State Estimation of Jump Parameter Systems with State-Dependent Observation Noise

By

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Abstract

A least-squares state estimation algorithm is obtained for a general class of stochastic systems having 1) system nonlinearity; 2) an unknown jump parameter and 3) state-dependent observation noise. The algorithm developed is consistent in the sense that for each special case with the properties 1) and 2) or 3), it reduces to the algorithm the authors have already developed. Illustrative examples of numerical computation are given for better understanding of the result.

1. Introduction

The object of this paper is to present an easily calculable method of least-squares state estimation for a general class of stochastic systems. The system for which we consider the state estimation is "general" in the following sense:

i) The system is nonlinear in the system state.
ii) The system has an unknown, abruptly changing parameter.
iii) The statistics of the observation noise are explicitly dependent on the system state.

For each class of systems having one of the properties i)-iii) but not all of them, there are a large number of studies.[1)-15)] As far as the authors are aware, however, there is no state estimation algorithm which is applicable for all the situations i)-iii).

For systems which are nonlinear in the system state, many studies have been devoted to obtain finite dimensional state estimators. These studies are summarized in the textbook of Jazwinski.[5)] On the other hand, for class ii), the exact and approximated filters have been studied rather independently of the nonlinear filtering problem. Nahi[6)] considered the filtering problem for the linear systems with time-varying uncertainty of the signals in the observation. Similar problems were discussed by Jaffer and Gupta[7),8)] via Bayesian approach. Ackerson and Fu[9)] considered the case where the statistics of the noise process

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change suddenly by the dependence on a finite state Markov chain. These problems were treated by Takeuchi[10),11)] for continuous-time systems in a unified formulation.

In contrast to the case of additive white Gaussian noise, there are a relatively small number of studies dealing with case iii), especially for continuous-time systems. For linear continuous-time systems, McLane[13)] obtained a linear filter by restricting the filter-structure to a linear class with respect to the observation. The corresponding result for nonlinear systems was obtained by Sunahara et al.[14)] The least-squares suboptimal filter for nonlinear systems was given by Takeuchi and Akashi[15)].

The approach used in this paper is based on the general nonlinear filtering formula developed by Takeuchi and Akashi[15)]. A feasible state estimation algorithm is constructed by employing the extended stochastic linearization technique developed by Takeuchi[10)]. Also, the results of the numerical experiments are shown to demonstrate the applicability of the algorithm.

Throughout this paper, column vectors are denoted by lower case letters, and matrices are denoted by capital letters. The prime denotes the transpose of a vector or a matrix. The Euclidean norm is denoted by $|\cdot|$. The trace of square matrix A is tr$[A]$. If A is nonsingular, A^{-1} denotes the inverse matrix of A. If A and B are symmetric, then $[A, B] \triangleq \sum_{i \leq j} a_{ij} b_{ij}$ and $\|A\| \triangleq \{[A, A]\}^{1/2}$. The triplet $(\Omega, \mathcal{F}, P)$ is a complete probability space. $E\{\cdot\}$ and $E\{\cdot \mid \mathcal{G}\}$, $\mathcal{G} \subset \mathcal{F}$ denote respectively the expectation and the conditional expectation, given $\mathcal{G}$, with respect to P. $\sigma\{\cdot\}$ is the minimal sub-σ-algebra of $\mathcal{F}$ with respect to which the family of $\mathcal{F}$-measurable random variables $\{\cdot\}$ is measurable.

2. Problem Formulation

We are concerned with the following dynamical system having an abruptly changing parameter α_t which takes values in $\boldsymbol{L} \triangleq \{1, 2, \cdots, l\}$.

$$(1) \qquad \begin{cases} dx_t = f(t, x_t, \alpha_t)dt + k(t, x_{t-}, \alpha_{t-})dn_t + G(t, x_t, \alpha_t)dw_t \\ x_0 = x^o, \qquad 0 \leq t \leq T, \end{cases}$$

where $x_t \in \boldsymbol{R}^n$ is the state vector; $w_t \in \boldsymbol{R}^{d_1}$ is a standard Brownian motion process; $x^o \in \boldsymbol{R}^n$ is a Gaussian random vector independent of α_t with mean $\hat{x}_0$ and covariance matrix Q_0; and n_t is a right-continuous counting process[16)] denfined by

$$(2) \qquad n_t \triangleq \{\text{number of jumps of } \alpha_s,\ 0 \leq s \leq t \text{ such that } i \rightarrow j,\ j \in S_i,\ i \in \boldsymbol{L}\},$$

and is adapted to

(3) $$\mathcal{G}_t \triangleq \sigma\{x^0, (w_s, \alpha_s), 0 \leq s \leq t\} .$$

The set S_i in (2) is a subset in $\boldsymbol{L}$-$\{i\}$ for each $i \in \boldsymbol{L}$. The transition of random parameter α_t is dependent on the state process x_t and has the property

(C-1) $$\lim_{s \downarrow 0} \frac{1}{s} \{P(t+s, j \mid t, x, i) - \delta_{ij}\} = \sigma_{ij}(t, x) \qquad \text{uniformly in } (t, x) ,$$

where

$$P(t+s, j \mid t, x, i) \triangleq P\{\alpha_{t+s} = j \mid (x_t, \alpha_t) = (x, i)\} .$$

It should be noted that due to the second term in the right-hand side of (1), the state process x_t is discontinuous at the time when n_t increases, namely, when α_t makes a jump: $i \rightarrow j$ such that $j \in S_i$, $i \in \boldsymbol{L}$. Clearly, combined process (x_t, α_t), $0 \leq t \leq T$ is a right-continuous Markov process with values in $\boldsymbol{R}^n \times \boldsymbol{L}$.

All the information which can be obtained about (x_t, α_t) is assumed to be contained in the observation y_s, $0 \leq s \leq t$ generated by

(4) $$\begin{cases} dy_t = h(t, x_t, \alpha_t)dt + R(t, x_t, \alpha_t)dv_t \\ y_0 = 0 , \qquad 0 \leq t \leq T , \end{cases}$$

where $v_t \in \boldsymbol{R}^{d_2}$ is a standard Brownian motion process independent of x^0, $\{w_t; 0 \leq t \leq T\}$ and $\{\alpha_t; 0 \leq t \leq T\}$. An important feature of (4) is that noise coefficient R is dependent on (x_t, α_t) and, therefore, the additive noise is the so-called state-dependent noise.

The state estimation problem is to compute the least-squares estimate of the state vector x_t based on the data y_s, $s \leq t$ given by (4), i.e., to compute the conditional expection:

(5) $$\hat{x}(t) \triangleq E\{x_t \mid \mathcal{Y}_t\} ,$$

where $\mathcal{Y}_t$ is the σ-algebra:

(6) $$\mathcal{Y}_t \triangleq \sigma\{y_s; 0 \leq s \leq t\} .$$

Introducing

(7) $$\hat{x}(t \mid i) \triangleq E\{x_t \mid \alpha_t = i, \mathcal{Y}_t\} , \qquad i \in \boldsymbol{L}$$

and

(8) $$\Pi(t, i) \triangleq P\{\alpha_t = i \mid \mathcal{Y}_t\} , \qquad i \in \boldsymbol{L} ,$$

we have the representation:

(9) $$\hat{x}(t) \triangleq \sum_{i=1}^{l} \hat{x}(t \mid i) \Pi(t, i) .$$

We will consider the problem of computing $\{\hat{x}(t \mid i), \Pi(t, i), i \in \boldsymbol{L}\}$ in order to compute $\hat{x}(t)$. For the purpose, we will assume the following conditions:

(C-2) For all $i\in \boldsymbol{L}$, $\Pi_0(i)>0$

(C-3) For all $i,j\in \boldsymbol{L}$ and $x\in \boldsymbol{R}^n$, $\sigma_{ij}(\cdot, x)$ is continuous on $[0,T]$

(C-4) For all $i,j\in \boldsymbol{L}$, $\sigma_{ij}(\cdot,\cdot)$ is the bounded function on $[0,T]\times \boldsymbol{R}^n$ such that either $\sigma_{ij}(\cdot,\cdot)\equiv 0$ or $0<|\sigma_{ij}(\cdot,\cdot)|\leq M$

(C-5) For all $i\in \boldsymbol{L}$, $f(\cdot,\cdot,i)$, $k(\cdot,\cdot,i)$, $G(\cdot,\cdot,i)$, $h(\cdot,\cdot,i)$ and $R(\cdot,\cdot,i)$ are continuous functions on $[0,T]\times \boldsymbol{R}^n$ satisfying the Lipschitz and the linear growth conditions

(C-6) There exists a positive constant C such that $\det|R_0(t,x,i)|\geq C$ for all $t\in[0,T]$ and $(x,i)\in \boldsymbol{R}^n\times \boldsymbol{L}$, where $R_0\triangleq RR'$

(C-7) For all $i,j\in \boldsymbol{L}$, and $t\in[0,T]$, the function k_{ji}^t: $\boldsymbol{R}^n\rightarrow \boldsymbol{R}^n$ defined by

$$k_{ji}^t(x)\triangleq x+\tilde{\chi}(j,i)k(t,x,i)\,, \qquad x\in \boldsymbol{R}^n$$

is one-to-one and invertible.

In (C-7), $\tilde{\chi}(i,j)$ is the indicator function given by

$$\tilde{\chi}(i,j)\triangleq\begin{cases}1\,, & j\in S_i\\ 0\,, & j\notin S_i\,,\end{cases} \qquad i\in \boldsymbol{L}\,.$$

3. Fundamental Formulas

In this section, we will give fundamental formulas for the estimation problem in the form of the stochastic differential equations. First, let us define process Z_t by

$$Z_t\triangleq\nu_t\nu_t'-\int_0^t\nu_s d\nu_s'-\left(\int_0^t\nu_s d\nu_s'\right)'\,,$$

where

$$\nu_t=y_t-\int_0^t\hat{h}(s)ds$$

and

$$\hat{h}(s)\triangleq E\{h(s,x_s,\alpha_s)\,|\,\mathcal{Y}_s\}\,.$$

Then, Z_t is the quadratic covariation process of the square integrable martingale ν_t, and is given by

$$dZ_t=R_0(t,x_t,\alpha_t)dt\,,\qquad Z_0=0\,. \tag{10}$$

As we have pointed out[15], this process Z_t plays an important role in the state estimation problem with the state-dependent observation noise, i.e., the optimal estimate is given in terms of the conditional distribution of $\{x_t;0\leq t\leq T\}$ subject to $\{Z_s;s\leq t\}$. Because of the singularity of this conditional distribution with respect to the a priori distribution of $\{x_t; 0 \leq t \leq T\}$, it is difficult to compute the optimal estimate $\hat{x}(t)$. An effective suboptimal scheme[15] is to approximate the conditional distribution subject to $\{Z_s; s \leq t\}$ by the one subject to $\{Z_s^\varepsilon, s \leq t\}$ defined by

$$Z_t^\varepsilon = Z_t + \sqrt{\varepsilon}\, V_t, \quad \varepsilon > 0, \tag{11}$$

where V_t is an $m \times m$-dimensional symmetric matrix-valued process whose triangular components are mutually independent standard Brownian motion processes. The straight-forward generalization of the result in Takeuchi and Akashi[15] yields the following statement:

Theorem 1. Let $\phi: \boldsymbol{R}^n \times \boldsymbol{L} \rightarrow \boldsymbol{R}$ be such that $E|\phi(x_t, \alpha_t)|^2 < \infty$ for all t. Then, approximate estimate $\hat{\phi}_t^\varepsilon$ of $\phi_t \triangleq \phi(t_t, \alpha_t)$ having the property

$$\hat{\phi}_t^\varepsilon \xrightarrow{\varepsilon \downarrow 0} E\{\phi(x_t, \alpha_t) \mid \mathcal{Y}_t\} \text{ in } P,$$

satisfies the stochastic differential equation:

$$d\hat{\phi}_t^\varepsilon = \widehat{\mathcal{A}\phi_t}^\varepsilon dt + \widehat{\phi_t \{h_t - \hat{h}^\varepsilon(t)\}'}^\varepsilon \{\hat{R}_0^\varepsilon(t) + \Gamma(\varepsilon)\}^{-1} \{dy_t - \hat{h}^\varepsilon(t)dt\} \tag{12}$$
$$+ (\varepsilon)^{-1}[(\widehat{\phi_t R_0}^\varepsilon - \hat{\phi}_t^\varepsilon \hat{R}_0^\varepsilon), (dZ^\varepsilon - \hat{R}_0^\varepsilon dt)],$$

where $h_t \triangleq h(t, x_t, \alpha_t)$, $R_0 \triangleq R_0(t, x_t, \alpha_t)$ and

$$\mathcal{A}\psi(t, x, i) \triangleq \sum_{j=1}^{l} \sigma_{ji}(t, x)\psi(t, k_{ji}^l(x), j) + f'(t, x, i)\psi_x(t, x, i) \tag{13}$$
$$+ \frac{1}{2}\, tr[G'(t, x, i)\psi_{xx}(t, x, i)G(t, x, i)],$$

and where ψ_x and ψ_{xx} respectively denote the gradient vector and the Hessian matrix of ψ. □

Based on (12) in Th. 1, we will derive fundamental formulas for the present problem. Let $\chi_i(j)$ be the indicator function on $\boldsymbol{L}$ defined by

$$\chi_i(j) = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i, \end{cases} \qquad i, j \in \boldsymbol{L}. \tag{14}$$

Then, by taking $\phi_t = \chi_i(\alpha_t)$ in (12), we have the following stochastic differential equation for $\Pi(t, i) \triangleq \widehat{\chi_i(\alpha_t)}^\varepsilon$†.

† It would be better to write $\Pi^\varepsilon(t, i)$, since this is an approximation of $\Pi(t, i)$. For the sake of simplicity, however, we will use this notation.

$$(15)\quad \begin{cases} d\Pi(t,i) = \sum_{j=1}^{l} \hat{\sigma}_{ji}(t\,|\,j)\Pi(t,j)dt \\ \qquad +\Pi(t,i)\{\hat{h}(t\,|\,i)-\hat{h}(t)\}'\{\hat{R}_0(t)+\Gamma(\varepsilon)\}^{-1}\{dy_t-\hat{h}(t)dt\} \\ \qquad +(\varepsilon)^{-1}\Pi(t,i)[\{\hat{R}_0(t\,|\,i)-\hat{R}_0(t)\},\ \{dZ_t^\varepsilon-\hat{R}_0(t)dt\}] \\ \Pi(0,i) = \Pi_0(i), \qquad t\in[0,T], \qquad i\in \boldsymbol{L}, \end{cases}$$

where

$$(16)\qquad \hat{\cdot}(t\,|\,i) \triangleq \hat{\cdot}^{(i)} \ \text{ and } \ \hat{\cdot}^{(i)} \triangleq \frac{\widehat{(\cdot)\chi_i(\alpha_t)}^\varepsilon}{\Pi(t,i)}.$$

On the other hand, taking $\phi_t=\rho(x_t)\chi_i(\alpha_t)$ in (12) and applying it with (15) and

$$(17)\qquad \hat{\rho}(t\,|\,i) = \frac{\widehat{\rho(x_t)\chi_i(\alpha_t)}^\varepsilon}{\Pi(t,i)},$$

it follows from Itô's stochastic differential formula that

$$(18)\qquad \begin{aligned} d\hat{\rho}(t\,|\,i) &= \sum_{j=1}^{l}\frac{\Pi(t,j)}{\Pi(t,i)}\{\widehat{\rho(k_{ji}^t)\sigma_{ij}}^{(j)}-\hat{\sigma}_{ji}(t\,|\,j)\hat{\rho}(t\,|\,i)\}dt \\ &+\widehat{\rho_x' f}^{(i)}dt+\frac{1}{2}\widehat{tr[G'\rho_{xx}G]}^{(i)}dt \\ &+\{\widehat{\rho h}^{(i)}-\hat{\rho}(t\,|\,i)\hat{h}(t\,|\,i)\}'\{\hat{R}_0(t)+\Gamma(\varepsilon)\}^{-1}\{dy_t-\hat{h}(t\,|\,i)dt\} \\ &+(\varepsilon)^{-1}[\{\widehat{\rho R_0}^{(i)}-\hat{\rho}(t\,|\,i)\hat{R}_0(t\,|\,i)\},\ \{dZ_t^\varepsilon-\hat{R}_0(t\,|\,i)dt\}]. \end{aligned}$$

Equations (15) and (18) are the basic (approximated) formulas for the estimation problem. Although the convergence of these estimates to the optimal estimates is guranteed by Th. 1, (15) and (18) generate an infinite dimensional filter. In order to obtain a finite dimensional filter, it is necessary to apply a certain moment truncation technique.

4. Derivation of a State Estimation Algorithm

In this section, we will construct a feasible state estimation algorithm by applying the extended stochastic linearization technique developed by Takeuchi[10]. According to this method, a nonlinear function is expanded into a linear from by selecting the coefficients in such a way that the mean square error due to the expansion becomes minimum. The result shows that

$$(19a)\qquad f(t,x_t,i)\cong\hat{f}(t\,|\,i)+F_2(t,i)\{x_t-\hat{x}(t\,|\,i)\}$$

$$(19b)\qquad k(t,x_t,i)\cong\hat{k}(t\,|\,i)+K_2(t,i)\{x_t-\hat{x}(t\,|\,i)\}$$

$$(19c)\qquad h(t,x_t,i)\cong\hat{h}(t\,|\,i)+H_2(t,i)\{x_t-\hat{x}(t\,|\,i)\}$$

(19d) $$G(t, x_t, i) \simeq \hat{G}(t|i) + \sum_{j=1}^{n} G_j(t, i)\,\{x_t^j - \hat{x}^j(t|i)\}$$

(19e) $$R(t, x_t, i) \simeq \hat{R}(t|i) + \sum_{j=1}^{n} R_j(t, i)\,\{x_t^j - \hat{x}^j(t|i)\}$$

and

(19f) $$\sigma_{ji}(t, x_t) \simeq \hat{\sigma}_{ji}(t|j) + \eta'_{ji}(t)\,\{x_t - \hat{x}(t|j)\}\ ,$$

where

(20) $$F_2(t, i) \triangleq \widehat{[f(t, x_t, i) - \hat{f}(t|i)]\,[x_t - \hat{x}(t|i)]'}^{(i)} Q^{-1}(t|i)\ ,$$

(21) $$Q(t|i) \triangleq \widehat{[x_t - \hat{x}(t|i)]\,[x_t - \hat{x}(t|i)]'}^{(i)}$$

and the other coefficients K_2, H_2, G_j, R_j and σ_{ji} are given by a form similar to (20). For the details of the extended stochastic linearization technique, see Takeuchi[10] and Akashi and Takeuchi[12].

Using (19) to (18), and taking $\rho(x_t) = x_t^k, k = 1, 2, 3, \cdots, l$, we can easily obtain

(22) $$\begin{cases} d\hat{x}(t|i) = \xi(t|i)dt + \hat{f}(t|i)dt \\ \qquad + Q(t|i)H_2'(t, i)\,\{\hat{R}_0(t) + \Gamma(\varepsilon)\}^{-1}\{dy_t - \hat{h}(t|i)dt\} \\ \qquad + (\varepsilon)^{-1}Q(t|i)d\lambda(t, i) \\ \hat{x}(0|i) = \hat{x}_0\ , \qquad t \in [0, T]\ , \qquad i \in \boldsymbol{L}\ , \end{cases}$$

where

(23) $$\xi(t|i) \triangleq \sum_{j=1}^{l} \frac{\Pi(t, j)}{\Pi(t, i)}\,\{\hat{\sigma}_{ji}(t|j)c_{ji}(t) + \tilde{Q}(t|j, i)\eta_{ji}(t)\}\ ,$$

(24) $$[d\lambda(t, i)]_k \triangleq [\{\hat{R}(t|i)R_k'(t, i) + R_k(t, i)\hat{R}'(t|i)\},\ \{dZ_t^* - \hat{R}_0(t|i)dt\}]\ , \quad (k = 1, 2, 3, \cdots, n)\ ,$$

(25) $$c_{ji}(t) \triangleq \hat{x}(t|j) + \tilde{\chi}(j, i)\hat{k}(t|j) - \hat{x}(t|i)\ ,$$

(26) $$\tilde{Q}(t|j, i) \triangleq [I + \tilde{\chi}(j, i)K_2(t, j)]Q(t|j)\ ,$$

and

(27) $$\hat{R}_0(t|i) = \sum_{k,l}^{n} [Q(t|i)]_{kl}R_k(t, i)R_l'(t, i) + \hat{R}(t|i)\hat{R}'(t|i)\ .$$

For $Q(t|i)$, since we can write

$$dQ(t|i) = \widehat{dx_t x_t'}^{(i)} - d\{\hat{x}(t|i)\hat{x}'(t|i)\}\ ,$$

it follows from (18), (19) and (22) that

(28) $$\begin{cases} dQ(t|i) = D(t|i)dt + F_2(t, i)Q(t|i)dt + Q(t|i)F_2'(t, i)dt \\ \qquad + \hat{G}_0(t|i)dt - Q(t|i)H_2'(t, i)\,\{\hat{R}_0(t) + \Gamma(\varepsilon)\}^{-1}H_2(t, i)Q(t|i)dt \\ \qquad - (\varepsilon)^{-1}Q(t|i)\,\{\textstyle\sum(t|i)dt - d\Lambda(t|i)\}\,Q(t|i) \\ Q(0|i) = Q_0\ , \qquad t \in [0, T]\ , \qquad i \in \boldsymbol{L}\ , \end{cases}$$

where

$$(29)\quad D(t\,|\,i) \triangleq \sum_{j=1}^{l} \frac{\Pi(t,j)}{\Pi(t,i)} [\hat{\sigma}_{ji}(t\,|\,j)\, \{Q^*(t\,|\,j,i) - Q(t\,|\,i) + c_{ji}(t)c'_{ji}(t)\} + \tilde{Q}(t\,|\,j,i)\eta_{ij}(t)c'_{ji}(t) + c_{ji}(t)\eta'_{ji}(t)\tilde{Q}'(t\,|\,j,i)]\,,$$

$$(30)\quad [dA(t\,|\,i)]_{kl} \triangleq [\{R_k(t,i)R'_l(t,i) + R_l(i,t)R'_k(t,i)\},\ \{dZ_t^e - \hat{R}_0(t\,|\,i)dt\}]\,, \quad (k, l = 1, 2, 3, \cdots, n)\,,$$

$$(31)\quad [\textstyle\sum(t\,|\,i)]_{kl} \triangleq [\{R_k(t,i)\hat{R}(t\,|\,i) + \hat{R}(t\,|\,i)R'_k(t,i)\},\ \{R_l(t,i)\hat{R}'(t\,|\,i) + \hat{R}(t\,|\,i)R'_k(t,i)\}]\,, \quad (k, l = 1, 2, 3, \cdots, n)\,,$$

and

$$(32)\quad Q^*(t\,|\,j,i) \triangleq [I + \tilde{\chi}(j,i)K_2(t,j)]Q(t\,|\,j)\,[I + \tilde{\chi}(j,i)K_2(t,j)]'\,,$$

Here note that the coefficients $\hat{f}(t\,|\,i)$, $F_2(t,i)$, $\hat{k}(t\,|\,i)$, $K_2(t,i)$, $\hat{h}(t\,|\,i)$, $H_2(t,i)$, $\hat{G}(t\,|\,i)$, $G_j(t,i)$, $\hat{R}(t\,|\,i)$, $R_j(t,i)$, $\hat{\sigma}_{ji}(t\,|\,j)$ and $\eta_{ji}(t)$ $(i, j \in \boldsymbol{L})$ are given as functions of $\hat{x}(t\,|\,i)$ and $Q(t\,|\,i)$[10]. Then, since

$$\hat{R}_0(t) = \sum_{i=1}^{l} \hat{R}_0(t\,|\,i)\,\Pi(t,i)$$

and

$$\hat{h}(t) = \sum_{i=1}^{l} \hat{h}(t\,|\,i)\,\Pi(t,i)\,,$$

we can compute $\{\Pi(t,i), \hat{x}(t\,|\,i), Q(t\,|\,i);\ i \in \boldsymbol{L}\}$ recursively by (15), (22) and (28). Hence, the suboptimal state estimate $\hat{x}(t)$ is given by (9). Thus, we have constructed a suboptimal state estimation algorithm.

Remark 1. For the case of linear systems where Eqs. (1) and (4) are linear in x_t, and σ_{ij} does not depend on x_t, we can construct a truncated moment estimator by setting

$$\widehat{[x_t - \hat{x}(t\,|\,i)]\,[x_t - \hat{x}(t\,|\,i)]'[x_t - \hat{x}(t\,|\,i)]^{k(i)}} = 0\,, \quad (k = 1, 2, \cdots, n)\,.$$

The result, however, coincides with the above algorithm, and the coefficients are directly given by the following relations: $\hat{f}(t\,|\,i) = f(t, \hat{x}(t\,|\,i), i)$, $F_2(t,i) = \left(\frac{\partial}{\partial x}\right)' f(t,x,i)$ and etc.

Remark 2. This algorithm, which is applicable in a wide class of state estimation problems, is consistent with the specializations. Namely, for each special case treated by Takeuchi[10] and Takeuchi and Akashi[15], the above algorithm coincides with the algorithm, developed by them. For the digital computer implementation of the state estimation, therefore, we only have to prepare a computer program of the above algorithm.

5. Numerical Examples

Consider the scalar case where the dynamical system and the observation are respectively given by

$$(33)\qquad \begin{cases} dx_t = a(\alpha_t)x_t dt + g dw_t \\ x_0 = x^o, \qquad t\in[0,2] \end{cases}$$

and

$$(34)\qquad \begin{cases} dy_t = h(\alpha_t)x_t dt + r_1 x_t dv_t^1 + r_2 dv_t^2 \\ y_0 = 0, \qquad t\in[0,2], \end{cases}$$

where the coefficients g, r_1 and r_2 are

$$(35)\qquad g = 0.25, \qquad r_1 = 0.15, \qquad r_2 = 0.10.$$

The initial distribution of x_t is given by

$$(36)\qquad \hat{x}_0 = 5.0, \qquad Q_0 = 10.0.$$

In this system, as we have noted, the important point is that (33) and (34) are "the jump parameter system with state-dependent observation noise". At present, there is no other state estimation algorithm applicable to this class of systems. In this section, we will give two examples of the numerical computation for (33) and (34).

Example 1. (System with interrupted observation and state-dependent observation noise). Let α_t be a binary-valued Markov chain which takes values in $\boldsymbol{L}=\{1,2\}$ and has the transition rates:

$$(37)\qquad \begin{cases} \sigma_{11} = -1.0 \qquad \sigma_{12} = 1.0 \\ \sigma_{21} = 1.0 \qquad \sigma_{22} = -1.0. \end{cases}$$

The initial distribution of α_t is given by

$$(38)\qquad \Pi_0(1) = \Pi_0(2) = 0.5.$$

The coefficients $a(i)$ and $h(i)$, $i\in\boldsymbol{L}$ are given by

$$(39)\qquad \begin{cases} a(1) = a(2) = -0.30 \\ h(1) = 4.0 \qquad h(2) = 0.0. \end{cases}$$

In this case, the event $\{\alpha_t=1\}$ describes the normal operation of the observation

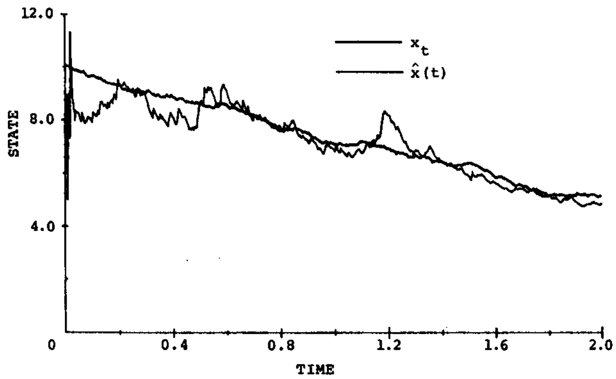

Fig. 1. One sample run results of x_t and $\hat{x}(t)$ (Example 1)

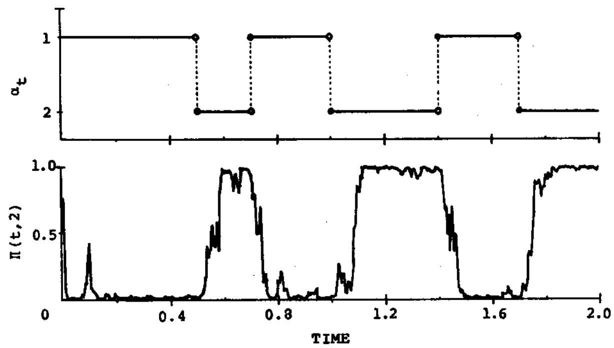

Fig. 2. On the sample run of α_t and the corresponding run of $\Pi(t, 2)$ (Example 1)

mechanism, whereas $\{\alpha_t=2\}$ is the fault situation where the observation signal, i.e., the first term in the right-hand side of (34), is interrupted. In the digital computer simulation, the step size dt is selected as 10^{-3} dividing the interval [0,2] into 2,000 steps.

In Fig. 1, one sample run of x_t for $x_0=10.0$ and the corresponding sample run of the suboptimal state estimate by the proposed algorithm are shown. For the same sample, the trajectories of α_t and $\Pi(t,2)$ are shown in Fig. 2, where we can see that the value of α_t is identified with good accuracy. An interesting fact in Fig. 1 is that even when $\Pi(t,2)$ is nearly equal to 1, for instance $1.1 \leq t \leq 1.4$, the transition of the suboptimal estimate is not so smooth as in the case of the additive wihte Gaussian noise. (See Ex. 5.1 in Takeuchi[10].) This implies that, due to the dependence on x_t of the noise term in (34), there is available information in y_t even when $h(\alpha_t)=0$, and that the proposed algorithm uses this information efficiently.

Example 2. Let $\boldsymbol{L}=\{1,2,3,\cdots,4\}$ and α_t be a Markov chain with values in

$\boldsymbol{L}$ having the transition rates:

$$\sigma_{ij}=\begin{cases}-2.0 & \text{for } i=j\\ 0.0 & \text{for } i+j=5\\ 1.0 & \text{otherwise,}\end{cases}$$

and the initial distribution:

$$\Pi_0(1)=\Pi_0(2)=\Pi_0(3)=\Pi_0(4)=0.25\,.$$

The system parameter $a(i)$ and $h(i)$, $i\in\boldsymbol{L}$ are

$$\begin{array}{llll} a(1)=-2.0 & a(2)=2.0 & a(3)=-2.0 & a(4)=2.0\\ h(1)=\ \ 4.0 & h(2)=4.0 & h(3)=\ \ 0 & h(4)=0\,. \end{array}$$

This case describes the situation where, under the existence of the state-dependent observation noise, abrupt changes of system parameter $a(\alpha_t)$ and interruptions of the observation take place independently.

In Fig. 3, one sample run results of the system state and the suboptimal estimate are given. For the same sample, the true value of the parameter $a(\alpha_t)$ and the a posteriori probability for $a(\alpha_t)=2.0$ are shown in Fig. 4. The true value of $h(\alpha_t)$ and the a posteriori probability for $h(\alpha_t)=0$ are shown in Fig. 5. As we have pointed out[10], the identification of a system parameter is more difficult in the dynamical system than in the observation. Comparing Figs. 4 and 5, we can confirm this property. The filter performance in Fig. 3, however, is good enough for application.

By these results, and by comparing them with the results for the case of the additive white Gaussian noise, one may well say that the performance of the pro-

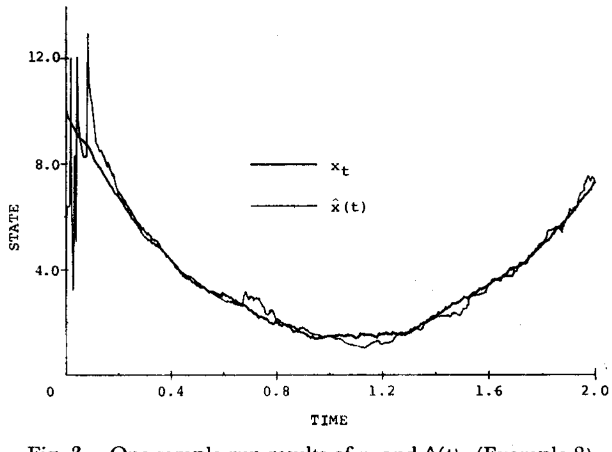

Fig. 3. One sample run results of x_t and $\hat{x}(t)$ (Example 2)

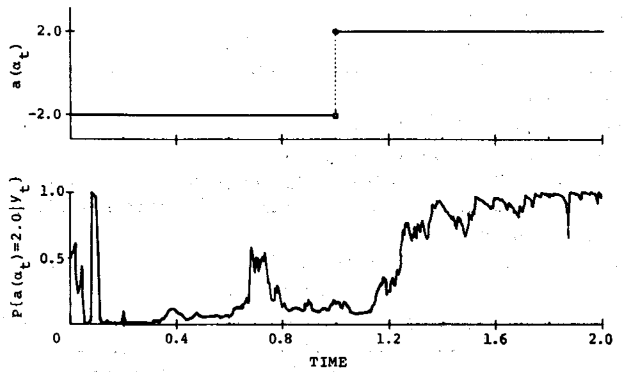

Fig. 4. One sample run of $a(\alpha_t)$ and the corresponding run of $P\{a(\alpha_t)=2.0|\mathcal{Y}_t\}$ (Example 2)

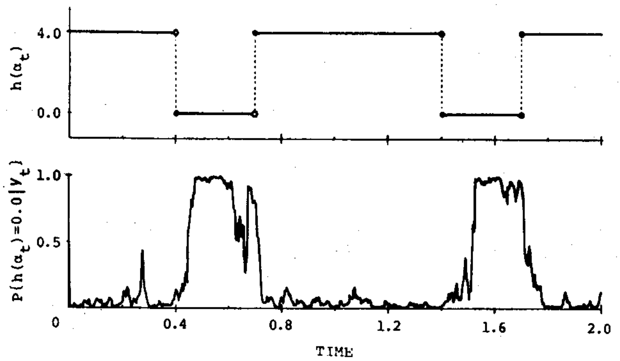

Fig. 5. One sample run of $h(\alpha_t)$ and the corresponding run of $P\{h(\alpha_t)=0.0|\mathcal{Y}_t\}$ (Example 2)

posed algorithm is satisfactory, and that the proposed algorithm is applicable in a wide class of state estimation problems.

6. Conclusions

A least-squares state estimation algorithm was derived for a general stochastic system with 1) system nonlinearity; 2) a jump parameter and 3) state-dependent observation noise. The performance of the algorithm was examined by numerical computations. Because of limited space, we cannot show all the results of computer simulation. It should be reported, however, that for each case, the proposed algorithm was shown to be effective.

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