Numerical Processing of The Two Dimensional Inverse Laplace Transform and Its Application to Equation of Heat Conduction

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Abstract

The numerical inversion of the Laplace transform is used effectively in many fields where analytical processing is difficult or impossible. The same situation occurs in the two dimensional inverse Laplace transform. To solve such a problem, a numerical processing of the two dimensional inverse Laplace transform is presented. The numerical inversion formulas and their computer algorithms are shown. As an example of the two dimensional inversion method, the equation of heat conduction is analysed for various conditions.

1. Introduction

The Laplace transform is used as a powerful tool to analyse the initial value problems for many systems whose properties are described by linear ordinary differential equations or partial differential equations with constant coefficients.

The reason why the Laplace transform is used so effectively is the fact that the differential operations of the variable to be transformed change to algebraic operations.

By the transformation, the ordinary differential equation changes to an algebraic equation, and the operational solution can be obtained fairly easily. The original function of the operational function is obtained by the inverse Laplace transform.

Today, many transformation pairs are tabulated in several publications, and in many cases the desired original function can be found easily.

By the same transformation, the partial differential equation with two variables changes to the ordinary differential equation of an untransformed variable. The operational solution can be obtained by solving this differential equation, and the original function can be obtained in the same way as before. To solve this equation is slightly difficult compared to the case of the algebraic equation. However, if the

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Laplace transform can be applied to the remaining variable, the procedure to obtain the operational solution can be done algebraically.

Instead of successive applications of the Laplace transform, the above problem can be solved by the application of the two dimensional Laplace transform defined by the double integral[1].

In order to apply the two dimensional Laplace transform to a two variable function $f(x, t)$, it must be defined in the interval $0 \leq x < \infty$, $0 \leq t < \infty$.

Furthermore, in order to solve the two variable partial differential equation algebraically by the two dimensional Laplace transform, strict restriction is imposed on the relation among the initial conditions and the forcing term of the equation. This point is the difference from the one dimensional Laplace transform, and this condition is called the compatible condition.

The principle and the fundamental theorems are detailed in Reference (1). There, many transformation pairs are tabulated. In many cases, when the operational function which satisfies the compatible condition is determined, its original function can be found in the table.

Let us consider the function of three variables of time t and point x, y defined in the interval $0 \leq t < \infty$, $0 \leq x < \infty$ and any finite interval of y. The partial differential equation of the initial value problem with respect to x and t and the boundary value problem with respect to y can be transformed into the ordinary differential equation of y. The boundary value problem is solved by the two dimensional Laplace transform and the operational solution can be obtained.

In many cases, the desired original function can be found in the table, but its form is expressed by the integral or the infinite series of the special functions, and it is difficult to know its property. In some cases, it is impossible to find the transformation pair in the table.

The same situation occurs in the one dimensional Laplace transform. However, in such a case, a sufficiently accurate numerical solution can be obtained by the numerical inversion of the Laplace transform[2),3)].

One purpose of this paper is to extend this method to the two dimensional inverse Laplace transform. Chapter 2 is allocated to this subject and there, numerical inversion formulas and computer algorithms are presented.

The other purpose is to examine the validity of the inversion method by practical application. The equation of heat conduction is a complex example, such as stated above. Chapter 3 is allocated to this subject and there, some examples of inversion and various techniques to obtain numerical solutions are presented. These examples indicate that this method is accurate enough for practical usage.

2. Numerical processing method of the two dimensional inverse Laplace transform[4),5)]

For a two dimensional function $f(x, t)$ which is defined by

$$f(x, t)=\begin{cases} f(x,t) & : 0\leq x<\infty, \quad 0\leq t<\infty \\ 0 & : \text{elsewhere} \end{cases} \tag{1}$$

the two dimensional Laplace transform is defined by the following formula:

$$F(s_1, s_2)=\int_0^\infty\int_0^\infty e^{-s_1x-s_2t}f(x, t)\,dxdt \tag{2}$$

where we call $f(x, t)$ an original function and $F(s_1, s_2)$ its operational function, if above integral exists on the complex s_1-plane and s_2-plane.

For a given operational function $F(s_1, s_2)$ which satisfies the compatible condition, the two dimensional inverse Laplace transfom is carried out by the following successive one dimensional transform.

$$\begin{aligned} f(x, t) &=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{s_1x}\Big[\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}e^{s_2t}\,F(s_1, s_2)\,ds_2\Big]ds_1 \\ &=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}e^{s_2t}\Big[\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{s_1x}\,F(s_1, s_2)\,ds_1\Big]ds_2 \end{aligned} \tag{3}$$

In this formula, $a\geq 0$, $b\geq 0$ are the real parts of the contours on the complex s_1-plane and s_2-plane.

When an operational function is given, its original function is known by the transformation table or by the residue theorem based on Eq. (3). However, in many cases, such an analytical procedure is very complicated or impossible.

In the one dimensional inversion, the numerical processing method is used effectively for such cases.

For the one dimensional inversion formula

$$f(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}F(s)\,e^{st}ds \tag{4}$$

its numerical inversion is carried out by the following formula[2)]:

$$\left.\begin{aligned} & f(t_n)\simeq\frac{e^{at_n}}{T}\Big[R_e\sum_{k=0}^{K-1}F(a+ik\pi/T)\,e^{i2\pi nk/K}-\frac{1}{2}F(a)\Big] \\ & t_n=n\cdot 2T/K \quad n=0,\ 1,\cdots\cdots,\ K-1 \end{aligned}\right\} \tag{5}$$

The subject of this chapter is to extend the numerical method to the two dimensional inverse transform. Now, let us consider the two dimensional inversion method.

For the first inverse transform, s_2 is fixed to a certain value s_{2m}.

Then, by putting $s_1=a+i\omega_1$, the following relation is obtained:

$$F^1(x,\ s_{2m}) = \frac{e^{ax}}{2\pi}\int_{-\infty}^{\infty} F(a+i\omega_1,\ s_{2m})\, e^{i\omega_1 x} d\omega_1 \tag{6}$$

Here, $F^1(x,\ s_{2m})$ is the intermediate inverse transform of $F(s_1,\ s_{2m})$ with respect to s_1.

By the series approximation

$$F^1(x,\ s_{2m}) = \frac{e^{ax}}{2\pi}\sum_{k=-\infty}^{\infty} F(a+ik\Delta\omega_1,\ s_{2m})\, e^{ik\Delta\omega_1 x} \Delta\omega_1 \tag{7}$$

and by putting $\Delta\omega_1 = \pi/X$, the following relation is obtained:

$$\left.\begin{array}{l} F^1(x,\ s_{2m}) + \sum_{n=1}^{\infty} e^{-2anX} F^1(x+2nX,\ s_{2m}) = \frac{e^{ax}}{2X}\sum_{k=-\infty}^{\infty} F(a+ik\pi/X,\ s_{2m})\, e^{ik(\pi/X)x} \\ 0 \leq x < 2X \end{array}\right\} \tag{8}$$

In this formula, the second term of the left hand side can be made sufficiently small by choosing the value of aX suitably, so that the following approximation is obtained.

$$\left.\begin{array}{l} F^1(x,\ s_{2m}) \simeq \frac{e^{ax}}{2X}\sum_{k=-\infty}^{\infty} F(a+ik\pi/X,\ s_{2m})\, e^{ik(\pi/X)x} \\ 0 \leq x < 2X \end{array}\right\} \tag{9}$$

Truncating the infinite series by the sufficiently large term K_1-1, and calculating the values of $F^1(x,\ s_{2m})$ at the K_1-1 points $x_l = l\cdot 2X/K_1$ $l=0,\ 1,\ \cdots\cdots,\ K_1-1$ in the interval $0 \leq x < 2X$, the following approximate relation is obtained.

$$\left.\begin{array}{l} F^1(x_l,\ s_{2m}) \simeq \frac{e^{ax_l}}{2X}\Big[\sum_{k=0}^{K_1-1} F(a+ik\pi/X,\ s_{2m})\, e^{i2\pi lk/K_1} + \sum_{k=0}^{K_1-1} F(a-ik\pi/X,\ s_{2m})\, e^{-i2\pi lk/K_1} \\ \qquad -F(a,\ s_{2m})\Big] \\ x_l = l\cdot 2X/K_1 \quad l=0,\ 1,\ \cdots\cdots,\ K_1-1 \end{array}\right\} \tag{10}$$

It is possible to reduce the execution time in calculation by applying the Fast Fourier Transform (F. F. T) to the series term of the above formula.

For the second inverse transform, the same procedure stated above is applicable. By fixing x to a certain value x_l, for the sequence $\{s_2\}_{k'} = b+ik'\pi/T$ $k'=0,\ 1,\cdots\cdots,\ K_2-1$, the following approximate relation is obtained.

$$\left.\begin{array}{l} f(x_l,\ t_m) \simeq \frac{e^{bt_m}}{2T}\Big[\sum_{k'=0}^{K_2-1} F^1(x_l,\ b+ik'\pi/T)\, e^{i2\pi mk'/K_2} \\ \qquad + \sum_{k'=0}^{K_2-1} F^1(x_l,\ b-ik'\pi/T)\, e^{-i2\pi mk'/K_2} - F^1(x_l,\ b)\Big] \\ t_m = m\cdot 2T/K_2 \quad m=0,\ 1,\ \cdots\cdots,\ K_2-1 \end{array}\right\} \tag{11}$$

Since x_l is real, by using the complex conjugate relation, the above relation becomes as follows:

$$\left.\begin{aligned} f(x_l,\ t_m) &\simeq \frac{e^{bt_m}}{T}\left[R_e \sum_{k'=0}^{K_2-1} F^1(x_l,\ b+ik'\pi/T)\, e^{i2\pi mk'/K_2} - \frac{1}{2}F^1(x_l,\ b)\right] \\ t_m &= m\cdot 2T/K_2 \quad m=0,\ 1,\ \cdots\cdots,\ K_2-1 \end{aligned}\right\} \tag{12}$$

In the numerical calculation using the above formulas, such parameters as a, X and K_1 with respect to $x(s_1)$ and b, T and K_2 with respect to $t(s_2)$ must be determined.

The values of X and T are determined by the necessary intervals to get the numerical solutions. To make the error caused by neglecting the second term of the left hand side of Eq. (8) minimum, aX is selected in the neighborhood of 3. 0. This fact is correct to the value of bT. The values of K_1 and K_2 are determined to make the errors caused by trucating the infinite series by finite in Eqs. (10) and (12) possibly small. In the numerical calculation, $2K_1 \times K_2$ complex regions are necessary to store the complex spectra. It is impossible to make the values of K_1 and K_2 too large by the restrictions imposed on the volume of store memories and execution time. In the numerical examples stated later, the value of $K_1 \times K_2$ is selected as 128×128, and practically sufficient results are obtained.

For the given value $K_1\times K_2$, the same number of numerical values are obtained in the interval $0\leqq x<2X$, $0\leqq t<2T$. However, to avoid the influence of a aliase effect, only the $K_1/2\times K_2/2$ numerical values in the interval $0\leqq x<X$, $0\leqq t<T$ are taken.

Next, we show the computer algorithms based on the above method.

S1: Set the values of the parameters of a, b, X, T, K_1 and K_2.

S2: Calculate the values of the spectra $F(s_1, s_2)$ at the points $s_1=a\pm ik\pi/X$ $k=0,\ 1,\ \cdots\cdots,\ K_1-1$ and $s_2=b+ik'\pi/T$ $k'=0,\ 1,\ \cdots\cdots,\ K_2-1$.

S3: Calculate the intermediate results for every value of s_2 by Eq. (10), using F. F. T.

S4: Calculate the final results for every value of x by Eq. (12), using F. F. T.

S5: Take only the $K_1/2\times K_2/2$ values in the interval $0\leqq x<X$, $0\leqq t<T$.

3. Applications to equation of heat conduction

3.1 Operational solution of equation of heat conduction[1)]

Here we consider a function $u(x,\ t,\ y)$, defined in the interval $0\leqq x<\infty$, $0\leqq t<\infty$ and $y\in D$. A linear partial differential equation of $u(x,\ t,\ y)$, with respect to three variables x, t and y, can be transformed into a linear ordinary differential equation with respect to one variable y by the two dimensional Laplace transform.

We consider the class of functions which satisfies the following two conditions where $\mathscr{L}^2$ means the two dimensional Laplace transform.

$$\text{I} \quad \mathscr{L}^2\left\{\frac{\partial u(x,\ t,\ y)}{\partial y}\right\} = \frac{\partial}{\partial y}\mathscr{L}^2\{u(x,\ t,\ y)\} = \frac{\partial}{\partial y}U\ (s_1,\ s_2,y) \tag{13}$$

$$\text{II} \quad \mathscr{L}^2\{\lim_{y\to R} u(x,\ t,\ y)\} = \lim_{y\to R}\mathscr{L}^2\{u(x,\ t,\ y)\} = \lim_{y\to R} U(s_1,\ s_2,y) \tag{14}$$

Here, R is a boundary point of the interval of D, and the capital letter means the operational form of the function of the small letter.

As an example of such a partial differential equation, we consider the equation of heat conduction. The temperature on the x, y-plane is expressed by the following equation, where t means the time on the assumption that the conductivity is equal to unity.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \qquad u = u(x,\ t,\ y) \tag{15}$$

Let us find out the temperature distribution at the time $t>0$ on the belt-shaped half plane $0 \leq x < \infty$, $0 \leq y < \pi$ (shown in Fig-1).

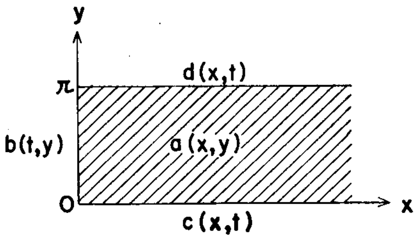


Fig-1 Belt-shaped half plane

The initial conditions are given by

$$\left.\begin{aligned} u(x,\ 0,\ y) &= a(x,\ y) \\ u(0,\ t,\ y) &= b(t,\ y) \end{aligned}\right\} \tag{16}$$

and the boundary conditions are given by

$$\left.\begin{aligned} u(x,\ t,\ 0) &= c(x,\ t) \\ u(x,\ t,\ \pi) &= d(x,\ t) \end{aligned}\right\} \tag{17}$$

Since the differential order with respect to the variable x is 2, the following boundary condition

$$u_x(0,\ t,\ y) = b_1(t,\ y) \tag{18}$$

is necessary, where u_x means the partial derivative about x. Later, this condition can be eliminated by introducing a compatible condition.

By the two dimensional Laplace transform, using the conditions (16), (17) and (18), the following relation is obtained.

$$\{s_1^2U(s_1,\ s_2,\ y)-s_1B(s_2,\ y)-B_1(s_2,\ y)\}-\{s_2U(s_1,\ s_2,\ y)-A(s_1,\ y)\}$$
$$+\frac{d^2U(s_1,\ s_2,\ y)}{dy^2}=0 \tag{19}$$

Let us rewrite the above relation as

$$\left.\begin{aligned}&\frac{d^2U}{dy^2}+(s_1^2-s_2)\,U=G\\&G(s_1\ s_2,\ y)=s_1B(s_2,\ y)+B_1(s_2\ ,y)-A(s_1,\ y)\end{aligned}\right\} \tag{20}$$

Then, the following solution is obtained:

$$U(s_1,\ s_2,\ y)=C(s_1,\ s_2)\frac{\sin\,(\pi-y)\sqrt{s_1^2-s_2}}{\sin\,\pi\sqrt{s_1^2-s_2}}+D(s_1,\ s_2)\frac{\sin y\sqrt{s_1^2-s_2}}{\sin\pi\sqrt{s_1^2-s_2}}$$
$$-\frac{\sin y\sqrt{s_1^2-s_2}}{\sqrt{s_1^2-s_2}\,\sin\pi\sqrt{s_1^2-s_2}}\int_0^y G(s_1,\ s_2,\ \xi)\ \sin\xi\sqrt{s_1^2-s_2}d\xi$$
$$-\frac{\sin y\sqrt{s_1^2-s_2}}{\sqrt{s_1^2-s_2}\,\sin\pi\sqrt{s_1^2-s_2}}\int_y^\pi G(s_1,\ s_2,\ \xi)\ \sin(\pi-\xi)\sqrt{s_1^2-s_2}d\xi \tag{21}$$

The process to obtain the final result is detailed in Reference (1), so we give its outline.

Since the factors $s_1=\sqrt{s_2+n^2}$ $n=0,\ 1,\ \cdots\cdots$ disturb the regularity of the above solution, they must be eliminated from the solution. That is to say, in order for the above solution to be a two dimensional Laplace transform of $u(x,\ t,\ y)$, the numerator must contain the same factors. Herce, the following relation must be kept.

$$-C(\sqrt{s_2+n^2},\ s_2)\,(-1)^n\sin ny+D(\sqrt{s_2+n^2},\ s_2)\ \sin ny$$
$$+\frac{(-1)^n}{n}\sin ny\int_0^\pi G(\sqrt{s_2+n^2},\ s_2,\ \xi)\ \sin n\xi d\xi=0 \tag{22}$$

This relation is nothing but the compatible condition, and by rewriting

$$B_1(s_2,\ y)=\frac{2}{\pi}\sum_{n=1}^{\infty}nC(\sqrt{s_2+n^2},\ s_2)\ \sin ny-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n nD(\sqrt{s_2+n^2},\ s_2)\ \sin ny$$
$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\sin ny\int_0^\pi A(\sqrt{s_2+n^2},\ \xi)\ \sin n\xi d\xi-\frac{2}{\pi}\sum_{n=1}^{\infty}\sqrt{s_2+n^2}\sin ny\int_0^\pi B(s_2,\ \xi)\sin n\xi d\xi \tag{23}$$

an additional condition $B_1(s_2,\ y)$ can be eliminated and the operational solution can be obtained perfectly.

The procedure of its derivation is very complicated, so we show the final result because there is not enough space for a detailed description.

$$\left.\begin{aligned}&U_A(s_1,\ s_2,\ y)=-\frac{2}{\pi}\sum_{n=1}^{\infty}\sin ny\int_0^\pi\frac{A(s_1,\ \xi)-A(\sqrt{s_2+n^2},\ \xi)}{s_1^2-(s_2+n^2)}\sin n\xi d\xi\\&U_B(s_1,\ s_2,\ y)\ =\ \frac{2}{\pi}\sum_{n=1}^{\infty}\sin ny\int_0^\pi\frac{B(s_2,\xi)}{s_1+\sqrt{s_2+n^2}}\sin n\xi d\xi\end{aligned}\right|$$

$$\left.\begin{aligned} U_C(s_1, s_2, y) &= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{C(s_1, s_2)-C(\sqrt{s_2+n^2}, s_2)}{s_1^2-(s_2+n^2)}n\sin ny \\ U_D(s_1, s_2, y) &= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{D(s_1, s_2)-D(\sqrt{s_2+n^2}, s_2)}{s_1^2-(s_2+n^2)}n\sin n(\pi-y) \\ U(s_1, s_2, y) &= U_A+U_B+U_C+U_D \end{aligned}\right\} \tag{24}$$

Analytical solutions of the original functions for U_A, U_B, U_C and U_D are shown in Reference (1), but they are very complex functions which contain integrals or infinite series of special functions. It is difficult to know their properties, so we obtain their numerical solutions by the method stated in the previous chapter.

3.2 Numerical Processing I (Operational function does not contain integral.)

Among the operational solutions given by Eq. (24), U_C and U_D do not contain any integral, although U_A and U_B contain an integral. For some initial conditions, integrals can be done and can be transformed into forms which do not contain an integral. Hence, the numerical processing of inversion can be made easier than before.

(i) Inversion of $U_B(s_1, s_2, y)$

This term is caused by a thermal source, and can be written as

$$U_B(s_1, s_2, y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sin ny\int_0^{\pi}\frac{B(s_2, \xi)}{s_1+\sqrt{s_2+n^2}}\sin n\xi d\xi \tag{25}$$

Here, we consider a case of $b(t, y)=1$, then $B(s_2, y)=1/s_2$. The integral in the above equation can be done analytically, obtaining the following solution:

$$U_B(s_1, s_2, y) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin[(2m-1)y]}{(2m-1)s_2[s_1+\sqrt{s_2+(2m-1)^2}]} \tag{26}$$

The original function can be known by its numerical inversion.

The numerical solutions are shown in Fgi-2, where the values of the used parame-ters are $X=2.56$, $T=3.2$, $aX=3.0$, $bT=3.0$ and $K_1=K_2=128$. These are the same for all the successive examples.

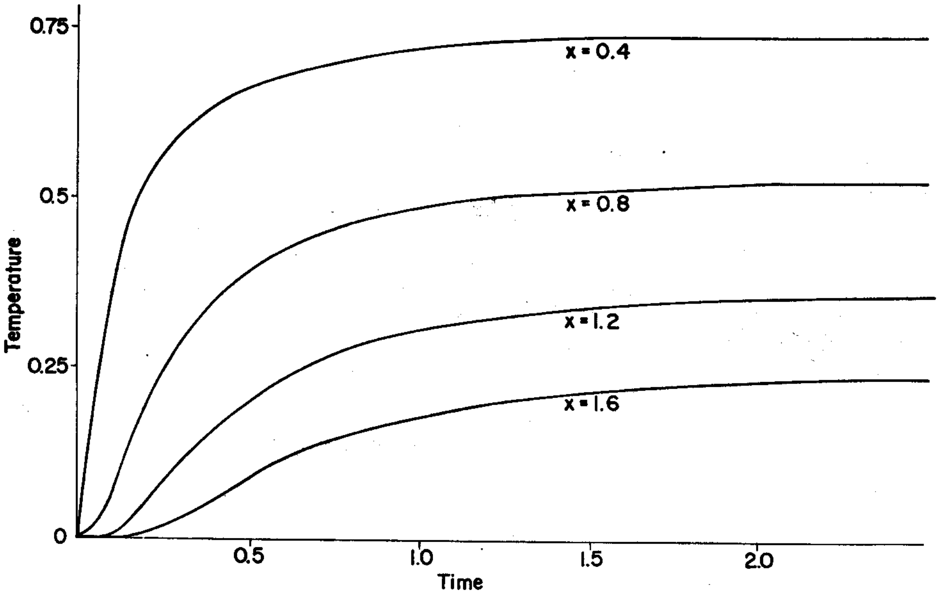
The infinite series of Eq. (25) is truncated by 50 terms, based on the result of previous inspection.

(ii) Inversion of $U_A(s_1, s_2, y)$

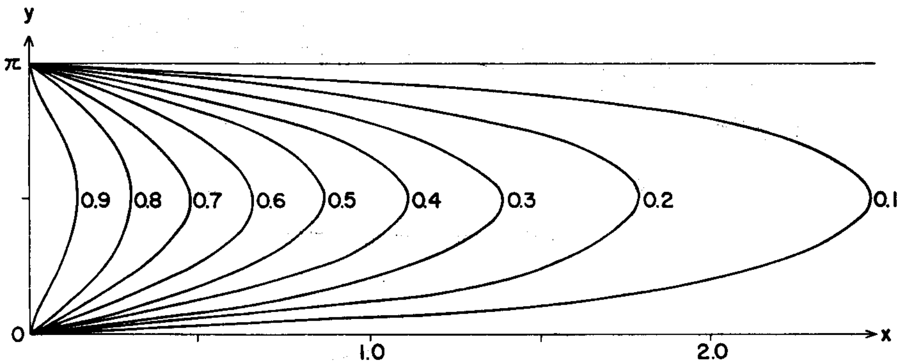
This term is caused by the initial distribution of the temperature, and can be written as

$$U_A(s_1, s_2, y) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\sin ny\int_0^{\pi}\frac{A(s_1, \xi)-A(\sqrt{s_2+n^2}, \xi)}{s_1^2-(s_2+n^2)}\sin n\xi d\xi \tag{27}$$

Here, we consider a case of $a(x, y)=e^{-0.5(x+y)}$, then $A(s_1, y)=e^{-0.5y}/(s_1+0.5)$, and obtain the following result.



(a) Temperature at $y=\pi/2$



(b) Temperature distribution at $t=3.0$

Fig-2 Inversion of U_B $B=1/s_2$

$$\left.\begin{aligned} U_A(s_1, s_2, y) &= -\frac{2}{\pi}\sum_{n=1}^{\infty}\sin ny\frac{1}{s_1^2-(s_2+n^2)}\left(\frac{1}{s_1+0.5}-\frac{1}{\sqrt{s_2+n^2}+0.5}\right)I \\ I &= \int_0^{\pi} e^{-0.5\xi}\sin n\xi d\xi = \begin{cases} \dfrac{n}{0.25+n^2}(1-e^{-0.5\pi}) & n:\ \text{even} \\ \dfrac{n}{0.25+n^2}(1+e^{-0.5\pi}) & n:\ \text{odd} \end{cases} \end{aligned}\right\} \quad (28)$$

The numerical solutions by inversion are shown in Fig-3, where the above infinite series is truncated by 20 terms.

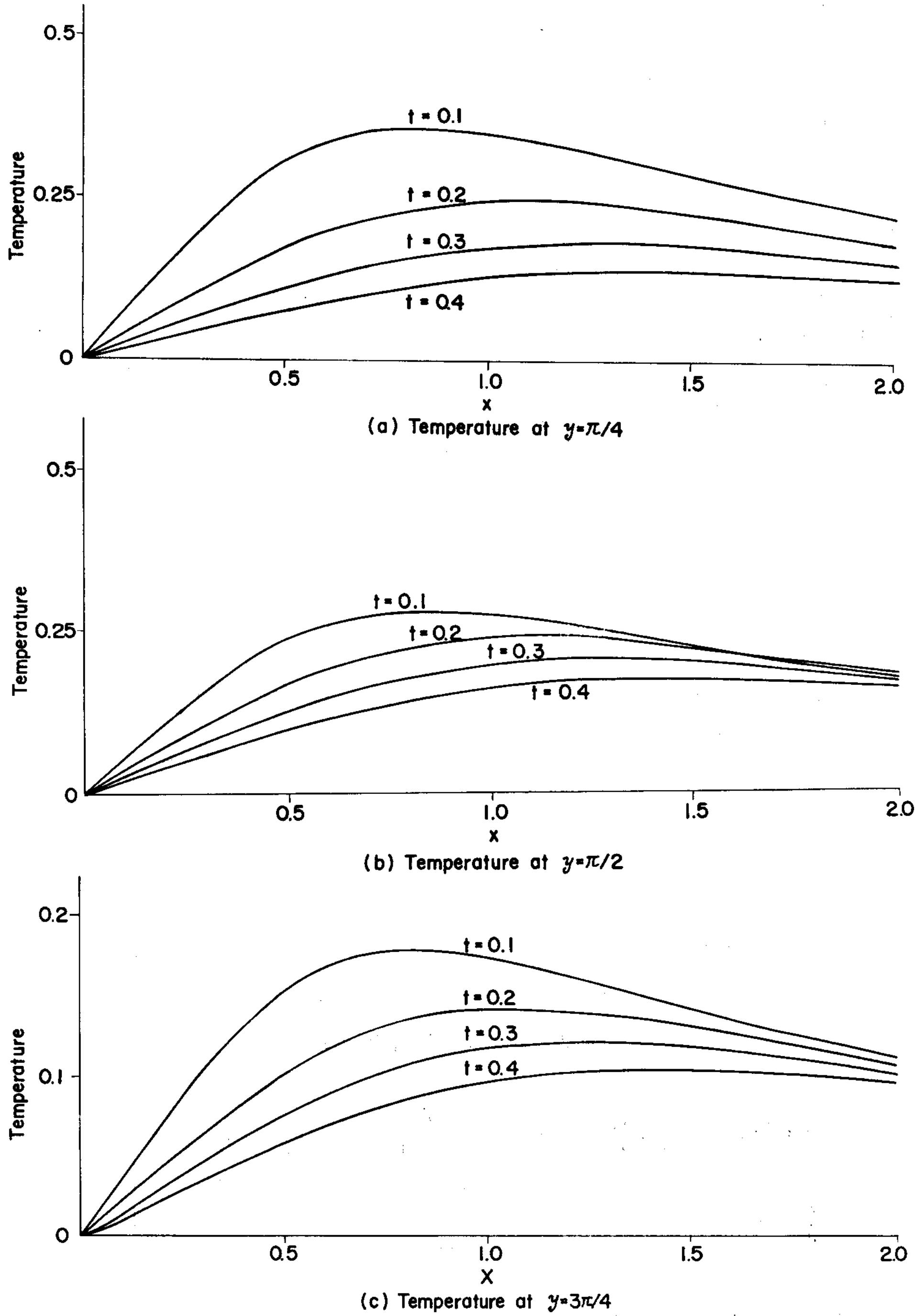


Fig-3 Inversion of U_A $A=e^{-0.5y}/(s_1+0.5)$

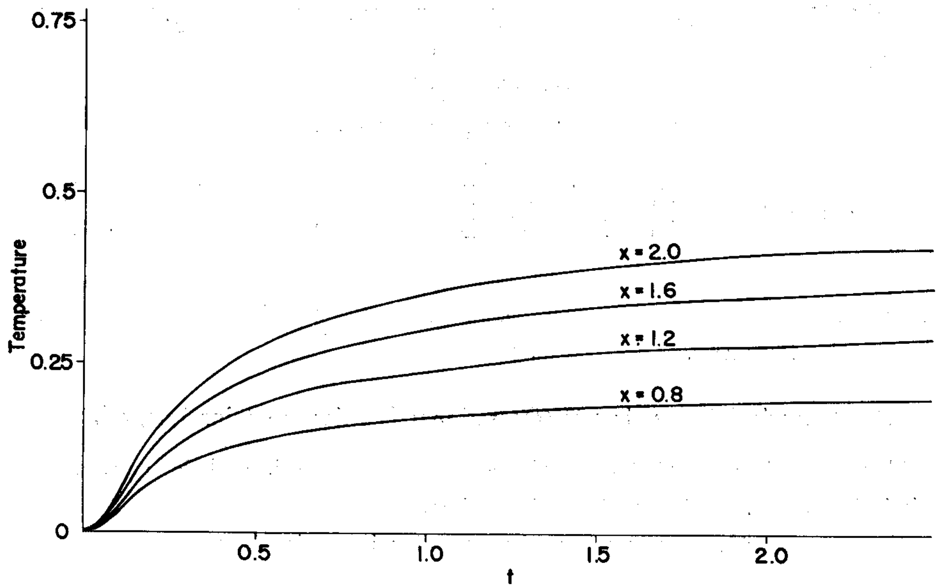
(iii) Inversion of $U_C(s_1, s_2, y)$ and $U_D(s_1, s_2, y)$

These terms are caused by the initial distributions of the temperature on the boundaries, and can be written as

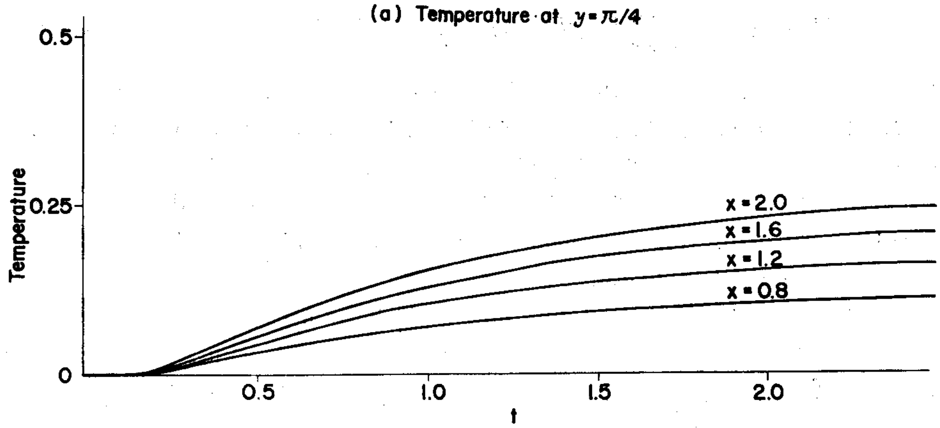
$$U_C(s_1, s_2, y) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{C(s_1, s_2) - C(\sqrt{s_2+n^2}, s_2)}{s_1^2-(s_2+n^2)}\, n \sin ny \tag{29}$$

$$U_D(s_1, s_2, y) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{D(s_1, s_2) - D(\sqrt{s_2+n^2}, s_2)}{s_1^2-(s_2+n^2)}\, n \sin n(\pi-y) \tag{30}$$

Since the above equations are symmetrical, about $y=\pi/2$, we only do the inversion of U_C. Then, we consider a case of $c(x, t) = 1-e^{-0.5x}$, then $C(s_1, s_2) = 0.5/s_1s_2(s_1+0.5)$, and obtain the following result.



(a) Temperature at $y=\pi/4$



(b) Temperature at $y=\pi/2$

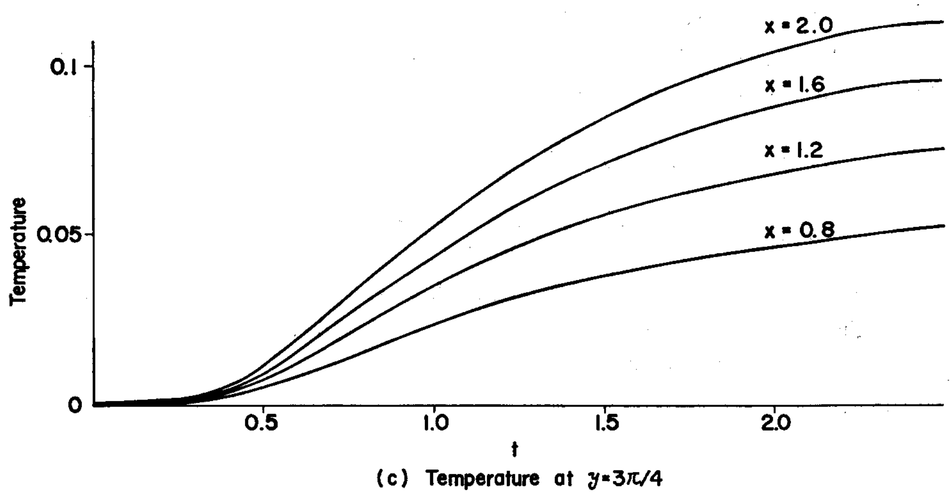


Fig-4 Inversion of U_C $C=0.5/s_1s_2(s_1+0.5)$

$$U_C(s_1, s_2, y) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n \sin ny}{s_2[s_1^2-(s_2+n^2)]}\left[\frac{1}{s_1(s_1+0.5)}-\frac{1}{\sqrt{s_2+n^2}(\sqrt{s_2+n^2}+0.5)}\right] \tag{31}$$

The numerical solutions by inversion are shown in Fig-4, where the above series is truncated by 100 terms.

3.3 Numerical Processing II (Operational function contains integral.)

The general form of U_A or U_B contains an integral, and in many cases, an analytical processing of integration is impossible.

Here, we consider a case where the operational function is given by

$$U(s_1, s_2, y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sin nyF(s_1, s_2, n)\int_0^{\pi}Y(\xi)\sin n\xi d\xi \tag{32}$$

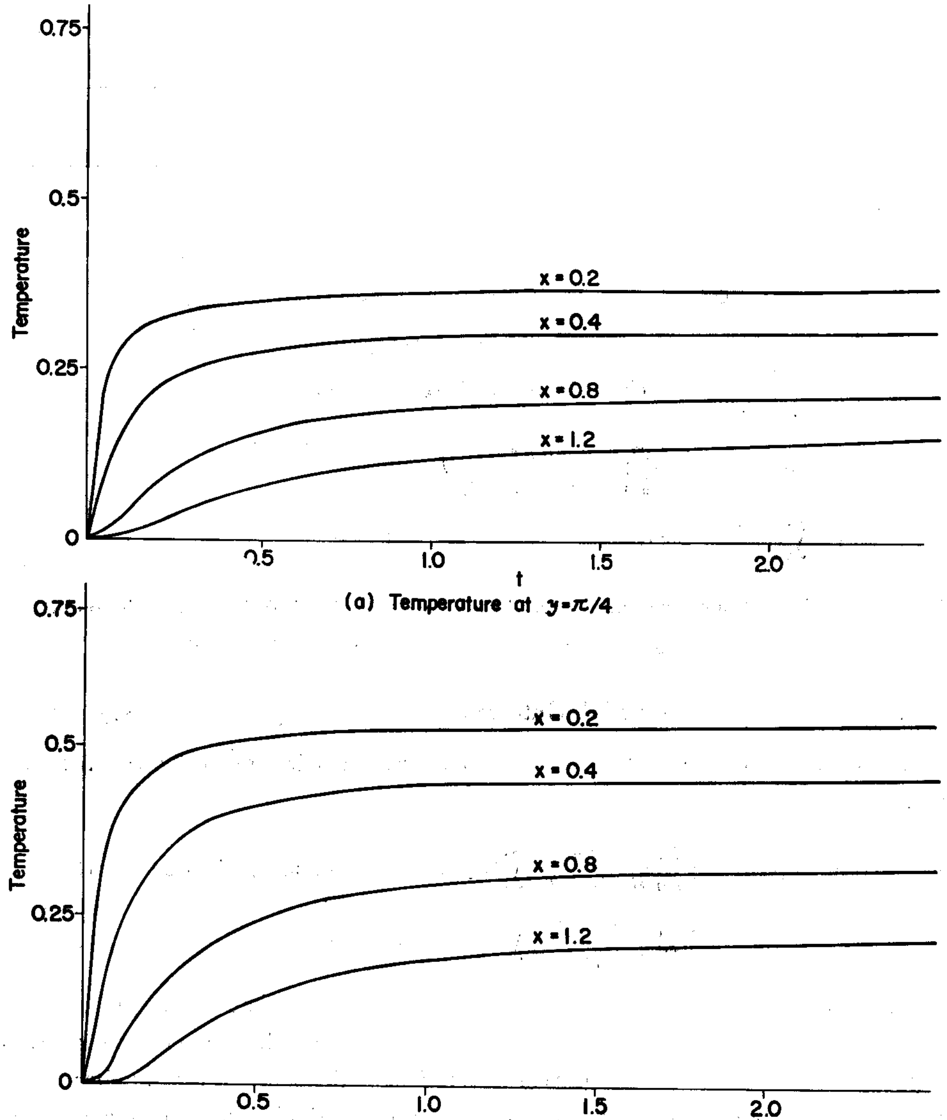
where $Y(\xi)$ does not contain the complex variables s_1 and s_2.

To obtain the spectra in the numerical inversion, we carry out a numerical integration by the Gauss integration formula which gives the value of a definite integral by the sum of the products of weight and the value of the integrand at the selected points in the interval. The sequences of weight and the selected point are tabulated for the number of selected points.

The integrand is the form of $Y(\xi)\sin n\xi$, which oscillates heavily as n increases. Therefore, it is necessary to increase the number of selected points. By example, for an integral $\int_0^{\pi}\sin n\xi d\xi$ $n=49$, the relative error is about 0.1 % for $i=48$ and 0.015 % for $i=100$, where i is the number of selected points. This fact indicates that this numerical integration can be regarded valid when i is selected about twice that of n. However, to fix the value of i, large forces unnecessarily increase the calculation for the small value of n. Since the total execution time is affected, we control the value of i by the value of n.

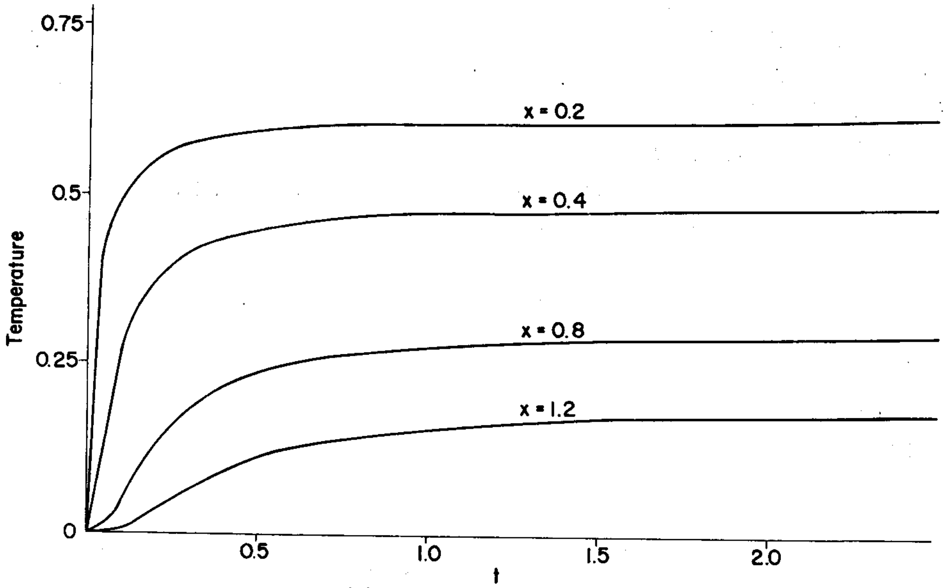
Inversion of $U_B(s_1, s_2, y)$

The complete operational function is given by Eq. (25). Now, let us consider a case of $b(t, y)=\sqrt{y/4}$, then $B(s_2, y)=\sqrt{y/4}/s_2$ and the obtained function is as follows.



(a) Temperature at $y=\pi/4$

(b) Temperatuer at $y=\pi/2$



(c) Temperature at $y=3\pi/4$

Fig-5 Inversion of U_B $B=\sqrt{y/4}/s_2$

$$\left.\begin{aligned} U_B(s_1, s_2, y) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin ny}{s_2(s_1+\sqrt{s_2+n^2})}I \\ I &= \int_0^{\pi}\sqrt{\frac{\xi}{4}}\sin n\xi d\xi \end{aligned}\right\} \quad (33)$$

The numerical solutions are shown in Fig-5, where the infinite series is truncated by 50 terms, and the values of I are calculated by Gaussian integration.

3.4 Numerical Prcessing III (Integrand contains complex variables.)

The operational functions of U_A and U_B are generally given by the following form:

$$U_A(s_1, s_2, y) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin ny}{s_1^2-(s_2+n^2)}\int_0^{\pi}\{F_1(s_1, \xi)-F_2(s_2, n, \xi)\}\sin n\xi d\xi \quad (34)$$

$$U_B(s_1, s_2, y) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin ny}{s_1+\sqrt{s_2+n^2}}\int_0^{\pi}G(s_2, \xi)\sin n\xi d\xi \quad (35)$$

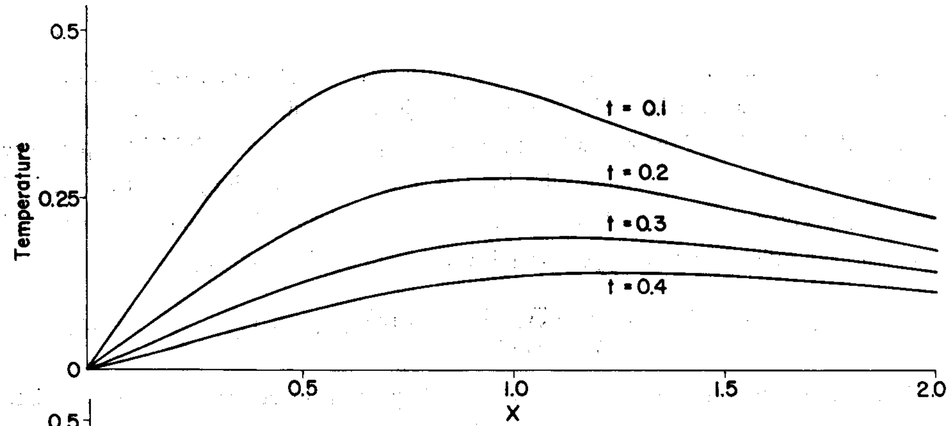
In these cases, numerical integration can be carried out the same as in the previous case. In the previous case, the integrand does not contain the complex variables s_1 and s_2. Also, the time of the numerical integrations is the same as the number of the terms of the truncated finite series. However, complex variables s_1 and s_2 cannot be separated from the integrands in Eqs. (34) and (35), so the integration must be done for every pair of $s_1 \times s_2$, whereby the total number of integrations becomes very large. For example, the inversion of Eq. (34) needs 5120 integrations when the series is truncated by 20 terms. However, this number can be reduced by special values of y where some terms vanish, for example $\sin n\pi/4 = 0$ for $n = 4m$ $m = 1, 2, \cdots\cdots$.

Inversion of $U_A(s_1, s_2, y)$

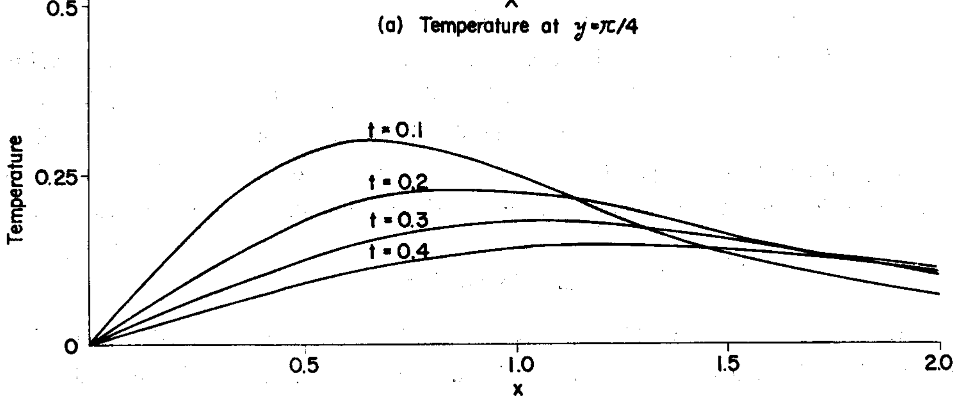
The complete operational function is given by Eq. (27). Now, let us consider a case of $a(x, y) = e^{-xy}$, then $A(s_1, y) = 1/(s_1 + y)$, and the operational function is obtained as follows.

$$\left.\begin{aligned} U_A(s_1, s_2, y) &= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin ny}{s_1^2-(s_2+n^2)} I \\ I &= \int_0^{\pi}\frac{1}{s_1+\xi}\sin n\xi d\xi - \int_0^{\pi}\frac{1}{\sqrt{s_2+n^2}+\xi}\sin n\xi d\xi \end{aligned}\right\} \quad (36)$$

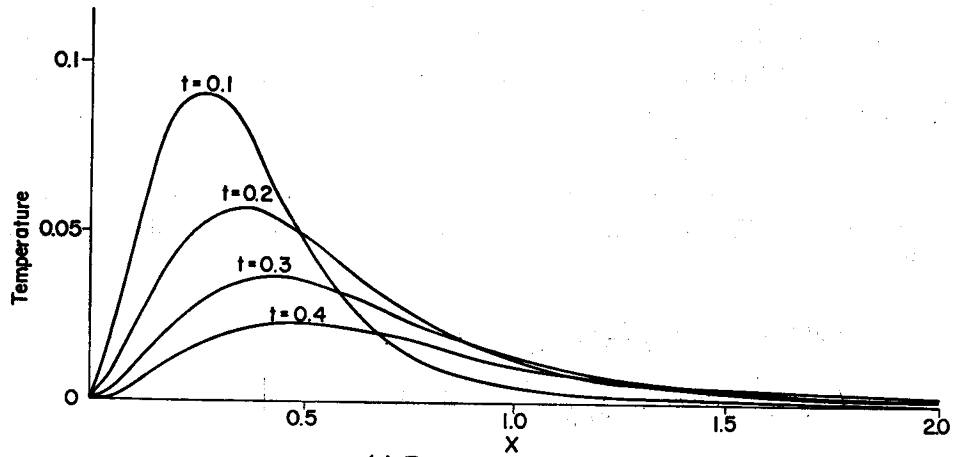
The numerical solutions are shown in Fig-6, where the series is truncated by 20 terms. The process of inversion is the same as in the previous case, except for doing too many integrations.



(a) Temperature at $y=\pi/4$



(b) Temperature at $y=\pi/2$



(c) Temperature at $y=3\pi/4$

Fig-6 Inversion of U_A $A=1/(s_1+y)$

3.5 Accuracy of numerical solution

As mentioned in 3.1 the analytical solutions of $u_a(x, t, y)$, $u_b(x, t, y)$, $u_c(x, t, y)$ and $u_d(x, t, y)$ are shown in Reference (1), but it is difficult to know their properties. However, the inversion of the operational function U_A, caused by the initial temperature distrfbution, can be done analytically by the residue calculation, and the original function u_a is given by the simple form when operational function $A(s_1, y)$ is a special form. For example, when $A(s_1, y)=1/s_1$ $[a(x, y)=1]$ u_a is given by

$$u_a(x, t, y)=\frac{4}{\pi}\ \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\sum_{n=0}^{\infty}\frac{1}{2n+1}\ e^{-(2n+1)^2 y}\sin(2n+1)y \tag{37}$$

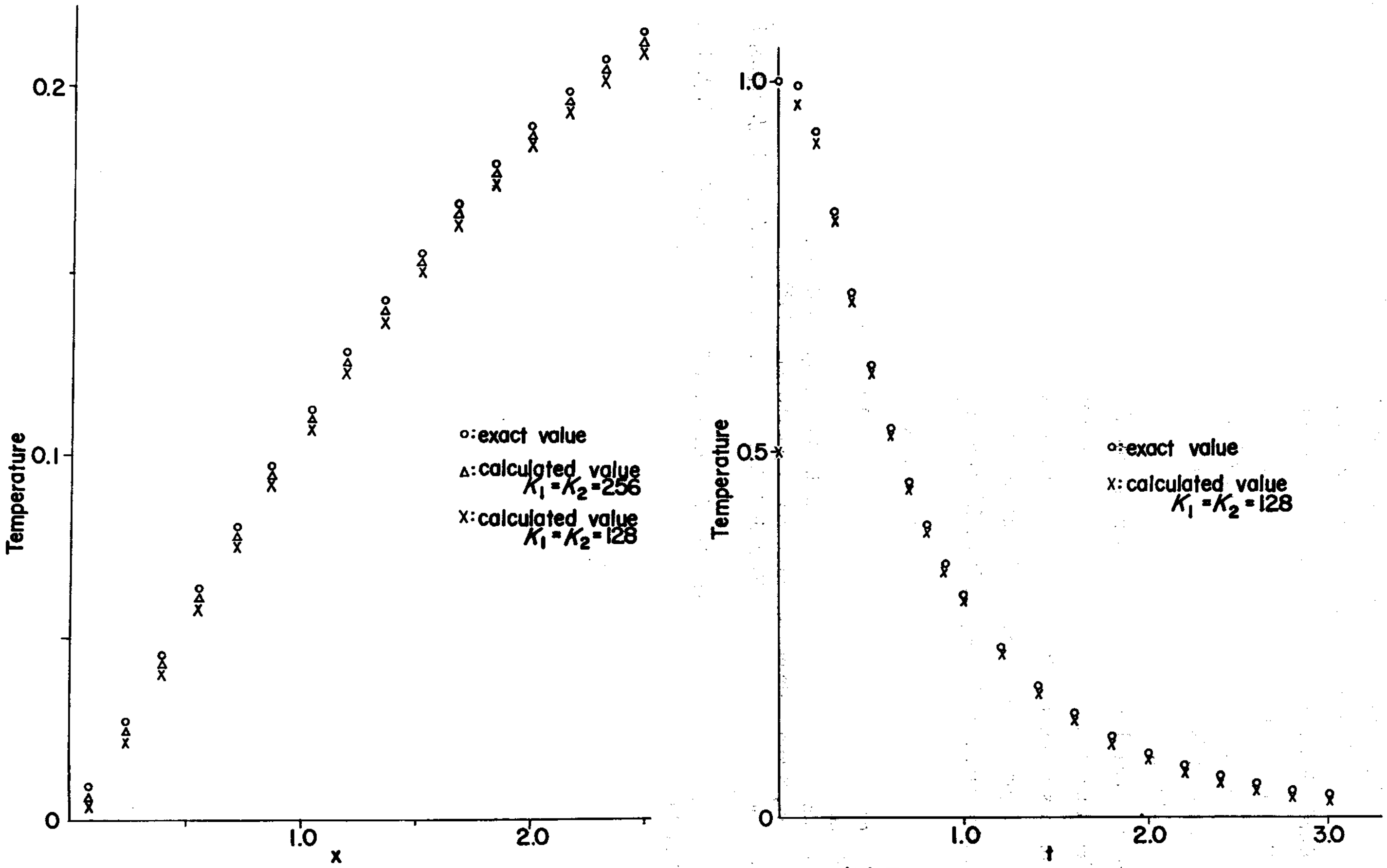
Where the value of the term decreases rapidly as n increases, we can calculate the exact values of the function at given points.

In Fig-7 we compare the differences at some points between the exact value and the numerical value obtained by the inversion of U_A at the point $y=\pi/2$.

This result shows that the two dimensional numerical inversion method gives practically sufficient accuracy. The process of inversion for other examples is based on the same policy, and their accuracy is regarded as being the same as in this case.

4. Conclusion

One of the purposes of this paper is to extend the numerical inversion method to the two dimensional inverse Laplace transform. The numerical inversion formulas and their computer algorithms are shown.



(a) Temperature at $t=1.6$ and $y=\pi/2$

(b) Temperature at $x=1.28$ and $y=\pi/2$

Fig–7 Difference between exact value and calculated value

The other purpose is to apply this method to practical problems. As an example, the temperature distribution on the belt-shaped half plane is considered. The temperature distribution is expressed by the partial differential equation of three variables. Its operational solution is a very complicated function, which contains integrals and the infinite series. An analytical processing of such an inversion is difficult or impossible.

Therefore, this problem provides a good opportunity to apply the numerical inversion method.

The numerical solutions for various conditions are obtained, and it may be concluded that the numerical processing of the two dimensional inverse Laplace transform has enough accuracy for practical usage.

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