Note on Weak and Short Waves in One-Dimensional Inhomogeneous Nonlinear Elastic Materials

By

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(Received December 25, 1982)

Abstract

The method of geometrical optics has been applied to the analysis of weak and short waves in one-dimensional deformed elastic materials [Tokuoka, J. Acoust. Soc. Am. **69**, 66–69 (1981)]. Since the conditions on the amplitude of the wave required for the order estimation may be violated after a time interval, in general, the obtained result holds only for a small time interval. This note improves the above result, and obtains a partial differential equation of a parabolic type for the displacement which remains valid for a longer time interval. The difference of the distortion of the waveform according to the original and the improved equations is discussed briefly for an example.

In a recent paper[1] Tokuoka analyzed, by use of the method of geometrical optics, the distortion of the waveform in one-dimensional inhomogeneous nonlinear elastic materials. We first summarize his result and show that, in general, it remains valid only for a small time interval.

The amplitude a and the wavelength λ of the wave are assumed to be small, respectively, in comparison with λ and the distance l, such that a and the inhomogeneities of the prestrain and the material property change considerably. Then the displacement u due to the wave motion can be expressed as

$$u(X, t)=a(X, t)f[\psi(X, t)], \tag{1}$$

where X denotes the co-ordinate in the reference configuration, t the time, f the waveform, and ψ the eikonal. The wavenumber k, the frequency ω, the wavelength λ, and the propagation velocity V are, respectively, defined by

$$k\equiv\frac{\partial\psi}{\partial X}, \qquad \omega\equiv-\dot{\psi}, \qquad \lambda\equiv\frac{2\pi}{k}, \qquad V\equiv\frac{\omega}{k}. \tag{2}$$

The constitutive equation of the inhomogeneous nonlinear elastic material is given by

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$$\sigma=\sigma(e, X), \tag{3}$$

where σ is the stress and e is the strain. Tokuoka estimated the order of the quantities concerned with a, k, ω and the strain and the stress ahead of the wave. Here we lay down only the estimations for a and its derivatives

$$0\left(\frac{a}{\lambda}\right)=\varepsilon\ll 1, \qquad 0\left(\frac{\partial a}{\partial X}\right)=\varepsilon^2, \qquad 0(\dot{a})=V\varepsilon^2,$$
$$0\left(\frac{\partial^2 a}{\partial X^2}\right)=\frac{\varepsilon^3}{\lambda}, \qquad 0(\ddot{a})=\frac{V^2\varepsilon^3}{\lambda}. \tag{4}$$

By applying the above estimations to the equation of motion, he obtained the propagation velocity and the amplitude equation:

$$V=\left(\frac{\sigma_0'}{\rho_\kappa}\right)^{1/2}, \tag{5}$$

$$\frac{\delta a}{\delta t}=-\alpha a+\gamma f'' a^2, \tag{6}$$

where

$$\alpha\equiv\frac{1}{4\sigma_0'}\left(\sigma_0''\frac{\partial e_0}{\partial t}+V\frac{\partial\sigma_0'}{\partial X}\right)+\frac{V}{4\rho_\kappa}\frac{\partial\rho_\kappa}{\partial X}, \tag{7*}$$

$$\gamma\equiv-\frac{\omega^2\sigma_0''}{2\sigma_0' V}, \tag{8}$$

ρ_κ is the mass density in the reference configuration, e_0, $\sigma_0'\equiv\partial\sigma/\partial e$, $\sigma_0''\equiv\partial^2\sigma/\partial e^2$ are evaluated values ahead of the wave, $f''\equiv\partial^2 f/\partial\psi^2$, and $\delta/\delta t$ denotes the displacement time derivative defined by

$$\frac{\delta\phi}{\delta t}=\dot{\phi}+V\frac{\partial\phi}{\partial X}. \tag{9}$$

In a special case where the prestrain is homogeneous and static, and the material is also homogeneous, we can put

$$\psi=kX-\omega t, \tag{10}$$

where k and ω are constants. Then α vanishes and γ becomes a constant. In this case the solution of (6) is given by

$$a=\frac{a(0)}{1-a(0)\gamma f'' t}, \tag{11}$$

where we have assumed that

$$a(X, 0)\equiv a(0)>0 \tag{12}$$

* Expression of α given in the preceding paper[1] is corrected here.

without loss of generality, and that a satisfies (4) at $t=0$. From (10) and (11) we can calculate

$$\frac{\partial^2 a}{\partial X^2}=k^2 a(0)^2 \gamma \frac{f''''t+a(0)\,\gamma(2f'''f'''-f''f'''')t^2}{[1-a(0)\,\gamma f''\,t]^3}. \tag{13}$$

For t of an order smaller than or equal to λ/V, by use of $(4)_1$ we can estimate the right hand side of (13) as

$$0\left(\frac{\partial^2 a}{\partial X^2}\right)=\frac{V\varepsilon^2}{\lambda^2}\,0(f''''t), \tag{14}$$

where we have also used the estimation $0(\gamma)=V/\lambda^2$. Thus, we find that if f'''' is a quantity of order 1, $(4)_4$ is fulfilled only for the time t of order $\varepsilon\lambda/V$. Hence, the amplitude equation (6) and the solution (11) hold only for a time interval of an order smaller than $\varepsilon\lambda/V$.

Next, we return to the general case and derive an equation for u which remains valid for a longer time interval than the above. Henceforth, we regard a and u as functions of ψ and t by transforming the independent variables X and t into ψ and t. Note that the displacement time derivative is identical with the partial derivative, with respect to the time for such functions of ψ and t. Taking the displacement time derivative of (1) and referring to (6) we have

$$\frac{\delta u}{\delta t}=-\alpha u+\gamma a f'' u. \tag{15}$$

Also, taking the second derivative of (1) with respect to ψ we have

$$u''=af''+2a'f'+a''f, \tag{16}$$

where a prime denotes the differentiation with respect to ψ. If a satisfies (4), we can estimate that

$$0(a)=\lambda\varepsilon, \quad 0(a')=\lambda\varepsilon^2, \quad 0(a'')=\lambda\varepsilon^3. \tag{17}$$

Neglecting the terms of the smaller order in (16), by use of (17) and substituting the result into (15), we obtain a partial differential equation of a parabolic type for u:

$$\frac{\delta u}{\delta t}=-\alpha u+\gamma u u''. \tag{18}$$

In general, the displacement u_1 governed by (1) and (6) does not coincide with the displacement u_2 governed by (18) for $t>t_0$, where we assume that they coincide at $t=t_0$. It is apparent, however, that if u_1 remains valid for a time, so does u_2 also within an error of the same order, because (18) is derived from (1) and (6) by using the estimations (4) assumed in the process of derivation of (6). Furthermore, we can show that (18) holds when the estimations, except (4), are fulfilled and when there exists a decomposition of u into f and a satisfying (4) at each instant. In other words, for (18) to hold, it is not required that a prescribed amplitude satisfies (4) for every t in a time interval. In fact, (18) is expressed in terms of u and the given functions α and γ, and this fact means that the expression does not depend on the manner of the decomposition of u into a and f. Thus, if the estimations, including (4), are satisfied at a time for an a, (18) remains valid at that time.

To demonstrate the difference between u_1 and u_2, we consider again a special case in which α vanishes identically and γ reduces to a constant, and we assume that u_1 and u_2 coincide at $t=0$. In this case, u_1 is given by (1) and (11), and u_2 is the solution of the equation:

$$\frac{\delta u}{\delta t}=\gamma u u''. \tag{19}$$

Henceforth we assume that $\gamma>0$. From (11) the absolute value of u_1 at any fixed

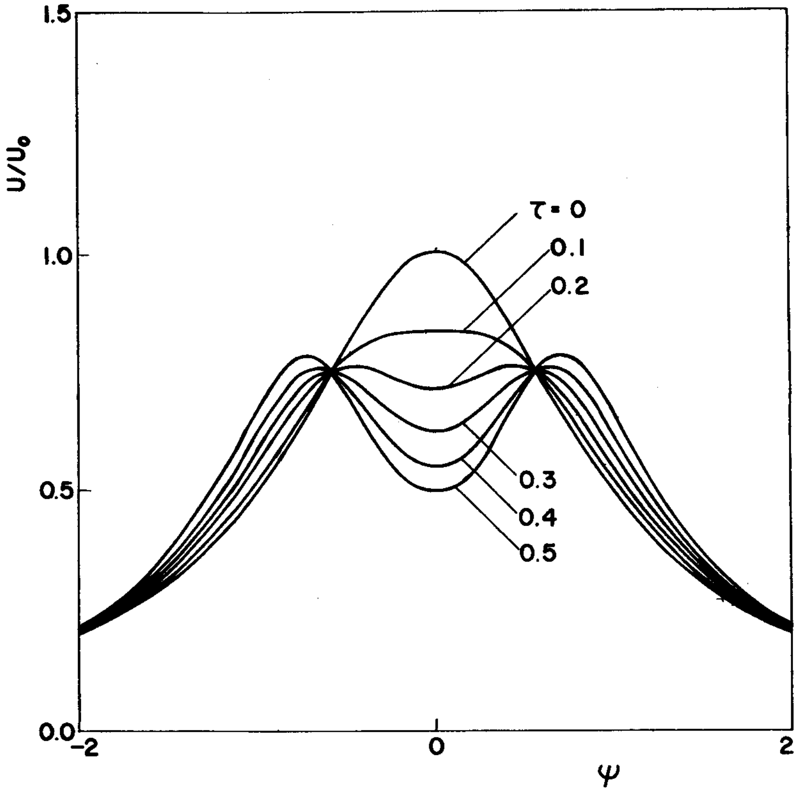

Fig. 1. Variation of displacement governed by original equations (1) and (6) for $\alpha=0$ and $\gamma>0$, where $\tau=\gamma u_0 t$.

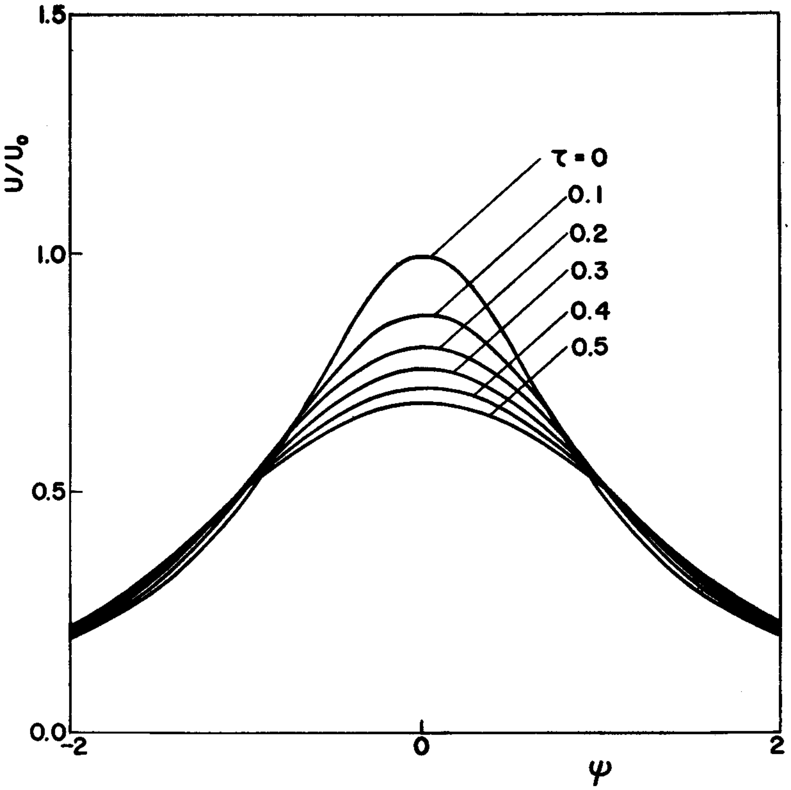

Fig. 2. Variation of displacement governed by improved equation (18) for $\alpha=0$ and $\gamma>0$, where $\tau=\gamma u_0 t$.

point ψ increases or decreases, respectively, if u_1'' *at the initial time* is positive or negative. On the other hand, from (19) the growth and decay of the absolute value of u_2 depend on the sign of u_2'' *at the present time*. This situation is illustrated in Figs. 1 and 2, where the initial condition is given by $u=u_0/(1+\psi^2)$, and we have solved (19) by a numerical calculation. In Fig. 1, u_1 at $\psi=0$ decreases even after u_1'' at the point becomes positive, but such a phenomenon never occurs for u_2.

It is worth noting that if u is always positive, (19) can be transformed into the equation of heat conduction with temperature-dependent thermal conductivity[2-7]. That is, by introducing a new variable $\theta\equiv\ln u$ we may rewrite (19) as

$$\frac{\delta\theta}{\delta t}=[\gamma\exp(\theta)\theta']'. \tag{20}$$

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