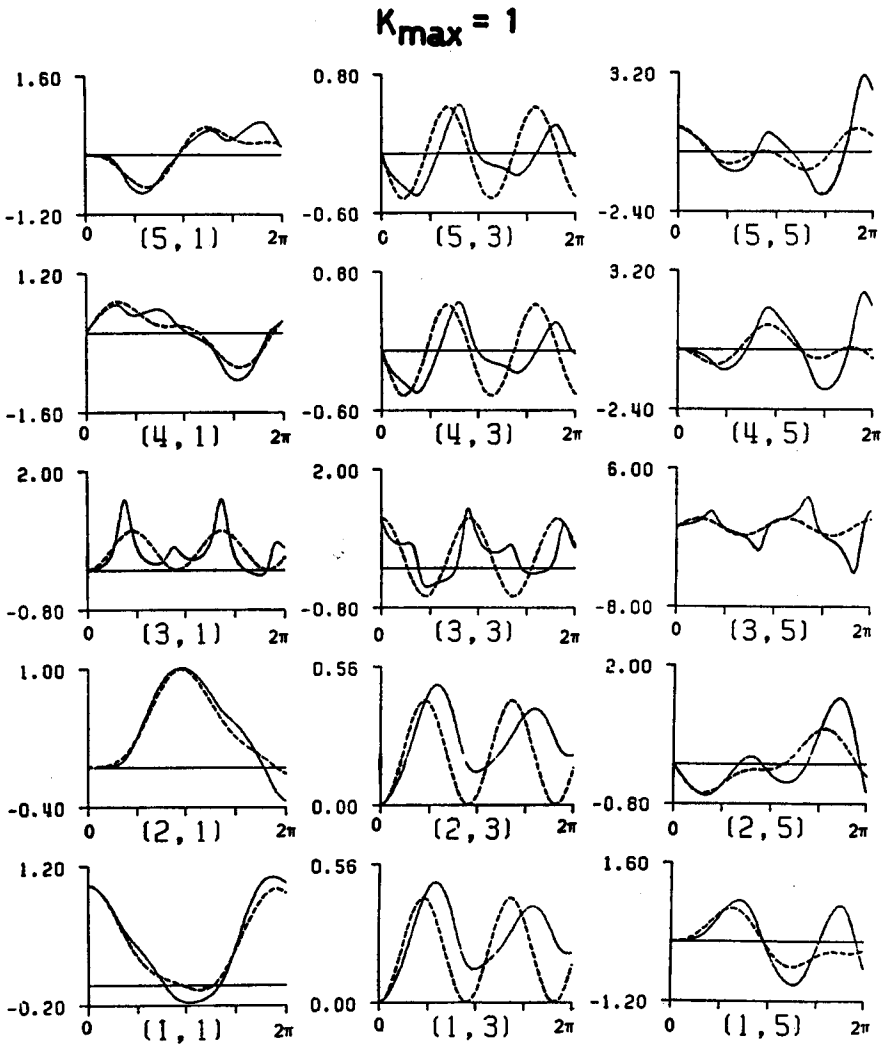

Fig. 4(a).

$$\boldsymbol{P}(\tau) =$$

$$\begin{pmatrix}
\cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \dfrac{-4\gamma_c \sin(4\tau+\theta)}{1-\gamma_c \cos(4\tau+\theta)} & \dfrac{k}{1-\gamma_c \cos(4\tau+\theta)} & \dfrac{k}{1-\gamma_c \cos(4\tau+\theta)} \\
\dfrac{1}{1-\gamma_L \cos(2\tau+\theta)} & \cdot & \dfrac{-1}{1-\gamma_L \cos(2\tau+\theta)} & \dfrac{-2\gamma_L \sin(2\tau+\theta)}{1-\gamma_L \cos(2\tau+\theta)} & \cdot \\
\cdot & \dfrac{1}{1-\gamma_L \cos(2\tau+\theta)} & \dfrac{-1}{1-\gamma_L \cos(2\tau+\theta)} & \cdot & \dfrac{-2\gamma_L \sin(2\tau+\theta)}{1-\gamma_L \cos(2\tau+\theta)}
\end{pmatrix}.$$

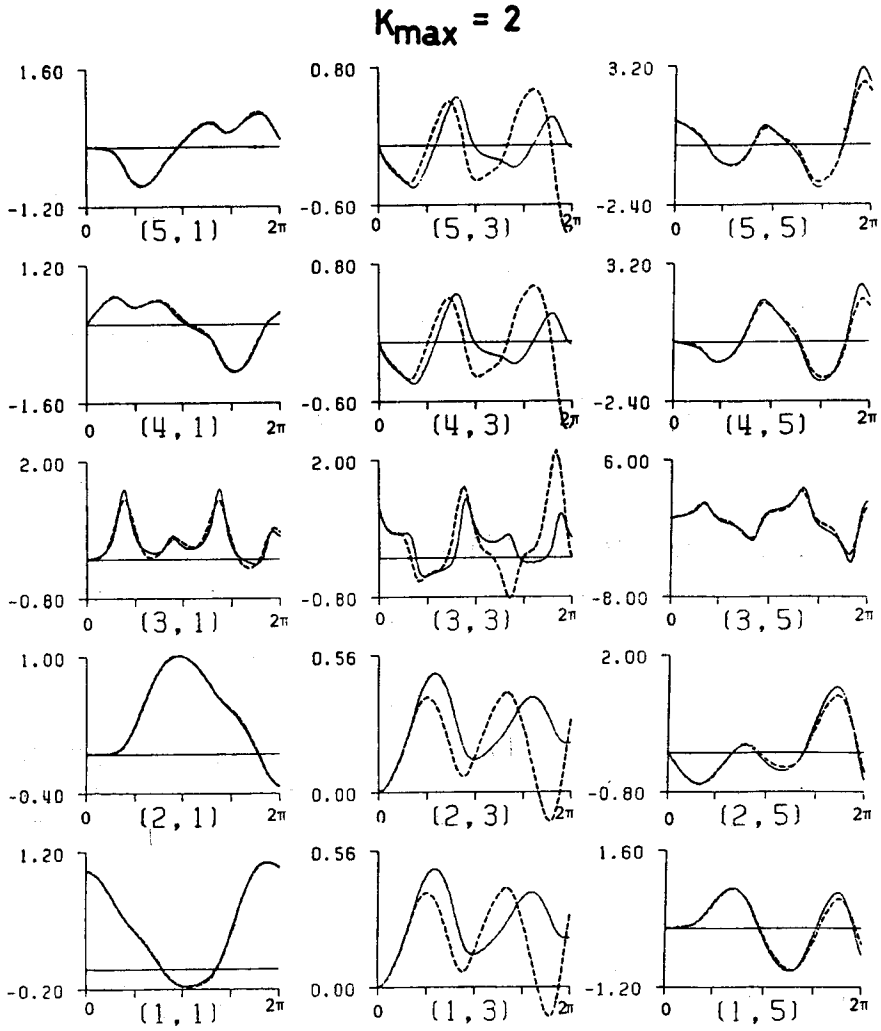

Fig. 4(b).

As an example, we take $k=1.5$, $\gamma_L=0.2$, $\gamma_c=0.6$ and $\theta=\phi=\pi/2$. The computational results are shown in Fig. 4 for each order of approximation. The real and dashed lines show the results by the RKG method and the perturbation method, respectively. In this example, we don't need to use the multistep perturbation method. As shown in the figures, the order of approximation is higher, and both results are in good agreement.

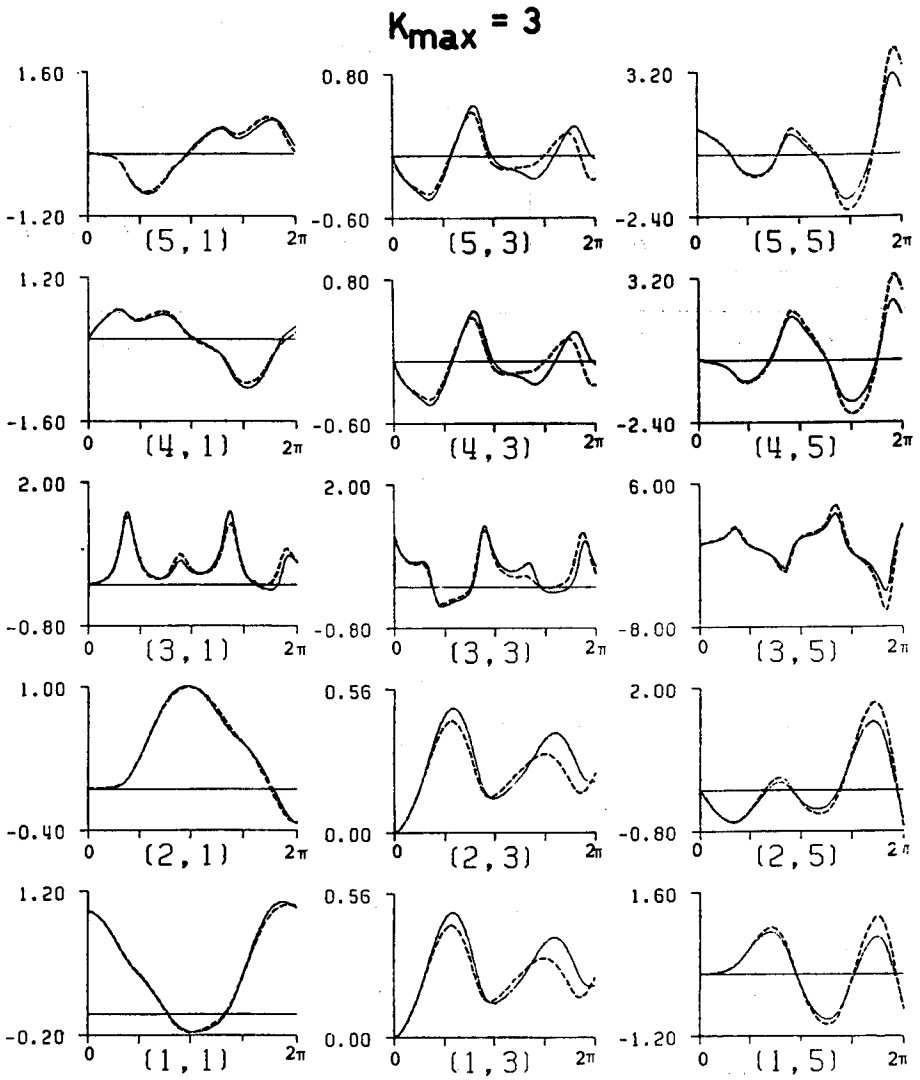

Fig. 4(c).

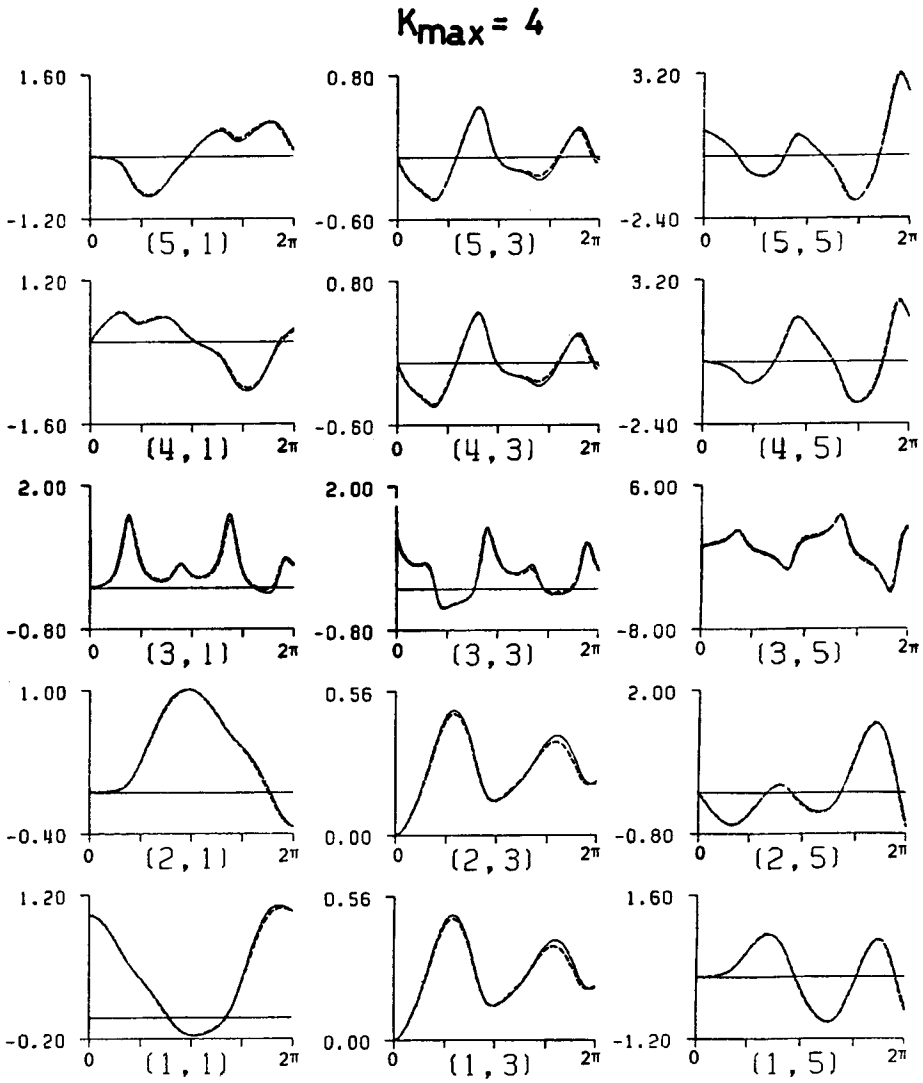

Fig. 4(d).

7. Conclusion

We have discussed the perturbation method for analysing the parametrically excited system of the higher order, and have shown the algorithm for the numerical computation. By this algorithm, we can know the wave form of the solution as well as the stability of the system. Furthermore, we have proposed the multistep perturbation method. This method is effective when the parameters defined as the perturbation terms are large. When we apply the multistep perturbation

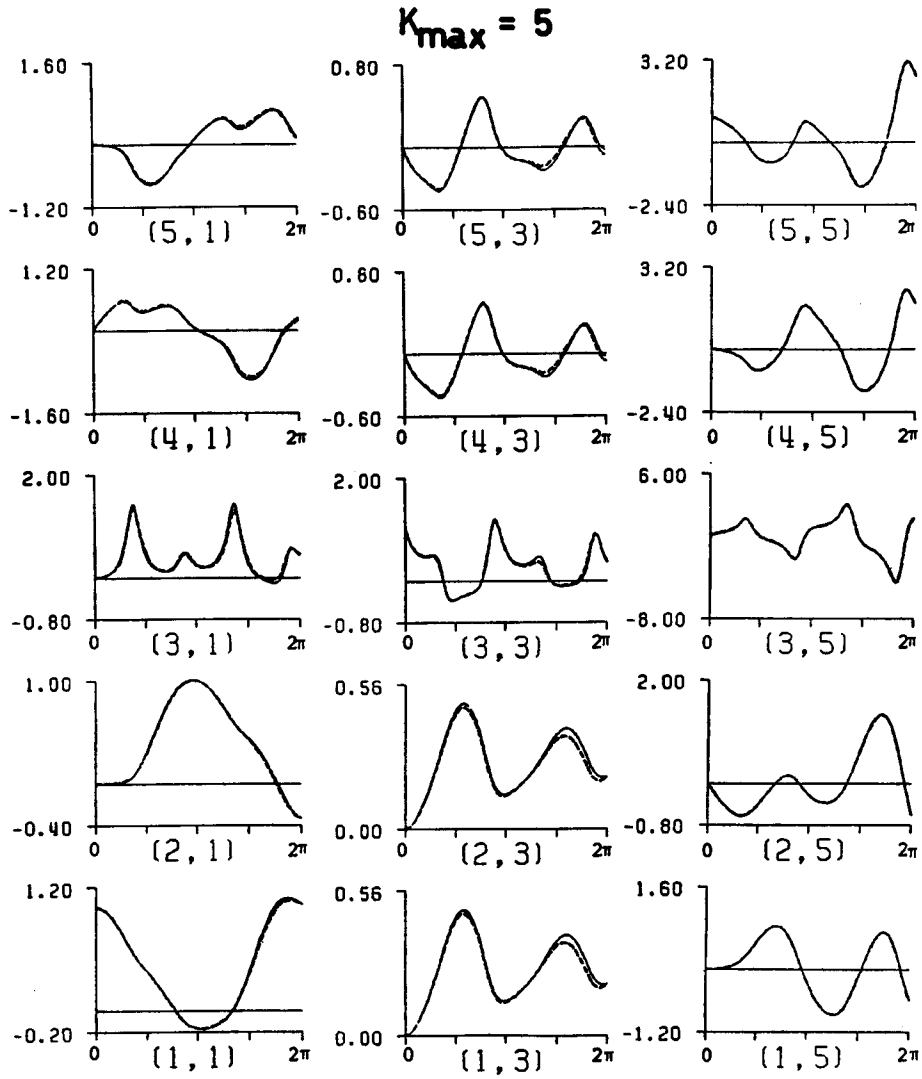

Fig. 4(e).

Fig. 4. The wave forms of $\boldsymbol{X}(\tau)$. The symbol $[i, j]$ denotes the ij element of $\boldsymbol{X}(\tau)$.

method, we must determine the step sizes of the parameters so that the errors of the solution may be as small as possible.

For all numerical examples we have used the computer FACOM-M200 at the Data Processing Center of Kyoto University.

Authors wish to express their gratitudes to a student of Kyoto University, Kenichi Ishii who helped to check our program.

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Appendix

We need some numerical techniques for $\boldsymbol{Z}(\tau)$ and $\boldsymbol{W}$ to be real matrices. If $\boldsymbol{X}(2^m\pi)$ $(m=1, 2, \cdots)$ has negative eigenvalues, then $\log \boldsymbol{X}(2^m\pi)$ $(m=1, 2, \cdots)$ becomes a complex matrix. Considering this, we can rewrite Eq. (2) as

$$\boldsymbol{X}(\tau) = \boldsymbol{Z}_{(m)}(\tau) \exp(\boldsymbol{W}_{(m)}\tau) \qquad m = 1, 2, \cdots \tag{A.1}$$

where $\boldsymbol{Z}_{(m)}(\tau)=\boldsymbol{Z}_{(m)}(\tau+2^m\tau)=\boldsymbol{Z}(\tau)$, $\boldsymbol{Z}_{(m)}(0)=\boldsymbol{1}$ $(m=1, 2, \cdots)$.
The matrix $\boldsymbol{W}_{(m)}$ is given by

$$\begin{aligned}\boldsymbol{W}_{(m)} &= \frac{1}{2^m\pi}\log \boldsymbol{X}(2^m\pi) \\ &\triangleq \boldsymbol{A}_{(m)}+j\frac{1}{2^m}\boldsymbol{B}_{(m)} \qquad m = 1, 2, \cdots\end{aligned} \tag{A.2}$$

where $\boldsymbol{A}_{(m)}$ and $\boldsymbol{B}_{(m)}$ are real commutative matrices. Here, $\boldsymbol{B}_{(m)}$ is of the form

$$\boldsymbol{B}_{(m)} = \boldsymbol{S}\boldsymbol{D}(0, 1)\boldsymbol{S}^{-1}. \tag{A.3}$$

where $\boldsymbol{D}(0, 1)$ is a diagonal matrix, the elements of which are equal to unity at the positions corresponding to the negative eigenvalues. They are zero at all other positions. Therefore, Eq. (A.1) can be written as

$$\boldsymbol{X}(\tau) = \boldsymbol{Z}_{(m+1)}(\tau) \exp(\boldsymbol{A}_{(m)}\tau) \qquad m = 1, 2, \cdots \tag{A.4}$$

where $\boldsymbol{Z}_{(m+1)}$ is a periodic matrix with period $2^{m+1}\pi$ and is given by

$$\boldsymbol{Z}_{(m+1)}(\tau) = \boldsymbol{Z}_{(m)}(\tau) \exp\left(j\frac{1}{2^m}\boldsymbol{B}_{(m)}\tau\right) \qquad m = 1, 2, \cdots. \tag{A.5}$$

We note that Eq. (A.4) can not be uniquely expressed when the eigenvalues of $\boldsymbol{A}_{(m)}$ are complex numbers. Hereafter, we are going to obtain the unique expression. We assume that $\boldsymbol{A}_{(m)}$ has a simple structure. The Jordan canonical form of $\boldsymbol{A}_{(m)}$ is given by

$$\boldsymbol{J}_{(m+1)} = \boldsymbol{S}^{-1}\boldsymbol{A}_{(m)}\boldsymbol{S} = R_e(\boldsymbol{J}_{(m+1)})+jI_m(\boldsymbol{J}_{(m+1)}) \tag{A.6}$$

where $Re(\ )$ and $Im(\ )$ denote the real and imaginary parts of $\boldsymbol{J}_{(m+1)}$, respectively.

Let the imaginary parts of the eigenvalues of $A_{(m)}$ be β_i $(i=1, 2, \cdots, I \leq n)$. We consider the decomposition of β_i such that

$$\beta_i = \frac{a_i}{2^m} + b_i \tag{A.7}$$

where a_i is an integer and b_i is a real number satisfying the inequality

$$|b_i| < \frac{1}{2^{m+1}} \tag{A.8}$$

Substituting Eq. (A.7) into Eq. (A.8), we have

$$2^m \beta_i - \frac{1}{2} < a_i < 2^m \beta_i + \frac{1}{2}. \tag{A.9}$$

From Eq. (A.9), we can determine one and only one integer a_i given by

$$\text{for} \quad \beta_i \geq 0 \qquad a_i = \left[2^m \beta_i + \frac{1}{2} \right] \tag{A.10}_1$$

$$\text{for} \quad \beta_i < 0 \qquad a_i = \left[2^m \beta_i - \frac{1}{2} \right] \tag{A.10}_2$$

where [ ] denotes the Gaussian symbal. Thus, the real number b_i is determined by

$$b_i = \beta_i - \frac{1}{2^m} \left[2^m \beta_i + \frac{1}{2} \right] \qquad \text{for} \quad \beta_i \geq 0 \tag{A.11}_1$$

$$b_i = \beta_i - \frac{1}{2^m} \left[2^m \beta_i - \frac{1}{2} \right] \qquad \text{for} \quad \beta_i < 0 \tag{A.11}_2$$

In section 2.3, including both cases, we define $R_m(\ )$ by

$$R_m(x) = \frac{1}{2^m} \operatorname{sgn}(x) \left[2^m |x| + \frac{1}{2} \right]. \tag{A.12}$$

Therefore, $J_{(m+1)}$ decomposes as

$$J_{(m+1)} = R_e(J_{(m+1)}) + j\hat{J}_{(m+1)} + j\hat{\hat{J}}_{(m+1)} \tag{A.13}$$

where $\hat{J}_{(m+1)}$ is a diagonal matrix, the elements of which correspond to $a_i/2^m$. $\hat{\hat{J}}_{(m+1)}$ is also a diagonal matrix, the elements of which correspond to b_i. Therefore, Eq. (A.4) can be rewritten as

$$X(\tau) = \hat{Z}_{(m+1)}(\tau) \exp(\hat{W}_{(m+1)}\tau) \tag{A.14}$$

where

$$\left.\begin{aligned}\hat{\boldsymbol{Z}}_{(m+1)}(\tau) &= \boldsymbol{Z}_{(m+1)}(\tau)\exp\left(j\boldsymbol{S}\hat{\boldsymbol{J}}_{(m+1)}\boldsymbol{S}^{-1}\tau\right)\\ \hat{\boldsymbol{W}}_{(m+1)} &= \boldsymbol{S}\{\boldsymbol{R}_e(\boldsymbol{J}_{(m+1)})+j\hat{\boldsymbol{J}}_{(m+1)}\}\boldsymbol{S}^{-1}.\end{aligned}\right\} \tag{A.15}$$

As is easily assured, $\hat{\boldsymbol{Z}}_{(m+1)}(\tau)$ is a periodic matrix with the period $2^{m+1}\pi$.

From the above consideration, we have the modification algorithm shown in section 2.3.