Note on the Energy Release Rate of Simple Crack Models with Finite Jump Discontinuities

By

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(Received December 27)

Abstract

A spherical flaw model in linear viscoelasticity and a simple double cantilever model were proposed by Williams and Freund respectively, with the hope that they would lead to a qualitatively similar phenomenon as a common crack model. Although both models were very simple, they had jump discontinuities of stress at the crack tip (or the like) at any time. In this paper, a general expression for the energy release rate in such crack models is derived and is evaluated for each model. Also, Nuismer's modified interpretation of Williams' model is discussed. Furthermore, the expressions of a partition for the energy release rate into the usual quasi-static part and a dynamic contribution are evaluated in Freund's model.

1. Introduction

Recently we[1] gave a unified treatment of dynamic energy release rate $\mathcal{E}$, for a sharp, straight crack. It was based on a classical smoothness hypothesis; for instance, the neighborhood of the crack tip is assumed to be free of shock waves.

On the other hand, Williams[2,3] proposed a spherical flaw model in linear viscoelasticity, and Freund[4] analysed a very simple model of the double cantilever beam with the hope that they would lead to a qualitatively similar phenomenon as a common crack model. Although both models were very simple, they had jump discontinuities of stress at the crack tip (or the like) at any time. This motivated us to examine the expressions of the energy release rate in such models.

After giving the basic equaitons in section 2, we define the dynamic energy release rate $\mathcal{E}$ and get the simple relation on $\mathcal{E}$ in Section 3. Using this general idea, we first discuss Williams' model[2,3] and the modified interpretation by Nuismer[5] in Section 4. We show that Nuismer's modified expression does not give the energy release rate correctly, except for the linear materials. In the final section, extending Freund's model to a two-dimensional one, we discuss and evaluate the partition $\mathcal{U}$ and $\mathcal{K}$ of $\mathcal{E}$ defined by Gurtin and Yatomi[1].

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2. Basic equations

We consider a three-dimensional regular body $\mathcal{B}$ which identifies with the regular region of $\mathbb{R}^3$ it occupies in a fixed reference configuration.

The displacement $\boldsymbol{u}(\boldsymbol{x}, t)$, the (Piola-Kirchhoff) stress $\boldsymbol{S}(\boldsymbol{x}, t)$ and the work $w(\boldsymbol{x}, t)$ obey the energy equation

$$\dot{w}=\boldsymbol{S}\cdot\nabla\dot{\boldsymbol{u}} \tag{2・1}$$

and the equation of motion

$$\operatorname{div}\boldsymbol{S}=\rho\ddot{\boldsymbol{u}} \tag{2・2}$$

with $\rho>0$ the density in the reference configuration. We assume throughout that ρ is constant. The above equations are appropriate to both the finite and infinitesimal theories. In the infinitesimal theory $\boldsymbol{S}$ is symmetric.

3. Energy-release rate

We assume that the regular sub-region $\mathcal{P}\subset\mathcal{B}$ contains a smooth moving surface $\mathcal{S}(t)$ with a velocity $\boldsymbol{c}=|\boldsymbol{c}|\boldsymbol{n}$, as indicated in Figure 1, where $\boldsymbol{n}$ is a continuous unit vector field normal to the surface $\mathcal{S}(t)$.

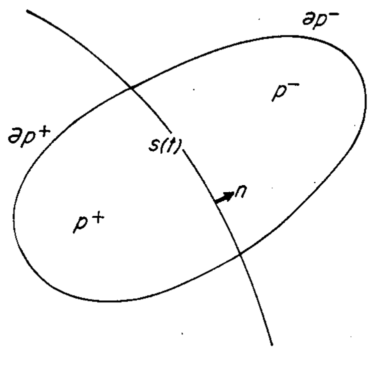

Fig. 1.

We also assume that the fields $\Phi(\boldsymbol{x}, t)$, such as w, $\boldsymbol{S}$ and $\dot{\boldsymbol{u}}$, suffer a finite jump discontinuity across $\mathcal{S}(t)$, but are smooth everywhere else. We write

$$\Phi^{\pm}(\boldsymbol{x}, t)=\lim_{\delta\to 0^+}\Phi(\boldsymbol{x}\mp\delta\boldsymbol{n}, t),\quad \boldsymbol{x}\in\mathcal{S}(t)$$

for the values of their fields on the back and front faces of $\mathcal{S}(t)$ and

$$[\Phi]=\Phi^{+}-\Phi^{-}$$

for the jump in Φ across $\mathcal{S}(t)$.

The function $\mathcal{E}$ defined by

$$\mathcal{E}(t)=\int_{\partial\mathcal{P}}\boldsymbol{s}\cdot\dot{\boldsymbol{u}}ds-\frac{d}{dt}\int_{\mathcal{P}}(w+k)dv \tag{3・1}$$

is called the dynamic energy release rate from $\mathcal{S}(t)$. *

Here, $k=\frac{\rho}{2}\dot{\boldsymbol{u}}^2$ is the kinetic energy per unit volume, and $\boldsymbol{s}$ in a boundary integral denotes the surface traction:

$$\boldsymbol{s}=\boldsymbol{Sn}.$$

Applying the divergence theorem to the two regions $\mathcal{D}^+$ and $\mathcal{D}^-$, bounded by $\partial\mathcal{D}^+ + \mathcal{S}(t)$ and $\partial\mathcal{D}^- + \mathcal{S}(t)$ respectively, and by (2・1) and (2・2), we have

$$\int_{\partial\mathcal{D}} \boldsymbol{s}\cdot\dot{\boldsymbol{u}}ds=\int_{\mathcal{D}}(\dot{w}+\dot{k})dv-\int_{\mathcal{S}(t)}[\boldsymbol{s}\cdot\dot{\boldsymbol{u}}]ds.$$

Since

$$\frac{d}{dt}\int_{\mathcal{D}}(w+k)dv=\int_{\mathcal{D}}(\dot{w}+\dot{k})dv+\int_{\mathcal{S}(t)}[w+k]\boldsymbol{c}\cdot\boldsymbol{n}ds$$

the relation (3・1) yields the simple formula:

$$\mathcal{E}(t)=-\int_{\mathcal{S}(t)}([\boldsymbol{s}\cdot\dot{\boldsymbol{u}}]+[w+k]|\boldsymbol{c}|)ds. \qquad (3\cdot 2)$$

4. Spherical flaw model

Because of its analytical simplicity and qualitatively similar results as a crack model, Williams[3] proposed a dynamic spherical flaw fracture model in infinitesimal theories of linear viscoelasticity. He considered an incompressible viscoelastic sphere $\mathcal{B}(t)$ of radius b with a concentric spherical cavity of radius $a(t)$. The sphere was subjected to a uniform radial loading at its outer bounary.

Then the rate of increase of surface energy $\dot{S}$ was computed by

$$\dot{S}=\int_{\partial\mathcal{B}_0}\boldsymbol{s}\cdot\dot{\boldsymbol{u}}ds-\frac{d}{dt}\int_{\mathcal{B}(t)}\int_0^t(\dot{w}+\dot{k})dtdv, \qquad (4\cdot 1)$$

where $\partial\mathcal{B}_0$ is the outer surface.

Regarding $\mathcal{S}(t)$ as the surface of the spherical flaw, that is, the inner surface, and taking $\mathcal{D}=\mathcal{B}(0)$, $\mathcal{D}^-=\mathcal{B}(t)$ and $\mathcal{D}^+=\mathcal{B}(0)/\mathcal{B}(t)$ in (3・1), we find that $\mathcal{E}(t)$ in the definition (3・1) is exactly the same as $\dot{S}$ in (4・1), since $\boldsymbol{s}=\boldsymbol{0}$ at the inner surface and $w=k=0$ in $\mathcal{B}^+$. Thus, (3・2) gives the simple formula for $\mathcal{E}$:**

$$\mathcal{E}(t)=\int_{\mathcal{S}(t)}(w^-+k^-)\dot{a}(t)ds.$$

Cherepanov[6] and Knauss[7] extended Williams' analysis[2] to the case of a Griffith crack in a quasi-static growth. Using a power balance as a fracture criterion, they obtained results similar to those of Williams[2]. Different conclusions were reached, however, by Graham[8] who used a virtual work argument, and by Nuismer[5] who used a thermodynamic power balance in a form used by Achenbach and Nuismer[9]. That is, the quantity like $\mathcal{E}$ which governs the initiation and growth of crack did

* Formally (3.1) is the same as (4.2) defined by Gurtin and Yatomi[1].

** Williams[3] evaluated (4.1) directly.

not involve the time-dependent part of the viscoelastic material behavior, but only the elastic portion. Then, Nuismer[5] gave an interesting interpretation of Williams' analysis, the results of which became identical to those obtained by Graham[8] and himself.

First of all, to eliminate the difficulties encountered by having a disappearing mass in $\mathcal{D}^+$, behind the advancing flaw surface $\mathcal{S}(t)$, he assumed that the strains $\boldsymbol{E}$ do not vanish in $\mathcal{D}^+$, although stress $\boldsymbol{S}$ vanishes there. As a result, the fields were given in the form*

$$\boldsymbol{S}(r,\ t)=\boldsymbol{\sigma}(r,\ t)H(r-a(t)), \tag{4・2}$$

$$\boldsymbol{E}(r,\ t)=\boldsymbol{e}(r,\ t)H(r-a(t))+\hat{\boldsymbol{E}}(r,\ t), \tag{4・3}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{e}$ and $\hat{\boldsymbol{E}}$ are continuous functions in the domain considered, $\hat{\boldsymbol{E}}\equiv\boldsymbol{0}$, if the material is elastic, and H is the unit Heaviside step function. Nuismer[5] then employed the power balance (4・1) in a slightly modified form

$$\dot{\Gamma}=\int_{\partial\mathcal{B}_0}\boldsymbol{s}\cdot\dot{\boldsymbol{u}}ds-\int_{\mathcal{B}(0)}\dot{w}dv, \tag{4・4}$$

where $\dot{w}$ was interpreted as the generalized function. Since $\dot{w}=\boldsymbol{S}\cdot\dot{\boldsymbol{E}}$, applying the divergence theorem and using (4・2), (4・3), and the relation used by Nuismer[5]:

$$\int_{\mathcal{B}(0)}f(r)h(r-a(t))\delta(r-a(t)dv=\frac{1}{2}\int_{\mathcal{S}(t)}f(a(t))ds,$$

where δ is the Dirac delta function and $f(r)$ is continuous in the neighborhood of $r=a(t)$, we have

$$\dot{\Gamma}=-\left(\int_{\mathcal{S}(t)}\frac{1}{2}\boldsymbol{\sigma}\cdot[\boldsymbol{E}]ds\right)\dot{a}(t),$$

$$=\left(\int_{\mathcal{S}(t)}\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{e}ds\right)\dot{a}(t). \tag{4・5}$$

It is easy to check that this gives exactly the same fracture condition as (23) in Nuismer[5], which does not involve the time-dependent part of the viscoelastic material behavior.

On the other hand, even in this modified interpretation of the fracture process, we consider that the simple application of (3・1):

$$\mathcal{E}(t)=\int_{\partial\mathcal{B}_0}\boldsymbol{s}\cdot\dot{\boldsymbol{u}}ds-\frac{d}{dt}\int_{\mathcal{B}(0)}wdv \tag{4・6}$$

gives the rate of increase of surface energy correctly. Then, since $\boldsymbol{s}=\boldsymbol{0}$ on $\mathcal{S}(t)$ and $w=0$ in $\mathcal{D}^+$, (3・2) yields

$$\mathcal{E}(t)=\left(\int_{\mathcal{S}(t)}w^{-}dt\right)\dot{a}(t),$$

where

$$w^{-}=\int_0^t\boldsymbol{S}^{-}\cdot\dot{\boldsymbol{E}}^{-}dt.$$

* These expressions may be derived from (12), (13) and (15) in Nuismer[5].

It is important to note the difference of the results between (4・4) and (4・6). Note that the expression (4・4) yields the form of a linear type just as (4・5) independently of the materials, as long as the fields are given in the forms of (4・2) and (4・3). However, since $w \neq \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{e}$ in general non-linear materials, the power balance relation (4・4), which was modified to account for all the materials by Nuismer[5], does not give $\mathcal{E}$ in (4・6) except for the linear materials. Interestingly, we find[10] that a similar restriction appears in Achenbach's expression[11] for $\mathcal{E}$ in the common crack problem.

5. Double cantilever beam model

Freund[4] analysed a very simple one-dimensional model of the double cantilever beam (dcb) dynamic crack propagation to understand the details of the physical process.

Extending his model to a two-dimensional one, we will discuss the dynamic energy release rate $\mathcal{E}$, and examine the partition $\mathcal{U}$ and $\mathcal{K}$ which was defined by Gurtin and Yatomi[1]*.

The finite specimen $\mathcal{B}$ contains a propagating plane crack along the center plane. (See Fig. 2.) The material ahead of the crack tip surface $\mathcal{S}(t)$, the cross section which is across the crack tip line $l(t)$, is assumed to be undeformed. The material behaind $\mathcal{S}(t)$ is constrained to deform as shear body symmetrically with respect to the center plane. That is to say, the cross-sections can translate only in

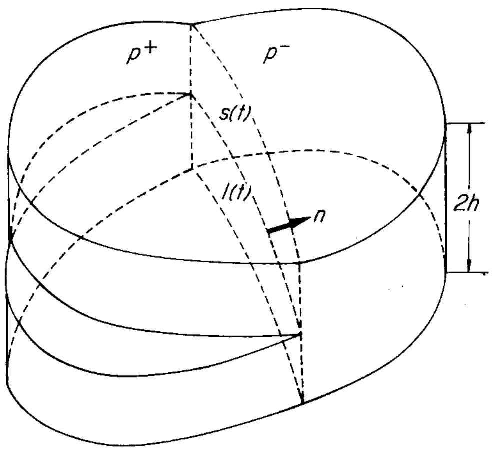

Fig. 2.

* Rice[12] made this partition first for the common crack model. Gurtin and Yatomi[1] reduced $\mathcal{K}$ to a simple form and proved $\mathcal{K}\leq 0$ assuming the neighborhood of the crack tip to be free of any shock wave.

the transverse direction. The amount of translation is denoted by $\omega(\boldsymbol{x}, t)$, where $\boldsymbol{x}$ is the point in the center plane. We assume the linear elastic constitutive equation:

$$\boldsymbol{\sigma}=\mu\boldsymbol{q}, \tag{5・1}$$

where $\boldsymbol{\sigma}(x, t)$ is the shear stress vector, μ the shear constant, and $\boldsymbol{q}=\nabla\omega$. Then we have

$$w=\frac{\mu}{2}|\boldsymbol{q}|^2, \tag{5・2}$$

$$k=\frac{\rho}{2}\dot{\omega}^2. \tag{5・3}$$

Assuming that $\dot{\omega}$ and $\boldsymbol{q}$ have a finite jump at $l(t)$, make $\mathcal{D}$ in the neighborhood of the crack tip surface sufficiently small to make $\boldsymbol{q}$ and $\dot{\omega}$ smooth in the center plane a in $\mathcal{D}$, except $l(t)$. By the assumption $\omega(\boldsymbol{x}, t)=0$ for all $\boldsymbol{x}\in a^- + l(t)$ and t, all the fields considered vanish in a^- and

$$\dot{\omega}^+ + \boldsymbol{q}^+\cdot\boldsymbol{c}=0, \tag{5・4}$$

where $\boldsymbol{c}=|\boldsymbol{c}|\boldsymbol{n}$ is the velocity of the crack tip surface with an unit normal $\boldsymbol{n}$.

Using (5・1)～(5・4), we have

$$-[\boldsymbol{s}\cdot\dot{\boldsymbol{u}}]=\mu(\boldsymbol{q}^+\cdot\boldsymbol{n})^2|\boldsymbol{c}|,$$

$$[w]|\boldsymbol{c}|=\frac{\mu}{2}|\boldsymbol{q}^+|^2|\boldsymbol{c}|,$$

and

$$[k]|\boldsymbol{c}|=\frac{\rho}{2}(\boldsymbol{q}^+\cdot\boldsymbol{n})^2|\boldsymbol{c}|^3.$$

Hence, the dynamic energy release rate from $\mathcal{S}(t)$ in (3・2) yields

$$\mathcal{E}(t)=2h\int_{l(t)}\left[\frac{\mu}{2}\{2(\boldsymbol{q}^+\cdot\boldsymbol{n})^2-|\boldsymbol{q}|^{+2}\}|\boldsymbol{c}|-\frac{\rho}{2}(\boldsymbol{q}^+\cdot\boldsymbol{n})^2|\boldsymbol{c}|^3\right]ds, \tag{5・5}$$

where $2h$ is the total height of $\mathcal{D}$.

Since $[\omega]=0$ and, therefore, the strain in the tangential direction of $\mathcal{S}(t)$ vanishes on the back face of $\mathcal{S}(t)$, we have

$$|\boldsymbol{q}|^+=\boldsymbol{q}^+\cdot\boldsymbol{n}.$$

Then (5・5) becomes

$$\mathcal{E}(t)=\mu h\int_{l(t)}(1-|\boldsymbol{c}|^2/v^2)|\boldsymbol{c}|\,|\boldsymbol{q}^+|^2ds, \tag{5・6}*$$

where $v=\sqrt{\mu/\rho}$ is the shear velocity.

On the other hand, by (3・1), we have

$$\mathcal{E}(t_0)=2h\left\{\int_{\partial a}(\boldsymbol{\sigma}_{t_0}\cdot\boldsymbol{n})\dot{\omega}_{t_0}ds-\left(\frac{d}{dt}\right)_{t_0}\int_a(w_t+k_t)da\right\}, \tag{5・7}$$

where $\left(\frac{d}{dt}\right)_{t_0}$ stands for the derivative with respect to t at $t=t_0$.

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* Cf. Freund[4], Eq. (3・3).

The first integral in (5・7) is equal to

$$\int_{\alpha}(\dot{w}_{t_0}+\dot{k}_{t_0})da-\int_{l(t_0)}(\boldsymbol{\sigma}_{t_0}{}^{+}\cdot\boldsymbol{n})\dot{\omega}^{+}{}_{t_0}ds.$$

Since $\boldsymbol{q}^{+}=|\boldsymbol{q}|^{+}\boldsymbol{n}$ and $\dot{\omega}^{+}+|\boldsymbol{q}||\boldsymbol{c}|=0$, this equals

$$\int_{\alpha}(\dot{w}_{t_0}+\dot{k}_{t_0})da+\int_{l(t_0)}(\boldsymbol{\sigma}_{t_0}{}^{+}\cdot\boldsymbol{q}_{t_0}{}^{+})|\boldsymbol{c}_0|ds$$

$$=\int_{\alpha}(\boldsymbol{\sigma}_{t_0}\cdot\dot{\boldsymbol{q}}_{t_0}+\rho\ddot{\omega}_{t_0}\dot{\omega}_{t_0})da+\lim_{t\to t_0}\int_{l(t)}[\boldsymbol{\sigma}_t\cdot(\boldsymbol{q}_t-\boldsymbol{q}_{t_0})$$

$$+\rho\ddot{\omega}_t(\omega_t-\omega_{t_0})]\boldsymbol{c}_t\cdot\boldsymbol{n}ds$$

$$=\left(\frac{d}{dt}\right)_{t_0}\int_{\alpha}\{\boldsymbol{\sigma}_t\cdot(\boldsymbol{q}_t-\boldsymbol{q}_{t_0})+\rho\ddot{\omega}_t(\omega_t-\omega_{t_0})\}da.$$

Hence, (5・7) yields

$$\mathcal{E}(t_0)=2h\left(\frac{d}{dt}\right)_{t_0}\int_{\alpha}\int_{t_0}^{t}\{(\boldsymbol{\sigma}_t-\boldsymbol{\sigma}_\lambda)\cdot\dot{\boldsymbol{q}}_\lambda+\rho(\ddot{\omega}_t-\ddot{\omega}_\lambda)\dot{\omega}_\lambda\}d\lambda da.$$

Then the partition of $\mathcal{E}$ is defined by[1]

$$\mathcal{U}(t_0)=2h\left(\frac{d}{dt}\right)_{t_0}\int_{\alpha}\int_{t_0}^{t}(\boldsymbol{\sigma}_t-\boldsymbol{\sigma}_\lambda)\cdot\dot{\boldsymbol{q}}_\lambda d\lambda da,$$

$$\mathcal{K}(t_0)=2h\left(\frac{d}{dt}\right)_{t_0}\int_{\alpha}\int_{t_0}^{t}\rho(\ddot{\omega}_t-\ddot{\omega}_\lambda)\dot{\omega}_\lambda d\lambda da.$$

For this simple model we may easily evaluate $\mathcal{U}(t_0)$ and $\mathcal{K}(t_0)$ as

$$\mathcal{U}(t_0)=2h\left\{\left(\frac{d}{dt}\right)_{t_0}\int_{\alpha}\boldsymbol{\sigma}_t\cdot(\boldsymbol{q}_t-\boldsymbol{q}_{t_0})da-\left(\frac{d}{dt}\right)_{t_0}\int_{\alpha}w\,da\right\}$$

$$=2h\left[\left\{\int_{\alpha}\dot{w}_{t_0}da+\int_{l(t_0)}(\boldsymbol{\sigma}_{t_0}{}^{+}\cdot\boldsymbol{q}_{t_0}{}^{+})|\boldsymbol{c}_0|ds\right\}-\left\{\int_{\alpha}\dot{w}_{t_0}da\right.\right.$$

$$\left.\left.+\frac{1}{2}\int_{l(t_0)}(\boldsymbol{\sigma}_{t_0}{}^{+}\cdot\boldsymbol{q}_{t_0}{}^{+})|\boldsymbol{c}_0|ds\right\}\right],$$

$$=\mu h\int_{l(t_0)}|\boldsymbol{q}_{t_0}{}^{+}|^2|\boldsymbol{c}_0|ds,\quad(\geq 0)\tag{5・8}$$

and

$$\mathcal{K}(t_0)=2h\left\{\left(\frac{d}{dt}\right)_{t_0}\int_{\alpha}\rho\ddot{\omega}_t(\omega_t-\omega_{t_0})da-\left(\frac{d}{dt}\right)_{t_0}\int_{\alpha}\frac{1}{2}\rho\dot{\omega}_t{}^2da\right\}$$

$$=2h\left\{\int_{\alpha}\dot{k}_{t_0}da-\left(\int_{\alpha}\dot{k}_{t_0}da+\int_{l(t_0)}\frac{1}{2}\rho\dot{\omega}_{t_0}{}^{+2}|\boldsymbol{c}_0|ds\right)\right\}$$

$$=-\rho h\int_{l(t_0)}\dot{\omega}_{t_0}{}^{+2}|\boldsymbol{c}_0|ds$$

$$=-\rho h\int_{l(t_0)}|\boldsymbol{q}_{t_0}{}^{+}|^2|\boldsymbol{c}_0|^3ds\quad(\leq 0).\tag{5・9}$$

Let A be the whole center plane of $\mathcal{B}$, α as before and W_A, W_α, K_A, and K_α designate the strain energy and kinetic energy in A and α respectively. When the power supplied by the enviroment of $\mathcal{B}$ vanishes, we have

$$\mathcal{E}=\mathcal{U}+\mathcal{K}=-\dot{W}_A-\dot{K}_A,$$

in general

$$\mathcal{U}\neq-\dot{W}_\alpha\neq-\dot{W}_A,$$

$$\mathcal{K} \neq -\dot{K}_{\alpha} \neq -\dot{K}_{A}.$$

It is important to note that the values of $\mathcal{U}$ and $\mathcal{K}$ are independent of the shape of α, althugh other quantities have a shape dependency in general.

Thus $\mathcal{U}$ and $\mathcal{K}$ together with $\mathcal{E}$ seem to be more intrinsic and important to the dynamic behavior of the crack tip than others. In Freund's model of the dcb with constant crack velocity c, both $\dot{W}_A$ and $\dot{K}_A$ have a sudden jump at the instant the stress wave reflects from the fixed end of the arms. ($\dot{K}_A$ changes the sign there.) This reflection should not affect the behavior of the crack tip until such information is transmitted to the tip by the stress wave. Then both $\mathcal{U}$ and $\mathcal{K}$ in (5・8) and (5・9) are constant from the initiation of fracture until the instant the reflected stress wave overtakes the crack tip.

Finally, for this simple model, we may easily show that

$$\mathcal{U}(t_0) = \lim_{\text{area of } \alpha \to 0} \dot{W}_{\alpha},$$

$$\mathcal{K}(t_0) = -\lim_{\text{area of } \alpha \to 0} \dot{K}_{\alpha}.$$

Hence, $\mathcal{U}$ is the relased rate of strain energy and $\mathcal{K}$ is the generating rate of kinetic energy in α in the limit, that is, from the crack tip surface.

References

1) M. E. Gurtin and C. Yatomi; Arch. Ration. Mech. Anal., **74**, 231 (1980).
2) M. L. Williams; Int. J. of Fracture, **1**, 292 (1965).
3) M. L. Williams; Int. J. of Fracture, **4**, 69 (1968).
4) L. B. Freund; J. of Mech. Phys. Solids, **25**, 69 (1977).
5) R. J. Nuismer; J. of Applied Mech., **41**, 631 (1974).
6) G. P. Cherepanov; Int. J. of Solid and Structure, **4**, 811 (1968).
7) W. G. Knauss; Int. J. of Fracture, **6**, 7 (1970).
8) G. A. C. Graham; Quarterly of Applied Math., **27**, 497 (1970).
9) J. D. Achenbach and R. J. Nuismer; Int. J. of Fracture, **7**, 77 (1971).
10) C. Yatomi; Int. J. of Fracture, **18**, 233 (1982).
11) J. D. Achenbach; Mechanics Today, **1**, 1 (1974).
12) J. R. Rice; Proc. of the first Int. Conference in Fracture, Sendai, 309 (1965).