A CAD Method of Multivariable Control Systems Using Generalized Gershgorin Bands

By

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Abstract

The design method proposed here is a frequency-domain method and uses a certain class of generalized Gershgorin bands mapped onto the gain-phase plane, which are referred to as the generalized Gershgorin pseudo-bands. The main advantages are that the generalized Gershgorin pseudo-bands have the same width for all loops, that the width of the pseudo-bands is invariant under the changes of the diagonal compensator and under the changes of the unit system at outputs and inputs, that no diagonal dominance is required at high frequencies, and that a quantitative guide line for pseudo-diagonalization is given based on the interaction index which is a satisfactory scalor measure of the cross interaction.

1. Introduction

During the last ten years, the INA (inverse Nyquist array) method of Rosenbrock[19),20)] has received much attention from engineers as a practical means for the computer-aided design of multivariable control systems. One of the distinctive features of the INA method is the use of the Gershgorin bands. Rosenbrock showed that each Gershgorin band includes the inverse polar-plot of the frequency response from one input to the corresponding output, if the controllers of the other loops satisfy certain constraints. Later, this result was generalized, and it was shown that there are many other bands which have properties parallel to the Gershgorin bands[2)]. In the same paper, it was also shown that the bands obtained from the maximum eigenvalue of a certain nonnegative matrix (referred to as the interaction index in the following), have favorable properties which enable us to apply Nichols' chart technique and work out a balanced design based on the ordinary (i.e. not inverse) transfer matrix. In addition, we can show that our interaction index is a reasonable scalor measure of the cross interaction among loops. The purpose

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of this paper is to present a design method which takes advantage of these new knowledges.

2. Bounds for the Transfer Functions of Multivariable Feedback Systems

Consider the n-input n-output system of Fig. 1. Here, r, e and y are n-vectors representing the reference, error and controlled variables, respectively. P is the permutation matrix corresponding to the renumbering of inputs to the plant, $G(s)=G'(s)P$ is the transfer matrix of the plant after the renumbering, $L(s)$ is the precompensator to make $Q(s)=G(s)L(s)$ nearly diagonal and $F(s)=\mathrm{diag}(f_j(s))$ is the diagonal main controller. (For the details of this structure, refer to Rosenbrock.[19),20)]) The $\Delta=\mathrm{diag}(\delta_j)$ is placed in the feedback path for the convenience of treating such situations where some loops are closed and the others open. Thus, the δ_j takes the value 1 or 0, and $\Delta=I$ if the system is in normal operation.

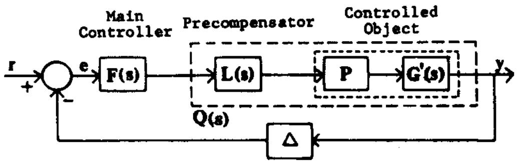


Fig. 1. Structure of the control system

Now, let $h_j(s|QF)$ be the transfer function from e_j to y_j when δ_k are

$$\delta_1=\ldots=\delta_{j-1}=1,\ \delta_j=0,\ \delta_{j+1}=\ldots=\delta_n=1 \tag{1}$$

In other words, $h_j(s|QF)$ is the open-loop transfer function of the j-th loop when the other loops are closed. If the off-diagonal elements $q_{kl}(s)$ $(k\neq l)$ of $Q(s)$ are all zero, $h_j(s|QF)$ is equal to $f_j(s)q_{jj}(s)$. When $q_{kl}(s)\neq 0$ $(k\neq l)$, the difference of $h_j(s|QF)$ and $f_j(s)q_{jj}(s)$ is bounded by the size of $|q_{kl}(s)|$ as given in the following. Define the *interaction matrix* $C(s|Z)=(c_{jk}(s|Z))$ of a square transfer matrix $Z(s)$ by

$$c_{jj}(s|Z)=0,\ c_{jk}(s|Z)=|z_{jk}(s)/z_{kk}(s)| \quad (j\neq k) \tag{2}$$

Define the *interaction index* $\lambda(s|Z)$ of $Z(s)$ as the maximum eigenvalue of $C(s|Z)$.† Then, the next theorem holds true.

Theorem 1: The difference of $h_j(s|QF)$ and $f_j(s)q_{jj}(s)$ is bounded as

$$\begin{aligned}|h_j(iw|QF)-f_j(iw)q_{jj}(iw)| &< \lambda(iw|Q)|f_j(iw)q_{jj}(iw)| \quad \text{if } \lambda>0\\ &= 0 \quad \text{if } \lambda=0\end{aligned} \tag{3}$$

† A nonnegative matrix (i.e. a matrix with nonnegative elements) has a nonnegative eigenvalue which is larger than, or equal to any other eigenvalue in magnitude.[6)] This nonnegative eigenvalue is called the maximum eigenvalue.

if the next inequalities are satisfied for $k=1, \ldots, j-1, j+1, \ldots, n$.

$$|1+f_k(iw)q_{kk}(iw)| > \lambda(iw|Q)|f_k(iw)q_{kk}(iw)| \tag{4}$$

The above theorem is a special case of the Theorem 1 of Araki & Nwokah[2]. Evidently, (3) holds true for all j if (4) is satisfied for all k. The inequalities (3) and (4) can be rewritten as, when $\lambda(iw/Q)\neq 0$,

$$\frac{|h_j(iw|QF)-f_j(iw)q_{jj}(iw)|}{|f_j(iw)q_{jj}(iw)|} < \lambda(iw|Q) \tag{3'}$$

$$\frac{|1+f_k(iw)q_{kk}(iw)|}{|f_k(iw)q_{kk}(iw)|} < \lambda(iw|Q) \tag{4'}$$

respectively. Based on the above expression, we can restate Theorem 1 as follows: "The relative deviation of $h_j(iw|QF)$ from $f_j(iw)q_{jj}(iw)$ is less than $\lambda(iw|Q)$ if the relative distance of $f_k(iw)q_{kk}(iw)$ to the point $(-1, 0)$ is larger than $\lambda(iw|Q)$." Here, we should note that the right-hand sides of (3′) and (4′) do not depend on $F(s)$, i.e. the assumption and the result of Theorem 1 are independent of the diagonal controller $F(s)$ so long as we consider the relative quantities. This property turns out to be very useful when we carry out our design procedure on the gain-phase plane.

Before proceeding to the study of the graphical implication of the theorem, let us clarify an important property of the interaction index. From (2), we obtain

$$C(s|QB)=C(s|Q) \tag{5}$$

$$C(s|BQ)=B\cdot C(s|Q)\cdot B^{-1} \tag{6}$$

for a diagonal matrix B with positive diagonal elements. Since the interaction indices of Q, QB and BQ are the maximum eigenvalues of $C(s|Q)$, $C(s|QB)$ and $C(s|BQ)$, respectively, we obtain

$$\lambda(s|QB)=\lambda(s|BQ)=\lambda(s|Q) \tag{7}$$

Thus, we can conclude that the interaction index is invariant under the multiplication of a diagonal matrix. This implies that the interaction index $\lambda(s|Q)$ is invariant under scaling at the output and input of $Q(s)$.† The above property of the interaction index is also the main reason for the fact that the right-hand sides of (3′) and (4′) do not depend on the diagonal controller $F(s)$.

Now, let us study the graphical implications of the theorem. Let $D_k(iw)$ be the disk with the center $f_k(iw)q_{kk}(iw)$ and the radius $\lambda(iw|Q)|f_k(iw)q_{kk}(iw)|$ on the complex plane. Also, let Γ_k be the band swept out by $D_k(iw)$ when w changes from

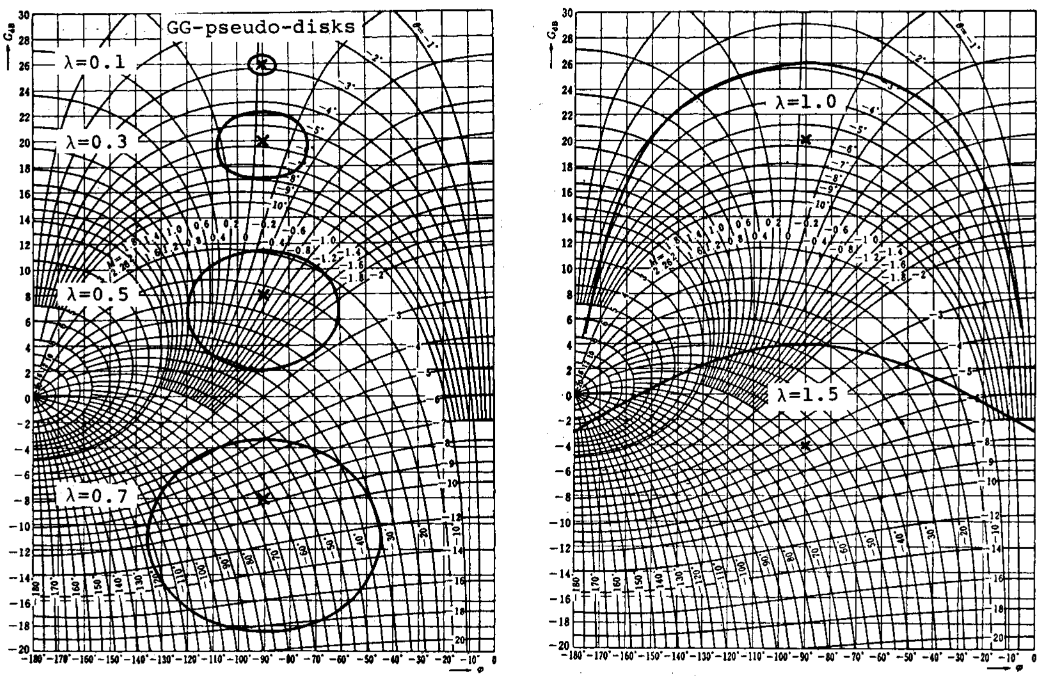
† Scaling at the input implies multiplication of a diagonal matrix from the right, and scaling at the output implies that from the left.

$-\infty$ to $+\infty$. We refer to $D_k(iw)$ and Γ_k as the k-th *GG disk* (*generalized Gershgorin disk*) and the k-th *GG band* (*generalized Gershgorin* band), respectively. As explained by Araki & Nwokah,[2] Theorem 1 implies that the polar plot of $h_j(iw|QF)$ lies inside the j-th *GG* band if the *GG* bands of the other loops do not include the point $(-1, 0)$. Here, let us consider the matter on the gain-phase plane (i.e. consider the log-modulus plots). Let $\bar{D}_k(iw)$ be the image of $D_k(iw)$ mapped on the gain-phase plane. It can be easily derived that the boundary of $\bar{D}_k(iw)$ is given by

$$\zeta = 20 \log|f_k(iw)q_{kk}(iw)| + 20 \log|1+\lambda(iw|Q) \exp(i\theta)| \qquad 0 \leqq \theta \leqq 2\pi \tag{8}$$

$$\xi = \arg[f_k(iw)q_{kk}(iw)] + \arg[1+\lambda(iw|Q) \exp(i\theta)] \qquad 0 \leqq \theta \leqq 2\pi \tag{9}$$

where ζ and ξ are the log-modulus gain (vertical coordinate) and the phase angle (horizontal coordinate), respectively. For $\lambda(iw|Q)<1$, (8) and (9) describe a closed curve surrounding the point $(20 \log|f_k(iw)q_{kk}(iw)|, \arg[f_k(iw)q_{kk}(iw)])$. The $\bar{D}_k(iw)$ is the inside of that curve (Fig. 2a). For $\lambda(iw|Q) \geqq 1$, (8) and (9) describe a curve (not closed) passing above the point $(20 \log|f_k(iw)q_{kk}(iw)|, \arg[f_k(iw)q_{kk}(iw)])$. The $\bar{D}_k(iw)$ is the lower-side of that curve (Fig. 2b). Note that the shape of $\bar{D}_k(iw)$ is determined completely by $\lambda(iw|Q)$ and is the same for all k. Also note that the



(a) $\lambda<1$ (b) $\lambda \geqq 1$

Fig. 2. Size of the *GG* Pseudo-disk

change of $f_k(iw)$ only causes parallel displacement of the $\bar{D}_k(iw)$ of the corresponding loop, and does not influence the $\bar{D}_j(iw)$'s of the other loops. Now, let $\bar{\Gamma}_k$ be the territory swept out by $\bar{D}_k(iw)$ when the frequency w changes from $-\infty$ to $+\infty$. We refer to $\bar{D}_k(iw)$ and $\bar{\Gamma}_k$ as the *GG pseudo-disk* (*generalized Gershgorin pseudo-disk*) and *GG pseudo-band* (*generalized Gershgorin pseudo-band*), respectively. Just like the case of the polar plots, we can obtain the following result from Theorem 1: "The log-modulus plot of $h_j(iw|QF)$ lies inside the j-th *GG* pseudo-band if the *GG* pseudo-bands of the other loops do not include the point (OdB, $-180°$)."

3. Stability and Generalized Diagonal Dominance

For simplicity, let us assume that $Q(s)F(s)$ has no unstable poles.† Define the *lower absolute-value matrix* $A(s|Z)=(a_{jk}(s|Z))$ of a square transfer matrix $Z(s)$ by

$$a_{jj}(s|Z)=|z_{jj}(s)|, \quad a_{jk}(s|Z)=-|z_{jk}(s)| \quad (j\neq k) \tag{10}$$

Then, we can restate the stability theorem of Araki & Nwokah[2] as follows.

Theorem 2: The system of Fig. 1 is stable for any Δ if

(a) the Nyquist locus of $f_j(s)q_{jj}(s)$ does not encircle $(-1, 0)$ for $j=1, \ldots, n$, and

(b) the lower absolute-value matrix $A(s|\mathrm{I}+QF)$ of the return difference matrix $\mathrm{I}+QF$ is an M-matrix everywhere on the Nyquist contour.††

The above theorem can be related to the *GG* bands and *GG* pseudo-bands by the next corollary.

Corollary 1: The system of Fig. 1 is stable for any Δ, if the conditions (a) and (b′) are satisfied where

(b′) each *GG* band does not include the point $(-1, 0)$ (, equivalently, each *GG* pseudo-band does not include the point (OdB, $-180°$)).

[Proof] We have only to prove that (b′) is a sufficient condition for (b). The condition (b′) implies that (4) is satisfied everywhere on the Nyquist contour for all k. From (4) we obtain, for the off-diagonal elements,

$$\begin{aligned} c_{jk}(iw|\mathrm{I}+QF) &\triangleq \frac{|f_k(iw)q_{jk}(iw)|}{|1+f_k(iw)q_{kk}(iw)|} \\ &< \frac{|f_k(iw)q_{jk}(iw)|}{\lambda(iw|Q)|f_k(iw)q_{kk}(iw)|}=\frac{c_{jk}(iw|Q)}{\lambda(iw|Q)} \quad j\neq k \end{aligned} \tag{11}$$

As for the diagonal elements of $C(s|\mathrm{I}+QF)$, we have

† When $Q(s)F(s)$ has unstable poles, the conclusion of Theorem 2 holds true for $\Delta=\mathrm{I}$ if (a) is replaced by "the total number of anti-clockwise encirclements of the Nyquist locus around $(-1, 0)$ is equal to the number of the unstable poles."

†† A square matrix is said to be an M-matrix if the off-diagonal elements are nonpositive and the principale minors are positive.

$$c_{jj}(iw|\mathrm{I}+QF)=c_{jj}(iw|Q)/\lambda(iw|Q)=0 \tag{12}$$

by definition. From (11) and (12) we can conclude that the maximum eigenvalue of $C(iw|\mathrm{I}+QF)$ is less than the $1/\lambda(iw|Q)$ times of the maximum eigenvalue of $C(iw|Q)$.[6] On the other hand, $\lambda(iw|Q)$ is nothing but the maximum eigenvalue of $C(iw|Q)$. Hence, the maximum eigenvalue of $C(iw|\mathrm{I}+QF)$ is less than 1. This implies that $1-C(iw|\mathrm{I}+QF)$ is an M-matrix.[4] Here, note that the matrix $A(s|\mathrm{I}+QF)$ is related to the matrix $C(s|\mathrm{I}+QF)$ by

$$A(s|\mathrm{I}+QF)=[1-C(s|\mathrm{I}+QF)]\ \mathrm{diag}\,(|1+f_j(s)q_{jj}(s)|) \tag{13}$$

Therefore, the matrix $A(s|\mathrm{I}+QF)$ is also an M-matrix on the Nyquist contour. Thus we obtained (b). [Q.E.D.]

Now, let us consider the relation of the above stability conditions to the diagonal dominance condition of Rosenbrock. A square transfer matrix $Z(s)$ is said to be *diagonally row dominant* at s, if

$$|z_{jj}(s)| > \sum_{k=1;\,k\neq j}^{n} |z_{jk}(s)| \qquad j=1, \ldots, n \tag{14}$$

Diagonal column dominance is defined in parallel.[20] It can be easily shown from the properties of M-matrices that $A(s|Z)$ is an M-matrix if $Z(s)$ is diagonally row or column dominant.[4] Thus, Rosenbrock's stability condition[20] can be viewed as a corollary to Theorem 2.

Corollary 2 (Rosenbrock): The system of Fig. 1 is stable for any Δ if the conditions (a) and (b″) are satisfied where

(b″) the return difference matrix $\mathrm{I}+Q(s)F(s)$ is diagonally row or column dominant on the Nyquist contour.

Considering the above relation, we say that $Z(s)$ is *G-diagonally dominant* (*diagonally dominant in the generalized sense*) *at s*, if $A(s|Z)$ is an M-matrix.† The G-diagonal dominance has 3 invariance properties: i.e. it is invariant under transpose operation, invariant under multiplication of a diagonal transfer matrix, and invariant under the change of the unit system (i.e. scaling) at the outputs and inputs.†† However, from the viewpoint of the feedback system design, it has the drawback that the condition is not loopwise and not graphical. This fact forces us to use Corollary 1 or 2 instead of Theorem 2 in the design procedure. Both corollaries use sufficient conditions for the G-diagonal dominance, and neither is implied by the other. In

† This terminology is also supported by the relation of the Gershgorin's theorem[19] and the Fan's theorem.[5]

†† The second and the third are the two different interpretations of the same mathematical fact. These invariance properties can be easily derived from the properties of the M-amtrices.[4]

this respect, the two corollaries are even. However, Corollary 1 has an advantage over Corollary 2 in that the condition (b′) succeeds the invariance properties of the G-diagonal dominance, whereas the condition (b″) does not. These invariant properties are the origin of the nice properties of the GG pseudo-bands described in the last section.

4. Cross Interaction of the Closed-Loop System

In multivariable control systems, it is often required that a change of one reference does not heavily influence the uncorresponding outputs; i.e. the "cross interaction of the closed-loop system" is required to be small. In this section, we show that the interaction index is a suitable measure for the cross interaction in the above sense, too. Then, we study the relation of the closed-loop and open-loop interaction indices.

Denote the transfer matrix of the closed-loop system of Fig. 1 for $\Delta=\mathrm{I}$ by $H(s)$; i.e.

$$H(s)=(\mathrm{I}+Q(s)F(s))^{-1}Q(s)F(s) \tag{15}$$

Consider the situation where each output y_j is expected to be moved within $-Y_j \leqq y_j \leqq Y_j$, and the operator manipulates the reference signals assuming that the system is completely decoupled (i.e. that the change of each reference signal causes only a change of the corresponding output). Then, in order to obtain the maximum change $y_j=\pm Y_j$ at the j-th output, the j-th reference signal should be set as

$$r_j=\pm Y_j/h_{jj}(0) \tag{16}$$

Thus, the maximum error $\Delta y_{k,max}$ at the k-th output caused by the operation of other reference signals turns out to be

$$\Delta y_{k,max}=\sum_{j=1;j\neq k}^{n}|h_{kj}(0)|\cdot|Y_j/h_{jj}(0)| \tag{17}$$

Hence the relative error ε_k defined as the ratio of the maximum error to the nominal maximum change is given by

$$\varepsilon_k(0)=\sum_{j=1;j\neq k}^{n}\left|\frac{h_{kj}(0)}{h_{jj}(0)}\right|\cdot\frac{Y_j}{Y_k} \tag{18}$$

In the above, we considered only stepwise inputs. Evidently, we can extend this consideration to the case of sinusoidal inputs, and show that the relative error $\varepsilon_k(iw)$ at the frequency w is given by

$$\varepsilon_k(iw)=\sum_{j=1;j\neq k}^{n}\left|\frac{h_{kj}(iw)}{h_{jj}(iw)}\right|\cdot\frac{Y_j}{Y_k} \tag{19}$$

where Y_j is the nominal maximal amplitude of the sinusoidal output.† Taking the minimax point of view, let us measure the degree of the deterioration of the control performance caused by the cross interaction with the *maximum relative error* $\varepsilon_{max}(iw)$ given by

$$\varepsilon_{max}(iw)=\max(\varepsilon_1(iw), \dots, \varepsilon_n(iw)) \tag{20}$$

Now, for the purpose of theoretical development, let us tentatively assume that we can choose Y_j arbitrarily. Then, we would naturally ask: which set of $Y_1, \dots, Y_n$ makes $\varepsilon_{max}(iw)$ minimum and how large is that minimum value. From (19) and (20), and by the definition (2) of the interaction matrix, we obtain

$$\sum_{j=1}^{n} c_{kj}(iw|H)Y_j \leqq \varepsilon_{max}(iw)Y_k \tag{21}$$

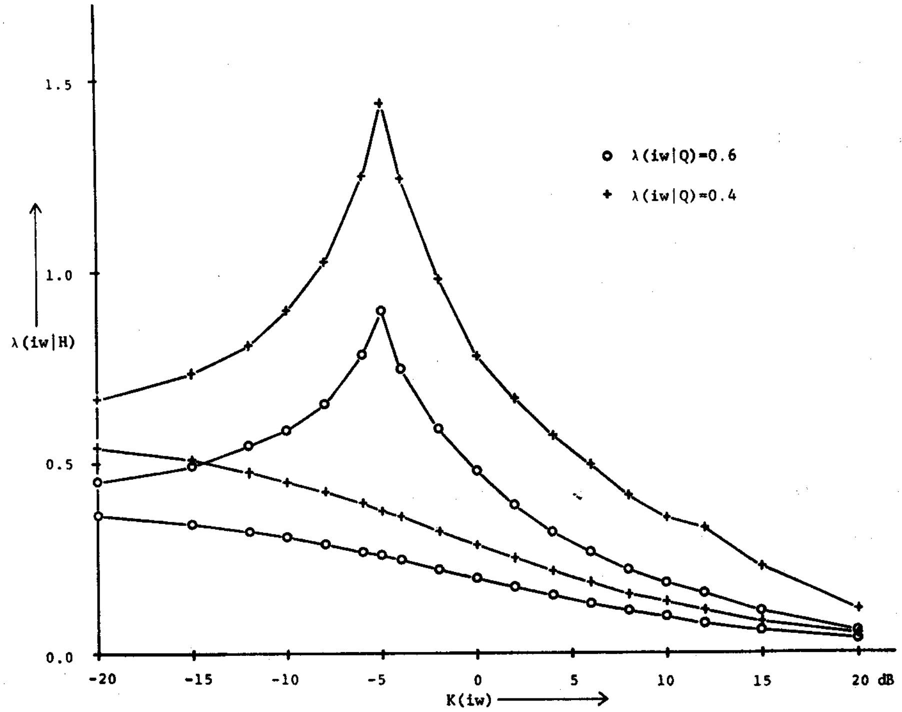
where $c_{kj}(s|H)$ are the elements of the interaction matrix of $H(s)$. Since $c_{kj}(iw/H)\geqq 0$, we can apply the theory of nonnegative matrices[6] and conclude that the minimum value of $\varepsilon_{max}(iw)$ is the maximum eigenvalue of $C(iw|H)$ and is attained when $(Y_1, \dots, Y_n)$ is the corresponding eigenvector. Here, the maximum eigenvalue of $C(iw|H)$ is nothing but the interaction index of $H(iw)$. Thus, we have shown that the interaction index of the closed-loop transfer matrix $H(iw)$ gives the minimum value of the maximum relative error $\varepsilon_{max}(iw)$ at the outputs. Here, we must note that we can not set $Y_1, \dots, Y_n$ at our will, and so the above minimum is not attained in most practical cases. But the interaction index remains as a useful measure of the cross interaction in the sense that a small interaction index is a "necessary condition" for a small cross interaction.

Now, let us study the relation of the closed-loop interaction index $\lambda(iw|H)$ to the open-loop interaction index $\lambda(iw|Q)$. When $f_j(iw)q_{jj}(iw)$ are small enough compared with 1, it is indicated by (15) that $\lambda(iw|H)\simeq\lambda(iw|Q)$. When $f_j(iw)q_{jj}(iw)$ are large enough, we can expect that the cross interaction is suppressed by the "high gain feedback" effect and so $\lambda(iw|H)$ becomes very small. But the relation of $\lambda(iw|H)$ and $\lambda(iw|Q)$ for the intermediate values of $f_j(iw)q_{jj}(iw)$ is not simple. To clarify this point, we made a numerical study, and the results are shown in Fig. 3 for the case of $n=4$. In this study, the same loop gain is chosen for all loops; i.e.

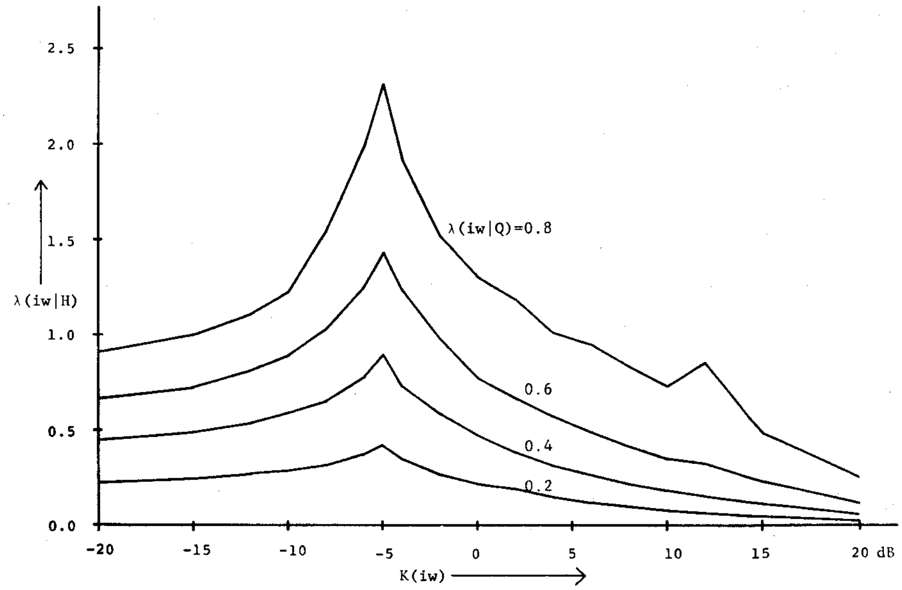
$$|f_1(iw)q_{11}(iw)|=\dots=|f_n(iw)q_{nn}(iw)|\underline{\underline{\Delta}}K(iw) \tag{22}$$

In Fig. 3a, each pair of lines gives the range of the values of the closed-loop index $\lambda(iw|H)$ obtained from 3000 examples with the values of $K(iw)$ and $\lambda(iw|Q)$ as in-

† Sinusoidal outputs would not be required in most situations. But the value of $\varepsilon_k(iw)$ is important for the transient interaction; i.e. a large $\varepsilon_k(iw)$ over a certain range means a strong transient interaction.



(a) Range of the closed-loop interaction index $\lambda(iw|H)$



(b) The maximum value of the closed-loop interaction index $\lambda(iw|H)$

Fig. 3. Relation of the closed-loop and open-loop interaction indices

dicated. The off-diagonal elements are chesen at random. The phase angles of $f_j(iw)q_{jj}(iw)$ are chosen at random in the region which avoids the inside of the $M_p=1.3$ curve and also avoid the part $\xi>180°$ and $|f_j(iw)q_{jj}(iw)|>1$. In Fig. 3b, only the maximum values of the closed loop index $\lambda(iw|H)$ are shown. From these figures, we can estimate how small the open-loop interaction index should be made for the intermediate range of frequencies at which $|f_j(iw)q_{jj}(iw)|$ is near 1.

5. Design Method

Here, we propose our design method; i.e. explain how to determine the compensators P, $L(s)$ and $F(s)$. The roles of these compensators are as described in §2. We determine these compensators in this order. Our basic policy is trial-and-error so that the earlier steps should be repeated whenever it is felt necessary. For the purpose of illustration and comparison, we use the gas-turbine example which was studied by McMorran[18] using the INA method. For this system, the transfer matrix $G'(s)$ is

$$G'(s)=\frac{1}{p(s)}\begin{bmatrix}14.96(s+1.7)(s+100) & 95150(s+1.898)(s+10)\\ 85.2(s+1.44)(s+100) & 124000(s+2.037)(s+10)\end{bmatrix} \tag{23}$$

$$p(s)=(s^2+3.225s+2.525)(s+10)(s+100) \tag{24}$$

The best method to determine P is to make use of any physical knowledge and to choose the most closely coupled input-output pairs. If such cannot be done, compute the interaction index $\lambda(iw|G'P)$ for all possible P, and choose the P which gives a small $\lambda(iw|G'P)$ as average. For our example, we have two possibilities for P:

$$P_1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad P_2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \tag{25}$$

The interaction indices of $G'P_1$ and $G'P_2$ are as shown in Fig. 4. Based on this information we choose P_2.†

To determine $L(s)$, we can use the variety of the pseudo-diagonalization methods proposed before. Above all, the successive cancelling using unimodular matrices[20),23)], the modified Hawkins' method[20)], the Bode diagram method[17)], and the integral measure method[12)] can be used in our context without any modification because they generally decrease the ratio of the off-diagonal elements to the diagonal elements. To apply the CARDIAD method[21)] and the function minimization method[9)] in our context, we must replace the diagonal dominance condition by a condition on the

† When $n=2$, the interaction index of $Z(s)$ is given by $\lambda(s|Z)=|z_{12}(s)z_{21}(s)/z_{11}(s)z_{22}(s)|^{1/2}$. Hence, we have $\lambda(s|G'P_1)=1/\lambda(s|G'P_2)$. Thus, we actually need either $\lambda(s|G'P_1)$ or $\lambda(s|G'P_2)$. When $n\geqq3$, the situation is not so simple.

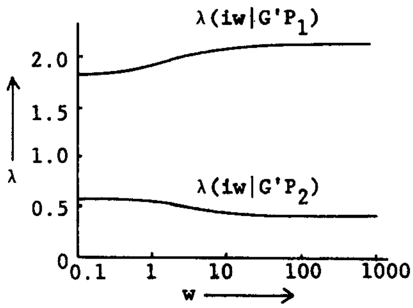


Fig. 4. Interaction index of the controlled object

interaction index. For the gas-turbine example, we applied the Bode diagram method[17] and obtained

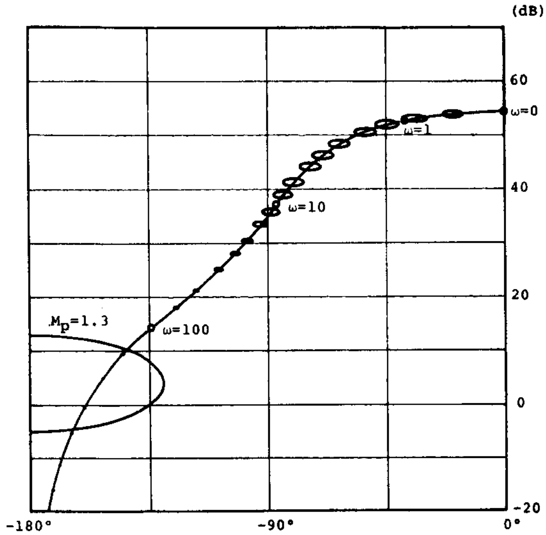
$$L(s)=\begin{bmatrix} 1 & -1 \\ -1450(s+12)/(s+100) & 6310(s+12)/(s+100) \end{bmatrix} \tag{26}$$

With this $L(s)$, $\lambda(iw/Q)\leqq 0.1$ is realized for $10\leqq w\leqq 1000$.

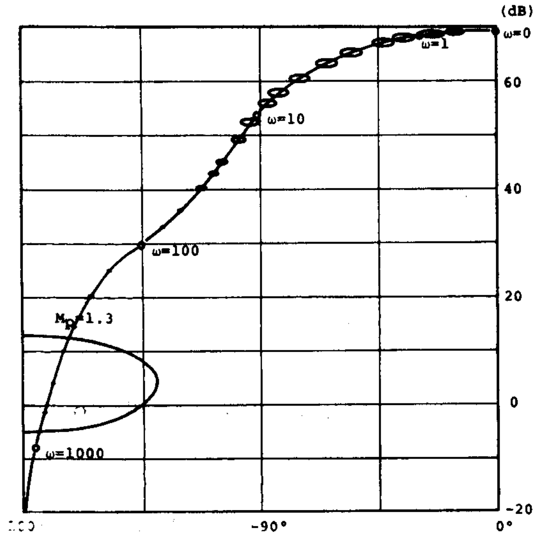
To design $f_j(s)$, draw the GG pseudo-band on the gain-phase plane, regard the upper boundary of the GG pseudo-band as the frequency response of the controlled object, and design each $f_j(s)$ applying the classical methods[8),18)] of cascade compensation. In this step, the properties of the GG pseudo-bands described in §2 become important. Note that the above procedure usually results in assigning conservative (i.e. smaller) values to the gain constants of $f_j(s)$. In order to obtain quick settling, it is recommended to increase the gain constants gradually checking the log-modulus plots of $h_j(iw|QF)$. If we keep the point (OdB, -180) outside the GG pseudo-bands in the last adjustment, the resulting system possesses integrity in the sense that stability is guaranteed for any Δ. The GG pseudo-bands of the example are shown in Fig. 5. From these figures, we could easily determine $F(s)$ as

$$F(s)=\text{diag}\ (0.18,\ 0.0096\ (1+\frac{1}{0.2s})) \tag{27}$$

The step responses of the closed-loop system obtained by our design procedure are shown in Fig. 6 together with those of the system designed by McMorran. The two results do not differ very much except that our design is a little more successful in suppressing the interaction from r_1 to y_2. This seems to have resulted from the fact that we concentrated our pseudo-diagonalizing efforts on the intermediate frequency range (see below). The main advantage of our procedure is that we can reach this result much easier with a clearer guideline; i.e. we do not need to scale the inputs and outputs in order to balance $G(s)$ because our GG pseudo-bands



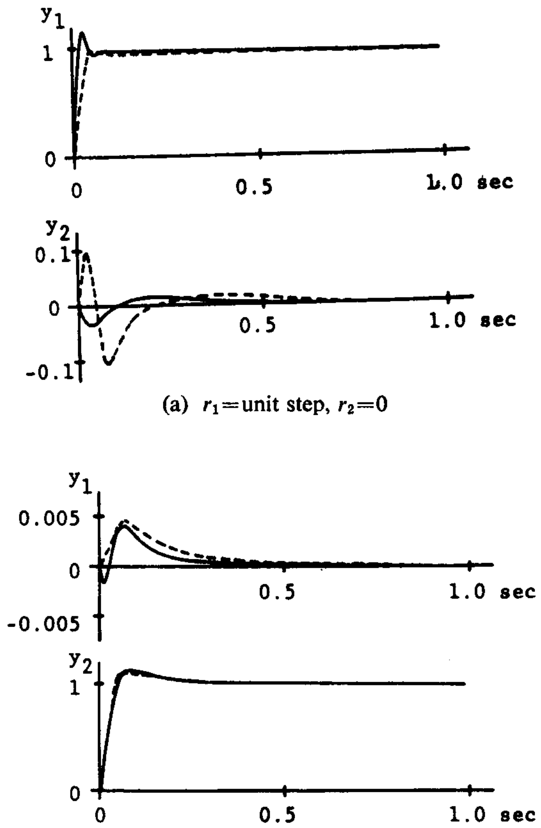
(a) The 1st loop



(b) The 2nd loop

Fig. 5. The *GG* pseudo-bands of the gas-turbine example after pseudo-diagonalizing compensation by $L(s)$

automatically have the same width for all loops. Also, we need not care about the interaction in the high frequency range (see below), and we know which frequencies our pseudo-diagonalizing efforts must be concentrated upon and how much of pseudo-diagonalization should be attained (see below).



(a) r_1=unit step, r_2=0

(b) r_1=0, r_2=unit step

Fig. 6. The responses of the closed-loop system
——; our design, ······: McMorran's design

Now, let us consider the problem: "How small should the interaction index $\lambda(iw|Q)$ be made at each frequency?". As explained in §2, the *GG* pseudo-disk covers the lower side of the curve given in Fig. 2b when $\lambda(iw|Q)\geqq 1$. Hence, if $\lambda(iw|Q)\geqq 1$ at low frequencies, it becomes impossible to increase the loop gain keeping the point (OdB, $-180°$) outside the *GG* pseudo-bands. Therefore, the interaction index $\lambda(iw|Q)$ must be made less than 1 in the low frequency range $[0, w_0]$. Here, the upper bound frequency w_0 is the frequency at which $f_j(iw)q_{jj}(iw)$ is expected to become sufficiently small and can be estimated as follows; i.e. when the main compensator is expected to be made up of the *P*, *PI* or phase-lag element w_0 can be chosen as the maximum w at which $\arg[q_{jj}(iw)]\simeq -180°$. When the main compensator is expected to include the *D* or phase-lead element w_0 should be chosen a few times larger than the above. For the high frequency range, we need not pay too much attention to the interaction index $\lambda(iw|Q)$ because $f_j(iw)q_{jj}(iw)$ becomes small in that frequency range and, therefore, the *GG* pseudo-bands lie much below the point (OdB, $-180°$).

The above reqirement is sufficient when the cross interaction of the closed-loop system does not matter; i.e. in the case of constant control systems. When the cross interaction of the closed-loop system must be made small, as is the case with the follow-up control systems, it is necessary to make the open-loop interaction index considerably small in the range of frequencies at which $|f_j(iw)q_{jj}(iw)|$ become near 1. This is because the closed-loop interaction index $\lambda(iw|H)$ can become much larger than the open-loop interaction index $\lambda(iw|Q)$ at these frequencies (see §4). Therefore, the efforts of pseudo-diagonalization must be mainly concentrated on the intermediate frequency range $[w_0', w_0]$ in which $|f_j(iw)q_{jj}(iw)|$ becomes near 1. Figures 3a and 3b give necessary information concerning about how small the open-loop interaction index should be made in this frequency range.† In the design of $L(s)$ for the gas-turbine example, the intermediate frequency was estimated as [10,1000] and $\lambda(iw|Q)\leqq 0.1$ was realized there as described above.

6. Remarks

The theoretical basis of the design method proposed here is the theorem of Araki & Nwokah.[2] Further research has been reported concerning this theorem.[15],[16] If the design of the main controller $f_j(s)$ is carried out based on the $M_p=\alpha$ (α is a constant usually around 1.3) criterion, the width of the *GG* pseudo-bands can be reduced as shown by Yamamoto et al.[24] The interaction index was already used by Araki & Nwokah[2] and Nwokah[14]. Its implication given in §4 is due to Araki[1]. The interaction functionals of Gray & Taylor[7] have close relation to the consideration given in §4. To be exact, by considering the mini-max problem we constructed a scalor measure, whereas Gray & Taylor used the vector quantities to give more specific information for each loop in a graphical form. The use of Nichols' chart for a multivariable design is also proposed by Pak et al.[17] and Leininger[11]. Further examples of the application of our method are reported by Araki et al.[3] and Shibata et al.[22]

If we consider the fact that the *GG* pseudo-bands give the maximum range of the deviation of $h_j(iw|QF)$ from $f_j(iw)q_{jj}(iw)$ caused by the off-diagonal terms of $Q(s)$, and that the influences of the off-diagonal terms, which are vector quantities, are seldom aligned in reality, we can expect, when $n\geqq 3$, to reach a reasonable result by using narrower bands. Our previous experience tells us that the band which is

† We usually use $\lambda(iw/Q)\leqq 0.2$ as our guide line for the intermediate frequency range.

$1/(n-1)$ times narrower can work fairly well, though this modification does not have any theoretical basis.

7. Conclusion

The main contribution of this paper is the proposal of a new design method. In this design method, the interaction index defined as the maximum eigenvalue of the interaction matrix (see (2)) plays a crucial roll. The interaction index is a suitable measure of the interaction in the sense that

(i) (concerning closing loops) it gives the upper bound of the relative deviation of the open-loop transfer function seen from one input-output pair which is caused by the feedback at the other input-output pairs (see §2), and

(ii) (concerning control operation) it gives the minimax optimal value of the relative error at the output caused by the operation of the uncorresponding reference signals (see §4).

The interaction index possesses satisfactory invariance properties: i.e. it is invariant

(iii) under the transpose operation,

(iv) against cascading a diagonal transfer matrix, and

(v) under the changes of the unit systems (i.e. scaling).

The interaction index is related to the generalized diagonal dominance condition by the proposition:

(vi) the transfer matrix $Z(s)$ is diagonally dominant in the generalized sense, if and only if its interaction index is less than 1.

The proposed design method succeeds Rosenbrock's idea as its principle policy, but differs from his method in that we use the generalized Gershgorin pseudo-bands of the ordinary transfer matrix instead of the Gershgorin bands of the inverse transfer matrix. Furthermore, we use the interaction index as the measure of pseudo-diagonalization. The main advantages of our method are

(a) that the generalized Gershgorin pseudo-bands are balanced (i.e. have the same width for all loops),

(b) that no generalized diagonal dominance is required at high frequencies, and

(c) that a quantitative guide line is given for the pseudo-diagonalization of the closed-loop system.

The item (a) releases us from the work of balancing the transfer matrix. Actually, McMorran needed to apply a preliminary operation K_a in order to balance the transfer matrix[18]. The dominance sharing operation by Leininger has a similar purpose[10]. Our method solves this balancing problem by using the generalized Gershgorin bands which are inherently balanced. As indicated by Araki & Nwokah,[2] it is possible to assign an arbitrary proportion to the widths of the bands among different loops.

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