A Discontinuity in Deformation Gradient Induced by Acceleration Waves in One-Dimensional Elastic Materials

By

Eiji MATSUMOTO*

(Received June 30, 1982)

Abstract

The discontinuity in the rate of the deformation gradient across an incident acceleration wave is partly reflected and partly transmitted at the boundary between two dissimilar one-dimensional elastic materials. The reflection and the transmission coefficients associated with those discontinuities are expressed in terms of the ratios of the densities and the speeds of the acceleration waves on both sides of the boundary. The rest of the discontinuity in the incident wave stays at the boundary, which induces either a discontinuity in the deformation gradient, or influences the growth and decay of such a discontinuity if it already exists.

1. Introduction

The usual propagation of acceleration waves in inhomogeneous elastic materials has been investigated in many works[1-4]. In general, acceleration waves are reflected and transmitted (refracted) at the boundary between two dissimilar materials. This subject has been studied for elastic materials by Chen and Gurtin[5], Wesołowski[6] and Borejko[7], and for elastic-plastic materials by Jahsman[8], and Cizek and Ting[9]. They obtained the relations between the speeds and the propagation directions of incident, reflected and refracted waves[7-9], and determined the amplitudes of reflected and transmitted waves[5),8),9]. The linear independence of the amplitudes of reflected waves was also discussed[6].

One can easily see that across the boundary there may exist a discontinuity in the deformation gradient. Therefore, it is of interest to know how the discontinuity is influenced by the incident acceleration wave. Such phenomena, however, have not been discussed in the above references.

This paper analyzes such phenomena in one-dimensional inhomogeneous elastic materials. The next section derives the propagation conditions for acceleration waves, and shows the existence of a discontinuity in the deformation gradient

* Department of Aeronautical Engineering

across the boundary between two materials which have different densities and elastic properties. In Section 3, it is shown that the jump of the rate of the deformation gradient across the incident acceleration wave is divided among the boundary, the reflected and the transmitted acceleration waves. In consequence, the jump of the rate of the deformation gradient across the boundary changes instantaneously, and, accordingly, so does the rate of the jump of the deformation gradient. In the case where the deformation gradient is initially continuous across the boundary, it becomes discontinuous after the arrival of the incident acceleration wave.

2. Acceleration waves and a discontinuity in the deformation gradient

A one-dimensional inhomogeneous elastic material is described by the constitutive equation:

$$T=\phi(F, X)\,, \tag{2.1}$$

where T is the stress, F the deformation gradient, and X the co-ordinate in the reference configuration. We assume that $X=0$ is the boundary of materials which have different densities and response functions (2.1). Suppose further that each response function is differentiable with respect to F and X, and that it and its derivatives have unique limits at the boundary. The equation of motion in the absence of external forces is

$$T_{,X}=\rho\dot{v}\,, \tag{2.2}$$

where ρ is the density in the reference configuration. Substituting (2.1) into (2.2) for $X\neq 0$, we have

$$EF_{,X}+G=\rho\dot{v}\,, \tag{2.3}$$

where

$$E\equiv\frac{\partial\phi}{\partial F}\,,\quad G\equiv\frac{\partial\phi}{\partial X}\,. \tag{2.4}$$

Suppose that $t\rightarrow y(t)$ is a smooth function. Let $y(t)$ denote the co-ordinate of a point in the material at the time t, and C the locus of the point in the $(X-t)$ space for an open time interval. We call the point $y(t)$ an N th order singular point ($N=1, 2, \cdots$), if the following conditions are satisfied:

(i) The displacement u and its derivatives up to and including $N-1$ th order are continuous in the neighbourhood of C of the $(X-t)$ space.

(ii) The Nth derivatives of u suffer finite jump discontinuities across the curve C.

Note that the density is discontinuous at $X=0$, even if the deformation gradient is continuous at the point. The velocity of the singular point is given by

$$U=\frac{dy}{dt}. \tag{2.5}$$

A singular point is called a material singular point or a wave, respectively, if $U=0$ or not. In particular, if $U\neq 0$ and $N=2$, it is called an acceleration wave.

We first consider a second order singular point which does not lie on the point $X=0$. The first order kinematical compatibility conditions for v and F across it become

$$[\dot{v}]+U[\dot{F}]=0, \tag{2.6}$$

$$[\dot{F}]+U[F_{,X}]=0, \tag{2.7}$$

where $[\cdot]$ is the jump of a quantity defined by

$$[f](t)\equiv\lim_{X\to y(t)-0}f(X,t)-\lim_{X\to y(t)+0}f(X,t). \tag{2.8}$$

Taking the jump of (2.3) across the singular point and using (2.6) and (2.7), we get

$$(\rho U^2-E)[F_{,X}]=0. \tag{2.9}$$

Since $[F_{,X}]$ should not vanish for $y(t)$ to be a second order singular point, from (2.9) it must hold that

$$U^2=\frac{E}{\rho}. \tag{2.10}$$

We may assume $E>0$ for usual cases, and hence, the singular point is an acceleration wave. Here, note that the speed of the acceleration wave depends on the density, the co-ordinate and the deformation gradient, so that in general it is discontinuous across $X=0$.

Next, we show that a first order material singular point may exist at the boundary. Since the stress must be continuous across the boundary, it follows that

$$[T]=0, \tag{2.11}$$

where $[\cdot]$ denotes the jump across the boundary. Applying the kinematical compatibility condition to T across the boundary, from $(2.4)_1$ and (2.11), we have

$$\frac{\delta}{\delta t}[T] = [\dot{T}] = [E\dot{F}] = 0\,. \tag{2.12}$$

Since E is discontinuous across $X=0$, there are two possibilities: (i) $\dot{F}$ is discontinuous across $X=0$. (ii) $\dot{F}$ is continuous at $X=0$ and vanishes identically. By this condition on first order singular point, u is continuous everywhere, and therefore the kinematical compatibility condition for u becomes

$$\frac{\delta}{\delta t}[u] = [v] = 0. \tag{2.13}$$

The kinematical compatibility conditions for v and F then become

$$\frac{\delta}{\delta t}[v] = [\dot{v}] = 0, \tag{2.14}$$

$$\frac{\delta}{\delta t}[F] = [\dot{F}]\,. \tag{2.15}$$

In view of (2.15), F may be discontinuous across the boundary for the above two possibilities. In fact, for the first possibility, F is necessarily discontinuous, and for the second one, F may have a constant discontinuity.

3. Reflection and transmission of acceleration waves

In this section, we call $[\dot{F}]$ across an acceleration wave the amplitude of it. Suppose that an acceleration wave is traveling in the direction of increasing X and arrives at the time $t=0$ and at the boundary $X=0$ between two dissimilar inhomogeneous elastic materials. As shown in the previous section, a first order material singular point may exist at the boundary before and after the arrival of the acceleration wave. For $t>0$, there may also exist reflected and transmitted acceleration waves. The loci of the above singular ponits in the $(X-t)$ space divide the neighbourhood of the origin into the five regions I—V as shown in Figure 1. Let $f(X, t)$ be a function which is continuous in the interior of each region, and which has a definite limit at the origin of each region. Henceforth, the superscripts I—V always denote such limits at the origin of the corresponding regions. Then the identity follows:

$$[f]_I+[f]_M = [f]_R+[f]_T+[f]_{M'}\,, \tag{3.1}$$

where

$$\begin{aligned} &[f]_I = f^{\mathrm{III}}-f^{\mathrm{II}}\,, \quad [f]_M = f^{\mathrm{II}}-f^{\mathrm{I}}\,, \quad [f]_R = f^{\mathrm{III}}-f^{\mathrm{IV}}\,, \\ &[f]_T = f^{\mathrm{V}}-f^{\mathrm{I}}\,, \quad [f]_{M'} = f^{\mathrm{IV}}-f^{\mathrm{V}}\,. \end{aligned} \tag{3.2}$$

Equation (3.1) implies that the total amount of the discontinuities of any quantity

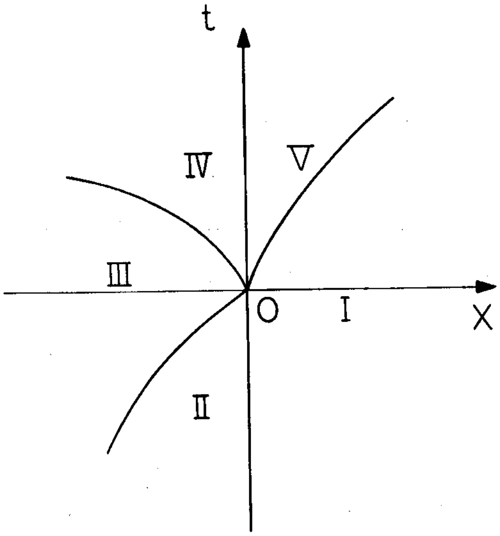

Fig. 1. Division of the neighbourhood of the origin in $(X-t)$ space.

across the considered singular points remains unchanged just before and just after the arrival of the incident acceleration wave.

Suppose that any one of v, F and their first derivatives in each region has a unique limit at the origin. The discussion in the previous section indicates that the velocities of the incident, the reflected and the transmitted acceleration waves in the vicinity of the origin are, respectively, equal to U^-, $-U^-$ and U^+ defined by

$$U^- \equiv \sqrt{\frac{E^-}{\rho^-}}, \quad U^+ \equiv \sqrt{\frac{E^+}{\rho^+}}, \tag{3.3}$$

where ρ^-, E^- and ρ^+, E^- denote, respectively, the limits of ρ, E at $X=0$ from the negative and the positive sides.

We shall next determine $[\dot{F}]_R$, $[\dot{F}]_T$ and $[\dot{F}]_{M'}$ in terms of $[\dot{F}]_I$ and $[\dot{F}]_M$. To do so, we take the limits of (2.6) at the origin for each acceleration wave:

$$\dot{v}^{\mathrm{III}} - \dot{v}^{\mathrm{II}} + U^-(\dot{F}^{\mathrm{III}} - \dot{F}^{\mathrm{II}}) = 0\,, \tag{3.4}$$

$$\dot{v}^{\mathrm{III}} - \dot{v}^{\mathrm{IV}} - U^-(\dot{F}^{\mathrm{III}} - \dot{F}^{\mathrm{IV}}) = 0\,, \tag{3.5}$$

$$\dot{v}^{\mathrm{V}} - \dot{v}^{\mathrm{I}} + U^+(\dot{F}^{\mathrm{V}} - \dot{F}^{\mathrm{I}}) = 0\,. \tag{3.6}$$

For the material singular point, by use of (2.10), the limits of $(2.14)_2$ and $(2.12)_3$ as $t \to -0$ and as $t \to +0$ are expressed in the form:

$$\dot{v}^{\mathrm{II}} - \dot{v}^{\mathrm{I}} = 0\,, \tag{3.7}$$

$$\rho^-(U^-)^2\dot{F}^{\mathrm{II}} - \rho^+(U^+)^2\dot{F}^{\mathrm{I}} = 0\,, \tag{3.8}$$

$$\dot{v}^{\mathrm{IV}} - \dot{v}^{\mathrm{V}} = 0\,, \tag{3.9}$$

$$\rho^-(U^-)^2\dot{F}^{\mathrm{IV}} - \rho^+(U^+)^2\dot{F}^{\mathrm{V}} = 0\,. \tag{3.10}$$

Eliminating $\dot{v}^{\mathrm{I}}$ to $\dot{v}^{\mathrm{V}}$ from (3.4) —(3.7) and (3.9), we get

$$U^-\dot{F}^{\mathrm{IV}} + U^+\dot{F}^{\mathrm{V}} - U^+\dot{F}^{\mathrm{I}} + U^-\dot{F}^{\mathrm{II}} - 2U^-\dot{F}^{\mathrm{III}} = 0\,. \tag{3.11}$$

Equations (3.10) and (3.11) can be solved for $\dot{F}^{\mathrm{IV}}$ and $\dot{F}^{\mathrm{V}}$ as

$$U^-(\rho^-U^- + \rho^+U^+)\dot{F}^{\mathrm{IV}} = \rho^+(U^+)^2\dot{F}^{\mathrm{I}} - \rho^+U^-U^+\dot{F}^{\mathrm{II}} + 2\rho^+U^-U^+\dot{F}^{\mathrm{III}}\,, \tag{3.12}$$

$$U^+(\rho^-U^- + \rho^+U^+)\dot{F}^{\mathrm{V}} = \rho^-U^-U^+\dot{F}^{\mathrm{I}} - \rho^-(U^-)^2\dot{F}^{\mathrm{II}} + 2\rho^-(U^-)^2\dot{F}^{\mathrm{III}} \tag{3.13}$$

Calculating $[\dot{F}]_R$, $[\dot{F}]_T$ and $[\dot{F}]_{M'}$ from (3.12) and (3.13), and representing the results in terms of $[\dot{F}]_I$ and $[\dot{F}]_M$ with the aid of (3.8), we finally obtain

$$[\dot{F}]_R = C_R[\dot{F}]_I\,, \tag{3.14}$$

$$[\dot{F}]_T = C_T[\dot{F}]_I\,, \tag{3.15}$$

$$[\dot{F}]_{M'} = C_M[\dot{F}]_I + [\dot{F}]_M\,, \tag{3.16}$$

where

$$C_R \equiv \frac{\rho^-U^- - \rho^+U^+}{\rho^-U^- + \rho^+U^+} = \frac{1-\xi\eta}{1+\xi\eta}\,, \tag{3.17}$$

$$C_T \equiv \frac{2\rho^-(U^-)^2}{U^+(\rho^-U^- + \rho^+U^+)} = \frac{2}{\eta(1+\xi\eta)}\,, \tag{3.18}$$

$$C_M \equiv \frac{2\{\rho^+(U^+)^2 - \rho^-(U^-)^2\}}{U^+(\rho^-U^- + \rho^+U^+)} = \frac{2(\xi\eta^2-1)}{\eta(1+\eta\xi)}\,, \tag{3.19}$$

and where

$$\xi \equiv \frac{\rho^+}{\rho^-}\,, \quad \eta \equiv \frac{U^+}{U^-}\,. \tag{3.20}$$

Then, it follows identically that

$$C_R + C_T + C_M = 1\,. \tag{3.21}$$

We may call C_R and C_T the reflection and the transmission coefficients associated with the jump of the rate of the deformation gradient.

Equations (3.14)–(3.16) together with (3.21) imply that the jump of the rate of the deformation gradient across the incident acceleration wave is divided among the reflected and the transmitted acceleration waves and the material singular point at the boundary. It is also seen that the jump $[\dot{F}]_M$ at the boundary is inherited, just as it is, by the boundary after the arrival of the incident acceleration wave. From (3.17)–(3.20), the ratio of the division is determined by the ratios of the densities and the speeds of the acceleration waves on the both sides of the boundary. In view of Figure 2, if η is small or large for a given value of ξ, then $c_T>1$ or $c_T<1$, e.g., the acceleration wave is amplified or damped at the boundary, respectively. Similarly, if ξ is small or large for a given value of η, the acceleration wave is amplifide or damped at the boundary, respectively. We can also see that if $\xi=1$ and $\eta=1$, $c_R=c_M=0$ and $c_T=1$. That is, as expected, the reflection does not occur in this case, and the incident acceleration wave passes through the boundary without changing its amplitude.

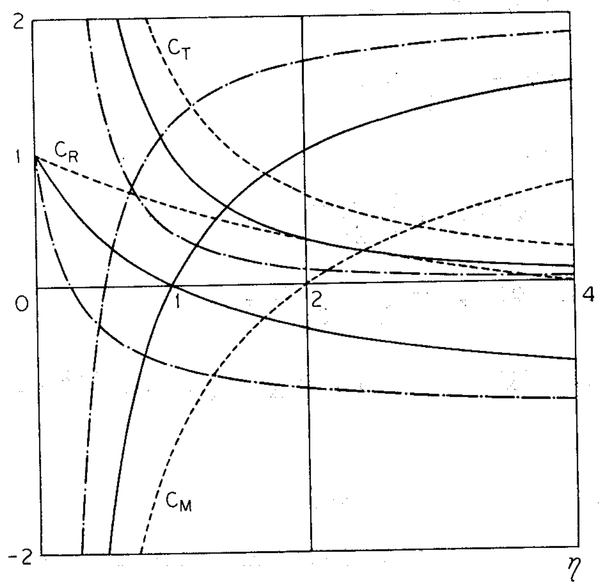

Fig. 2. Variation of c_R, c_T and c_M for $[\dot{F}]$. $\eta=0.25$: broken lines. $\eta=1.0$: solid lines. $\eta=4.0$ dot-dash lines.

Next, let us pay attention to the first order material singular point which may exist at the boundary. By this condition, it follows that

$$[F]_{M'} = [F]_M \, . \tag{3.22}$$

The time derivative of $[F]$ at the material singular point equals $[\dot{F}]$, from (2.15). In view of (3.16), $[\dot{F}]$ at the boundary changes discontinuously just before and just after the arrival of the incident acceleration wave. Thus we find that $[F]$ changes roughly when the incident acceleration wave reaches there. Figure 2 indicates that if η is small, the absolute value of c_M is very large, and hence, the above effect is then remarkable. In the special case where the deformation gradient is continuous for $t<0$, $[F]_M=[F]_{M'}=0$, and the discussion in the previous section implies that $[\dot{F}]_M=0$. Since $[\dot{F}]_{M'}$ does not vanish in general, from (2.15) we conclude that in this case a discontinuity in the deformation gradient is induced after the arrival of the incident acceleration wave.

For comparison, let us obtain the jump relations for $\dot{v}$ and $F,_X$ corresponding to (3.14)–(3.16). To do so, for the acceleration waves we may rewrite (3.14) and (3.15) in terms of $[\dot{v}]$ and $[F,_X]$ by use of (2.6) and (2.7). For the material singular point, we may apply (3.1) to the obtained results. As a result, we obtain

$$[\dot{v}]_R = C'_R[\dot{v}]_I \equiv \frac{\xi\eta-1}{1+\xi\eta}[\dot{v}]_I \,, \tag{3.23}$$

$$[\dot{v}]_T = C'_T[\dot{v}]_I \equiv \frac{2}{1+\xi\eta}[\dot{v}]_I \,, \tag{3.24}$$

$$[\dot{v}]_{M'} = [\dot{v}]_M = 0 \,, \tag{3.25}$$

$$[F,_X]_R = C''_R[F,_X]_I \equiv \frac{\xi\eta-1}{1+\xi\eta}[F,_X]_I \,, \tag{3.26}$$

$$[F,_X]_T = C''_T[F,_X]_I \equiv \frac{2}{\eta^2(1+\xi\eta)}[F,_X]_I \,, \tag{3.27}$$

$$[F,_X]_{M'} = C''_M[F,_X]_I+[F,_X]_M \equiv \frac{2(\eta^2-1)}{\eta^2(1+\xi\eta)}[F,_X]_I+[F,_X]_M \,, \tag{3.28}$$

where

$$C'_R+C'_T = 1 \,, \quad C''_R+C''_T+C''_M = 1 \,. \tag{3.29}$$

In view of (3.17), (2.23) and (3.26), the absolute values of the reflection coefficients associated with $[\dot{v}]$, $[\dot{F}]$ and $[F,_X]$ are the same, but the absolute values of the three transmission coefficients are different from each other. It is worth noting that the reflection and the transmission coefficients for $[\dot{v}]$ coincide, respectively, with those for the harmonic waves in the case where the two materials are homogeneous and linearly elastic. Equations (3.23)–(3.25) together with $(3.29)_1$ show that the jump of the acceleration in the incident wave is divided between the reflected and the transmitted acceleration waves. From (3.28), a part of the jump $[F,_X]$ across the incident wave stays at the boundary, as in the case of $[\dot{F}]$. The variation of the coefficients with ξ and η is shown in Figures 3 and 4.

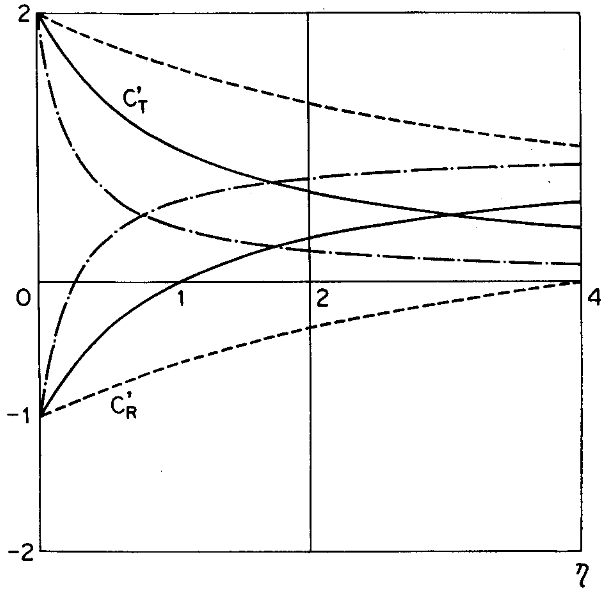

Fig. 3. Variation of c_R' and c_T' for $[\dot{v}]$. $\eta=0.25$: broken lines. $\eta=1.0$: solid lines. $\eta=4.0$: dot-dash lines.

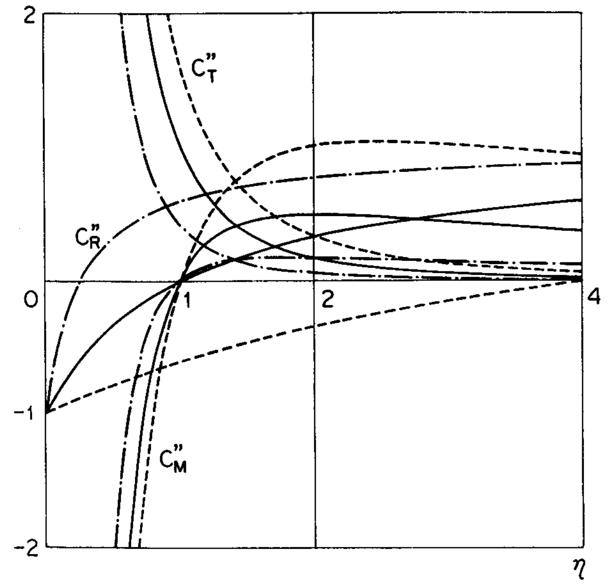

Fig. 4. Variation of c_R'', c_T'' and c_M'' for $[F, x]$. $\eta=0.25$: broken lines. $\eta=1.0$ solid lines. $\eta=4.0$: dot-dash lines.

References

1) B. D. Coleman; Arch. Rational Mech. Anal., **22**, 333 (1966).
2) R. M. Bowen and C.-C. Wang; Arch. Ratonal Mech. Anal., **38**, 13 (1970).
3) P. J. Chen; "Growth and Decay of Waves in Solids," Encyclopedia of Physics, VIa/3, ed. by C. Truesdell, Springer-Verlag Berlin (1973).
4) P. J. Chen; Acta Mech., **17**, 17 (1973).
5) P. J. Chen and M. E. Gurtin; J. Appl. Mech. **40**, 1055 (1973).
6) Z. Wesołowski; Bull. Acad. Polon. Sci., Serie Sci. Tech., **23**, 547 (1975).
7) P. Borejko; Arch. Mech., **31**, 373 (1979).
8) W. E. Jahsman; J. Appl. Mech., **41**, 117 (1974).
9) J. C. Cizek and T. C. T. Ting; J. Appl. Mech., **45**, 51 (1978).

ERRATUM

A Discontinuity in Deformation Gradient Induced by Acceleration Waves in One-Dimensional Elastic Materials

by

Eiji Matsumoto

p. 479, p.481: In the captions of Figs. 2, 3 and 4 η should be replaced with ξ .