Steady State Analysis of Oscillator by Volterra Series

By

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Abstract

This paper describes the numerical method to obtain a steady state oscillation wave of oscillator by the Volterra series expansion method. An amplitude and a period of a fundamental component are determined by solving algebraic equations, after which the harmonic components can be calculated. The method is applied to the analvsis of the van der Pole oscillator and phase shift oscillator.

1. Introduction

In a linear circuit, it is impossible to generate a periodic oscillation which does not depend on the initial state, when the external force does not exist. Therefore, a self oscillator, which generates a nearly sinusoidal periodic oscillation wave with constant amplitude, must necessarily contain nonlinear elements. In such a self oscillator, the period and amplitude of a periodic oscillation wave are unknown. Furthermore, a periodic wave contains many harmonic components which are generated by nonlinear elements.

The periodic oscillation of the self excitation circuit has been investigated in detail for a long time by many reserchers. Most of the methods of analysis have been carried out by deriving the nonlinear differential equation which contains the small parameter, and solving it by such methods as the averaging method, the perturbation method and the asymptotic method[1].

Other methods for the study on the analysis of the nonlinear circuits have been reported. These methods are based on the theory of the Volterra series expansion[2],[3]. As an application, a method to determine a period and an amplitude of fundamental component of a self oscillation wave has been presented[4]. By this method, it is unnecessary to derive and solve the nonlinear equations of the circuit. A period and an amplitude of fundamental component of a periodic oscillation wave are determined by solving the algebraic equations derived from the block diagram of the circuit.

The above method is limited to determine the fundamental component. In this paper, we will extend the above method in order to get many harmonic components. A numerical method to get a self oscillation wave composed by many harmonic components will be shown. This method is applied to the analysis of the van der Pole oscillator and phase shift oscillator.

2. Steady state analysis of Volterra series

2.1 Volterra series expansion of nonlinear system

Here, we shall consider a system whose relation between the input function $x(t)$ and the output function $y(t)$ is given by the following equation.

$$y(t)=\int_{-\infty}^{\infty}h_1(\tau_1)x(t-\tau_1)d\tau_1+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2+\cdots\cdots \quad (1)$$

This equation is called the Volterra series expansion of the system, h_i is called the i-th Volterra kernel, and i-th integral term which contains h_i is called the i-th term.

The Volterra series whose kernels, except h_1 are 0, is the input-output representation of the linear system. There, the Laplace or Fourier transform is used as a powerful tool to analyse the system.

In the same way, a multi-dimensional Laplace or Fourier transform can be applied to the Volterra series expansion of Eq. (1).

Let us note the i-th term of Eq. (1) as $y_i(t)$ and define the following i-dimensional function as

$$y_i(t_1,t_2,\cdots,t_i)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_i(\tau_1,\tau_2,\cdots,\tau_i)$$
$$x(t_1-\tau_1)x(t_2-\tau_2)\cdots x(t_i-\tau_i)d\tau_1 d\tau_2\cdots d\tau_i \quad (2)$$

Then, we have the following relation:

$$y_i(t)=y_i(t_1,t_2,\cdots,t_i)\big|_{t_1=t_2=\cdots=t_i=t} \quad (3)$$

By the multi-dimensional Laplace transform we have

$$Y_i(s_1,s_2,\cdots,s_i)=H_i(s_1,s_2,\cdots,s_i)X(s_1)X(s_2)\cdots X(s_i) \quad (4)$$

where

$$\left.\begin{aligned} Y_i(s_1,s_2,\cdots,s_i)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}y_i(t_1,t_2,\cdots,t_i)e^{-s_1t_1-s_2t_2-\cdots-s_it_i}dt_1dt_2\cdots dt_i\\ H_i(s_1,s_2,\cdots,s_i)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_i(t_1,t_2,\cdots,t_i)e^{-s_1t_1-s_2t_2-\cdots-s_it_i}dt_1dt_2\cdots dt_i\\ X(s_k)&=\int_{-\infty}^{\infty}x(t_k)e^{-s_kt_k}dt_k\\ k&=1,2,\cdots,i \end{aligned}\right\} \quad (5)$$

The same relation can be obtained by the multi-dimensional Fourier transform, substituting $i\omega_i$ for s_i, in the above formulas.

In the field of system analysis, it is convenient to treat the problem in the domain of the Laplace or Fourier transform (We call it the s or ω domain.) rather than in the original time domain. (We call it the t domain.)

For the nonlinear system composed of the fundamental connections (summation, product and cascade connections) of several subsystems, the relation between the kernels of the total system and the kernels of the individual systems are given as the formulas in the s or ω domain[3]. Let us show the following two formulas used in this paper.

F. 1 When the input-output relation of system H is represented by the following polynomial

$$y(t) = a_1x(t) + a_2x^2(t) + \cdots + a_Nx^N(t) \tag{6}$$

the i-th order kernel of H is

$$\left.\begin{array}{l} H_i(s_1, s_2, \cdots, s_i) = a_i \\ i=1, 2, \cdots, N \end{array}\right\} \tag{7}$$

F. 2 When the two subsystems F and G are cascaded as shown in Fig. 1, where system F has i-th kernel $F_i(s_1, s_2, \cdots, s_i)$ $(i \geq 1)$, and G is a linear system which has only the lst kernel $G_1(s_1)$, i-th kernel H_i of the system H is

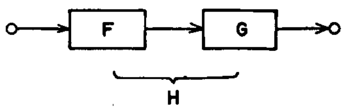

Fig. 1 Cascade connection

$$\left.\begin{array}{l} H_i(s_1, s_2, \cdots, s_i) = F_i(s_1, s_2, \cdots, s_i)G_1(s_1+s_2+\cdots+s_i) \\ i \geq 1 \end{array}\right\} \tag{8}$$

2.2 Calculation of periodic response[5]

Here we shall consider the steady state response substituting $i\omega_i$ for s_i. For the lst term

$$Y_1(i\omega_1) = H_1(i\omega_1)\,X(i\omega_1), \tag{9}$$

its inverse Fourier transform is given by

$$y_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_1(i\omega_1)\,X(i\omega_1)\,e^{i\omega_1 t}d\omega_1. \tag{10}$$

When the input functon is expressed by

$$x(t) = A\cos\omega t = \frac{A}{2}(e^{i\omega t} + e^{-i\omega t}) \tag{11}$$

we can write

$$X(i\omega_1) = X^{+}(i\omega_1) + X^{-}(i\omega_1) \tag{12}$$

where

$$X^{+}(i\omega_1) = \int_{-\infty}^{\infty} \frac{A}{2} e^{i\omega t} e^{-i\omega_1 t} dt = \pi A \delta(\omega_1 - \omega) \tag{13}$$

$$X^{-}(i\omega_1) = \int_{-\infty}^{\infty} \frac{A}{2} e^{-i\omega t} e^{-i\omega_1 t} dt = \pi A \delta(\omega_1 + \omega) \tag{14}$$

and δ is the delta function.

Substitution of Eqs. (13), (14) into Eq. (10) yields

$$\begin{aligned} y_1(t) &= \frac{A}{2} \int_{-\infty}^{\infty} H_1(i\omega_1) \{\delta(\omega_1 - \omega) + \delta(\omega_1 + \omega)\} e^{i\omega_1 t} d\omega_1 \\ &= \frac{A}{2} \{H_1(i\omega) e^{i\omega t} + H_1(-i\omega) e^{-i\omega t}\} \\ &= A \,\mathrm{Real}[H_1(i\omega) e^{i\omega t}] \end{aligned} \tag{15}$$

where the relation of the complex conjugate is used.

In the same way we have the following formulas:

$$\begin{aligned} y_{2k}(t) &= 2\left(\frac{A}{2}\right)^{2k} \mathrm{Real}\Big[\sum_{i=0}^{k-1} H_{2k}(\overbrace{i\omega, \cdots, i\omega}^{2k-i}, \overbrace{-i\omega, \cdots, -i\omega}^{i})\, {}_{2k}C_i \exp\{i(2k-2i)\omega t\} \\ &\quad + \frac{1}{2} {}_{2k}C_k H_{2k}(\overbrace{i\omega, \cdots, i\omega}^{k}, \overbrace{-i\omega, \cdots, -i\omega}^{k})\Big] \end{aligned} \tag{16}$$

$$\begin{aligned} y_{2k+1} &= 2\left(\frac{A}{2}\right)^{2k+1} \mathrm{Real}\Big[\sum_{i=0}^{k} H_{2k+1}(\overbrace{i\omega, \cdots, i\omega}^{2k+1-i}, \\ &\quad \overbrace{-i\omega, \cdots, -i\omega}^{i})\, {}_{2k+1}C_i \exp\{(2k-2i+1)\omega t\}\Big] \end{aligned} \tag{17}$$

where H_k means the symmetrical form defined as follows:

$$\mathop{\mathrm{Sym}}_{\omega_1, \cdots, \omega_k} \{H_k(i\omega_1, \cdots, i\omega_k)\} = \frac{1}{k!} \sum_{\text{permutation of } \omega_1, \cdots, \omega_k} H_k(i\omega_1, \cdots, i\omega_k) \tag{18}$$

3. Analysis of oscillator by Volterra series

3.1 Volterra series of oscillator[4)]

An oscillator which generates a nearly sinusoidal wave can be represented by the following block diagram shown in Fig. 2. Here, f is the function which gives the

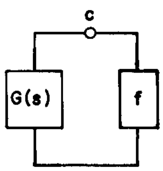

Fig. 2 Block diagram of an oscillator

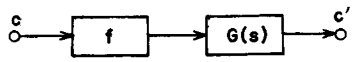

Fig. 3 Open loop system of an oscillator

property of the nonlinear element, and $G(s)$ is the transfer function of the linear part.

Since this circuit is a closed loop system which has no input function, it is impossible to analyse it by the Volterra series of Eq. (1) without some transformaton. Hence, we shall consider the following open loop system where the closed loop is cut at point c, as shown in Fig. 3.

Let us suppose that the nonlinear function f is given by the following polynomial.

$$f(t)=a_1u(t)+a_2u^2(t)+\cdots+a_Nu^N(t) \tag{19}$$

The Volterra kernels of the open loop system in the s domain are

$$\left.\begin{aligned} &H_1(s_1)=a_1\\ &H_2(s_1, s_2)=a_2\\ &\quad\vdots\\ &H_N(s_1, s_2, \cdots, s_N)=a_N \end{aligned}\right\} \tag{20}$$

and this system can be represented by the finite Volterra series.

We first separate the open loop system of Fig. 3 into the linear part F_L and the nonlinear part F_{NL}, as shown in Fig. 4. The problem is that when an input function $u(t)=A\cos\omega t$ with an unknown amplitude A and an angular velocity ω is imposed at point c, how we can determine them. In the following discussion, capital letters are used for the quantities in the s or ω demain and small letters are used for the corresponding quantities in the t domain.

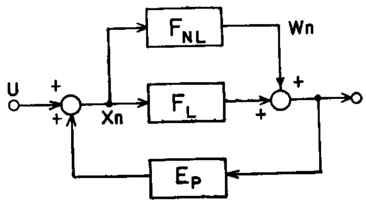

Fig. 4 Block diagram of separated system

X_n means the n-th term with respect to the input U, and W_n means the n-th output term generated by the nonlinear part F_{NL}. Furthermore, E_P means the filter which removes the fundamental component with the angular velocity ω generated by n-th output W_n. Therefore, this system is the open loop system for the fundamental component, and the closed loop system for the harmonic components.

To determine the unknown quantities A and ω, we equate the sum of the fundamental components generated from every odd term W_n at point c' to the input u at point c and solve the equation.

For the system of Fig. 4, we have the following relations:

$$X_1=U \tag{21}$$

$$X_n=E_P\{W_n+F_L(X_n)\} \tag{22}$$

Because F_L is a linear part, Eq. (22) becomes as follows:

$$X_n=E_P\{W_n+H_1(X_n)\} \tag{23}$$

Next, we shall show the procedures to determine every term W_n to the input

$X_1=U$. To avoid complexity on notation, we will omit the effect of E_P.

The higher terms which generate the fundamental component are only odd terms. Let us consider that the input function for F_{NL} is the the sum of the lst, 2nd, 3rd,$\cdots$ terms

$$x=x_1+x_2+x_3+\cdots \tag{24}$$

then, the output created by the nonlinear part F_{NL} is

$$\iint h_2(\tau_1, \tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 + \iiint h_3(\tau_1, \tau_2, \tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 + \cdots\cdots \tag{25}$$

From the combinations which create the 3rd term, we have

$$\begin{aligned}&\iint h_2(\tau_1, \tau_2)x_1(t-\tau_1)x_2(t-\tau_2, t-\tau_2)d\tau_1 d\tau_2\\ &\quad+\iint h_2(\tau_1, \tau_2)x_2(t-\tau_1, t-\tau_1)x_1(t-\tau_2)d\tau_1 d\tau_2\\ &\quad+\iiint h_3(\tau_1, \tau_2, \tau_3)x_1(t-\tau_1)x_1(t-\tau_2)x_1(t-\tau_1)d\tau_1 d\tau_2 d\tau_3\end{aligned} \tag{26}$$

By the multi-dimensional Laplace transform, we have

$$\begin{aligned}&\iiiint\!\!\int h_2(\tau_1, \tau_2)x_1(t_1-\tau_1)x_2(t_2-\tau_2, t_3-\tau_2)d\tau_1 d\tau_2 e^{-s_1t_1-s_2t_2-s_3t_3}dt_1dt_2dt_3\\ &\quad=\iint h_2(\tau_1, \tau_2)e^{-s_1\tau_1-(s_2+s_3)\tau_2}d\tau_1 d\tau_2\int x_1(t_1)e^{-s_1t_1}dt_1\iint x_2(t_2, t_3)e^{-s_2t_2-s_3t_3}dt_2dt_3\\ &\quad=H_2(s_1, s_2+s_3)X_1(s_1)X_2(s_2, s_3).\end{aligned} \tag{27}$$

In the same way, we have

$$\begin{aligned}&\iiiint\!\!\int h_2(\tau_1, \tau_2)x_2(t_1-\tau_1, t_2-\tau_1)x(t_3-\tau_2)d\tau_1 d\tau_2 e^{-s_1t_1-s_2t_2-s_3t_3}dt_1dt_2dt_3\\ &\quad=H_2(s_1+s_2, s_3)X_2(s_1, s_2)X_1(s_3)\end{aligned} \tag{28}$$

$$\begin{aligned}&\iiiint\!\!\iint h_3(\tau_1, \tau_2, \tau_3)x_1(t_1-\tau_1)x_1(t_2-\tau_2)x_1(t_3-\tau_3)d\tau_1 d\tau_2 d\tau_3 e^{-s_1t_1-s_2t_2-s_3t_3}dt_1dt_2dt_3\\ &\quad=H_3(s_1, s_2, s_3)X_1(s_1)X_1(s_2)X_1(s_3).\end{aligned} \tag{29}$$

and finally we have

$$\begin{aligned}W_3(s_1, s_2, s_3)=&H_2(s_1, s_2+s_3)X_1(s_1)X_2(s_2, s_3)\\ &+H_2(s_1+s_2, s_3)X_2(s_1, s_2)X_1(s_3)\\ &+H_3(s_1, s_2, s_3)X_1(s_1)X_1(s_2)X_1(s_3).\end{aligned} \tag{30}$$

Repeating the same procedures, we have

$$\begin{aligned}W_n(s_1, s_2, \cdots, s_n)=&H_2(s_1, s_2+\cdots+s_n)X_1(s_1)X_{n-1}(s_2, \cdots, s_n)\\ &+H_2(s_1+s_2, s_3+\cdots+s_n)X_2(s_1, s_2)X_{n-2}(s_3, \cdots, s_n)\end{aligned}$$

$$+\cdots+H_2(s_1+\cdots+s_{n-1},\ s_n)X_{n-1}(s_1,\ \cdots,\ s_{n-1})X_1(s_n)$$
$$+\sum_{\text{all permutation}} H_m(s_1+\cdots s_{k_1},\ \cdots,\ s_{n-k_m+1}+\cdots+s_n)$$
$$(\sum_{i=1}^{m} k_i=n \quad k_i\geq 1) \quad (3\leq m\leq n)$$
$$\times X_{k_1}(s_1,\ \cdots,\ s_{k_1})\cdots\cdots X_{k_m}(s_{n-k_m+1},\ \cdots,\ s_n)+\cdots$$
$$+H_n(s_1,\ s_2,\ \cdots,\ s_n)X_1(s_1)X_1(s_2)\cdots X_1(s_n). \tag{31}$$

Application of formula F. 1 to Eq. (23) yields

$$X_n(s_1,\ s_2,\ \cdots,\ s_n)=W_n(s_1,\ s_2,\ \cdots,\ s_n)$$
$$+X_n(s_1,\ s_2,\ \cdots,\ s_n)H_1(s_1+s_2+\cdots+s_n) \tag{32}$$

so, we have the following relation:

$$X_n(s_1,\ s_2,\ \cdots,\ s_n)=\frac{W_n(s_1,\ s_2,\ \cdots,\ s_n)}{1-H_1(s_1+s_2+\cdots+s_n)} \tag{33}$$

To the 2nd term we have

$$X_2(s_1,\ s_2)=\frac{H_2(s_1,\ s_2)X_1(s_1)X_1(s_2)}{1-H_1(s_1+s_2)} \tag{34}$$

$$X_2(s_2,\ s_3)=\frac{H_2(s_2,\ s_3)X_1(s_2)X_1(s_3)}{1-H_1(s_2+s_3)}. \tag{35}$$

Substituting Eqs. (34), (35) into Eq. (30), we have

$$W_3(s_1,\ s_2,\ s_3)=K_3(s_1,\ s_2,\ s_3)X_1(s_1)X_1(s_2)X_1(s_3) \tag{36}$$

where

$$K_3(s_1,\ s_2,\ s_3)=\frac{H_2(s_2,\ s_3)H_2(s_1,\ s_2+s_3)}{1-H_1(s_2+s_3)}+\frac{H_2(s_1,\ s_2)H_2(s_1+s_2,\ s_3)}{1-H_1(s_1+s_2)}$$
$$+H_3(s_1,\ s_2,\ s_3) \tag{37}$$

In the same way we have

$$W_n(s_1,\ s_2,\ \cdots,\ s_n)=K_n(s_1,\ s_2,\ \cdots,\ s_n)X_1(s_1)X_1(s_2)\cdots X_1(s_n) \tag{38}$$

where K_n is determined from H_1, H_2, $\cdots$, H_n.

3.2 Calculation of fundamental component[4)]

In this section, we consider the steady state response, so we use the Fourier transform instead of the Laplace transform substituting $i\omega_i$ for s_i. For the input function, we consider

$$u=x_1=A\cos\omega t=\frac{A}{2}(e^{i\omega t}+e^{-i\omega t}) \tag{39}$$

As shown in **2.2**, the Fourier transform of Eq. (39) is the sum of two δ functions.

The inverse transform is reduced to the substitution.

The fundamental component generated by this input function is the sum of H_1X_1 and all the fundamental components generated from W_1, W_3, $\cdots$, W_{2n+1}. The unknown quatities A and ω are determined to equate this fundamental component to X_1.

In order to inverse the n-th term W_n, it is sufficient to substitute $ik_i\omega$ for every $s_i(i=1, 2, \cdots, n)$ of $K_n(s_1, s_2, \cdots, s_n)$ where $k_i=1$ or $k_i=-1$.

To get the fundamental component, let us define

$$s(n)=\sum_{i=1}^{n} k_i \tag{40}$$

Then, by the relation of the complex conjugate, it is sufficient to take all the combinations for $s(n)=1$. In this case, we must take account of the effect of E_P, namely, for all $X_n(n\geq 2)$. The fundamental components are removed by E_P, and so, the following relation must be held:

$$X_n(ik_1\omega, ik_2\omega, \cdots, ik_n\omega)=\begin{cases}\dfrac{W_n(ik_1\omega, ik_2\omega, \cdots, ik_n\omega)}{1-H_1(is(n)\omega)} & : s(n)>1 \text{ or } s(n)=0\\ 0 & : s(n)\leq -1 \text{ or } s(n)=1\end{cases} \tag{41}$$

For the n-th kernel $K_n(s_1, s_2, \cdots, s_n)$, we use the notation K_{nm} when $s(n)=m$. Then, for $n=2N$, m must be $0, 2, \cdots, 2N$, and for $k=2N+1$, m must be $1, 3, \cdots, 2N+1$.

By the following definitions, we can get the same system as shown in Fig. 4 to determine K_{nm}.

$$K_{1m}=K'_{1m}=1 \quad (m=\pm 1) \tag{42}$$

$$K'_{nm}=\frac{K_{nm}}{1-a_1G(im\omega)} \quad (n\geq 2, m=0, 2, 3, \cdots, n) \tag{43}$$

K_{nm} is used for the term W_n, and K'_{nm} is used for the term X_n. K_{nm} is given by the following formula:

$$\left.\begin{aligned}&K_{nm}=\sum_{i=2}^{n} a_iG(in\omega)\sum_{\text{all permutation}} K'_{n_1m_1}\cdots K'_{n_im_i}\\ &(\sum_{l=1}^{i} n_l=n,\ n_l\geq 1,\ \sum_{l=1}^{i} m_l=s(i)=m) \quad (n\geq 2)\end{aligned}\right\} \tag{44}$$

The fundamental component is generated from $K_{2n+1\,1}$ $(n=1, 2, \cdots, N)$ and H_1. The coefficients for the factor $e^{i\omega t}$ are

$$\Omega_{2n+1}=\left(\frac{A}{2}\right)^{2n+1} K_{2n+1\,1} \quad (n\geq 1) \tag{45}$$

and

$$\frac{A}{2}H_1(i\omega) \tag{46}$$

The coefficient of the fundamental component of the output is the sum of all the above quantities, and is equal to the coefficient of the input $A/2$. Hence, if we define

$$d_N(A,\ \omega) \triangleq \frac{A}{2}H_1(i\omega) + \sum_{n=1}^{N} \Omega_{2n+1} - \frac{A}{2} \tag{47}$$

then

$$d_N(A,\ \omega) = 0. \tag{48}$$

The unknown quantities A and ω are the solution of Eq. (48). Since d_N is a complex quantity, Eq. (48) is resolved to

$$\left.\begin{array}{l} \text{Real}\ \ d_N(A,\ \omega) = 0 \\ \text{Imag}\ \ d_N(A,\ \omega) = 0 \end{array}\right\} \tag{49}$$

and solved by the Newton method. The total fundamental component of the oscillation wave is given by $A\cos\omega t$.

3.3 Calculation of harmonic components

After the fundamental component is determined as stated above, we can get the direct component and harmonic components generated from the terms which concern the determining of the fundamental component.

Since the fundamental component is determined from Ω_1, Ω_3, $\cdots$, Ω_{2N+1} by Eq. (47) the harmonic components to be calculated are generated from every kernel K_{nm} which is used to determine up to Ω_{2N+1}.

From Eqs. (42), (43) and (44), the kernels to be used to determine up to $K_{2N+1\,1}$ are

$$\left.\begin{array}{ll} n=2l & K'_{n0},\ K'_{n2},\ \cdots,\ K'_{nl^*} \\ n=2l+1 & K'_{n3},\ K'_{n5},\ \cdots,\ K'_{nl^*} \\ \multicolumn{2}{l}{\ \ l^* = \min\{n,\ 2N-n+2\}} \end{array}\right\}. \tag{50}$$

The coefficient of m-th harmonic $e^{im\omega t}$ from n-th term W_n is

$$\left(\frac{A}{2}\right)^n K_{nm} \tag{51}$$

and by the following formulas, we can get a direct component up to $(N+1)$-th harmonic component.

$$m=0\quad \sum_{n=1}^{N} 2\left(\frac{A}{2}\right)^{2n}\text{Real}[K_{2n\,0}] - \sum_{n=1}^{N}\left(\frac{A}{2}\right)^{2n}\text{Real}[{}_{2n}C_n a_{2n} G(0)] \tag{52}$$

$$m=2l \quad \sum_{n=l}^{N-l+1} 2\left(\frac{A}{2}\right)^{2n} \mathrm{Real}[K_{2n\,m}e^{im\omega t}] \quad (1\leq m\leq N+1) \tag{53}$$

$$m=2l+1 \quad \sum_{n=l}^{N-l} 2\left(\frac{A}{2}\right)^{2n+1} \mathrm{Real}[K_{2n+1\,m}e^{im\omega t}] \quad (1\leq m\leq N+1) \tag{54}$$

4. van der Pole oscillator

Let us consider the LC oscillator shown in Fig. 5 which has a voltage controlled nonlinear registor $N.L.$ with the relation

$$i=\frac{1}{3}v^3 \tag{55}$$

If we assume that R is a negative resistor of the value -1 and $L=1/C=\varepsilon$, then the voltage across $N.L.$ can be described by the following van der Pole equation.

$$\frac{d^2v}{d^2t}-\varepsilon(1-v^2)\frac{dv}{dt}+v=0 \tag{56}$$

This circuit is represented by the block diagram shown in Fig. 6 where

$$Z(i\omega)=\frac{\varepsilon}{-\varepsilon+i\left(\omega-\frac{1}{\omega}\right)} \tag{57}$$

The open loop system for this block diagram has only a 3rd kernel

$$H_3(i\omega_1,\ i\omega_2,\ i\omega_3)=-\frac{1}{3}Z\{i(\omega_1+\omega_2+\omega_3)\} \tag{58}$$

Thus, from Eqs. (43), (44) we have

$$\left.\begin{array}{l} K'_{2n+1\ 2m+1}=K_{2n+1\ 2m+1} \\ \quad n=0:\ m=0 \text{ or } -1 \\ \quad 1\leq n\leq N:\ 0\leq m\leq n \end{array}\right\} \tag{59}$$

$$\left.\begin{array}{l} K_{2n+1\ 2m+1}=-\frac{1}{3}Z\{i(2m+1)\omega\} \sum\limits_{\text{all permutation}} K'_{2n_1+1\ 2m_1+1}K'_{2n_2+1\ 2m_2+1}K'_{2n_3+1\ 2m_3+1} \\ \quad n_1+n_2+n_3=n \quad m_1+m_2+m_3=m \quad n\geq 1 \end{array}\right\}. \tag{60}$$

From the above equations, this system generates only the odd order harmonics.

By the method stated in **3.**, we obtained the amplitude A, the angular velocity ω of the fundamental component and the amplitude $Amp.$ of the total oscillation wave which consisted of the harmonic components for the various values of ε.

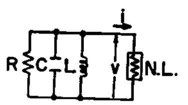

Fig. 5 van der Pole oscillator

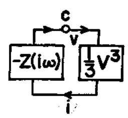

Fig. 6 Block diagram of van der Pole oscillator

Table. 1 Steady state solution of van der Pole oscillator

ε	*A*			*Amp.*			ω		
	Volterra $R=15$	Volterra $R=25$	R.K.G.	Volterra $R=15$	Volterra $R=25$	R.K.G.	Volterra $R=15$	Volterra $R=25$	S.E.M.
0.1	2.0005	2.0005	2.0002	2.0004	2.0004	2.0001	0.9993755	0.9993755	0.9993756
0.2	2.0019	2.0019	2.0006	2.0017	2.0017	2.0004	0.9975078	0.9975078	0.9975089
0.3	2.0042	2.0042	2.0014	2.0037	2.0037	2.0009	0.9944128	0.9944147	0.9944198
0.4	2.0074	2.0075	2.0025	2.0066	2.0067	2.0017	0.9901082	0.9901253	0.9901417
0.5	2.0112	2.0116	2.0039	2.0103	2.0104	2.0025	0.9845922	0.9846772	0.9847210
0.6	2.0151	2.0166	2.0056	2.0145	2.0151	2.0035	0.9778241	0.9780866	0.9782169
0.7	2.0172	2.0218	2.0075	2.0183	2.0209	2.0047	0.9697905	0.9702350	0.9707011
0.8	2.0148	2.0240	2.0097	2.0187	2.0265	2.0057	0.9608365	0.9607766	0.9622563
0.9	2.0048	2.0138	2.0112	2.0124	2.0204	2.0073	0.9522757	0.9509402	0.9529747
1.0	1.6371	1.4561	2.0149	9.1400	14.819	2.0086	0.3731043	0.3324072	0.9429559
10.0	1.3429	1.2050	2.1325	4.2945	8.4364	2.0142	0.4356221	0.3349921	0.3297066

Table. 2 Harmonic components of steady state solution

Harmonics	$\varepsilon=0.1$		$\varepsilon=0.5$		$\varepsilon=0.9$	
n	a_n	b_n	a_n	b_n	a_n	b_n
1	2.00046858856	0.0	2.01166	0.0	2.005	0.0
3	−0.00093706735	−0.02498241430	−0.02315	−0.12282	−0.010	−0.217
5	−0.00051955396	0.00003685820	−0.01217	0.00448	−0.005	0.052
7	0.00000124480	0.00001209078	0.00078	0.00138		
9	0.00000029527	−0.00000003970	0.00011	−0.00008		
11	−0.00000000129	−0.00000000741				
13	−0.00000000019	−0.00000000003				

The values of *A* and *Amp.* are compared to the results calculated by the Runge-Kutta-Gill method (R.K.G.), and ω is compared to the result given by the series expansion method (S.E.M.)[6]. We regard the results of R.K.G. and S.E.M. as the exact solution.

The results are listed in Table. 1 where R means the maximum order $2N+1$ of Eq. (47). The results of Table. 1 show that this method gives reasonable values for ε less than 0.9, but extremely different values for $\varepsilon \geq 1$.

The steady state oscillation wave can be expressed by the following form.

$$\sum_{n=1}^{L} a_n \cos n\omega t + \sum_{n=1}^{L} b_n \sin n\omega t \tag{61}$$

The values of a_n and b_n for $\varepsilon=0.1$, 0.5, 0.9 are listed in Table. 2, and the total oscillation wave form is shown in Fig. 7.

For the values of $\varepsilon=0.1$ and 0.5, the total oscillation wave can be recognized as a reasonable result. However, for the value of $\varepsilon=0.9$, the result is not so good.

The van der Pole equation has been investigated in detail by many researchers and accurate solutions are given for the various values of ε. Therefore, it is not our purpose here to analyse the van der Pole equation. The van der Pole oscillator is used as an example to investigate the validity of this method.

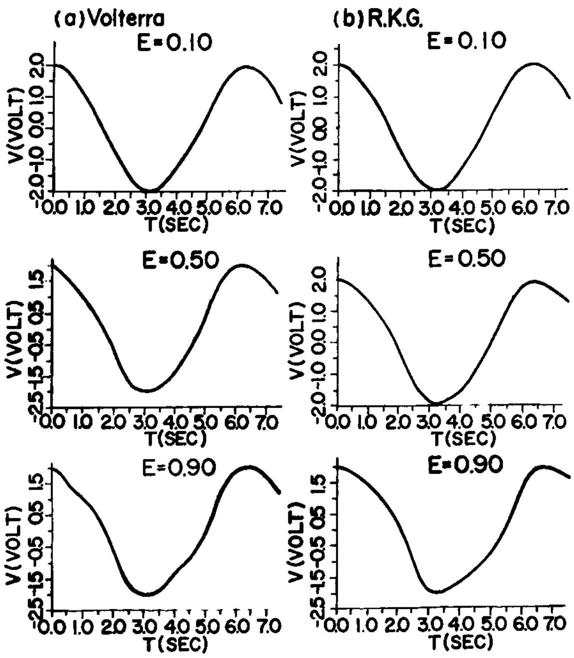

Fig. 7 Steady state oscillation wave of van der Pole oscillator

By the results given above, it may be well to conclude that this method gives reasonable results for nearly sinusoidal oscillations, but does not give good results for oscillations distorted by heavy nonlinearlity.

5. Phase shift oscillator

5.1 *CR* low pass phase shift oscillator

Here, we will apply the method stated in **3.** to the analysis of the *CR* low pass phase shift oscillator shown in Fig 8. The input-output relation of an amplifier has a saturation property, as shown by the solid line in Fig. 9. We will approximate this

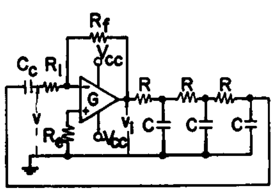

Fig. 8 *CR* low pass phase shift oscillator

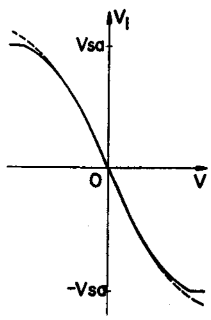

Fig. 9 Property of ampifier

property by dotted line expressed by the following equation.

$$v_1 = -A(v - \beta v^3) \tag{62}$$

For the open loop block diagram of this circuit, we have the following equations:

$$f(v) = -A(v - \beta v^3) \tag{63}$$

$$G(i\omega) = \frac{1}{1 - 5(CRn\omega)^2 + iCRn\omega\{6 - (CRn\omega)^2\}} \tag{64}$$

Here, we calculated the steady state oscillation wave when $R = 10k\Omega$, $C = 0.01\mu F$, $C_c = 10\mu F$, $R_1 = R_e = 1k\Omega$, $R_f = 31k\Omega$ and $V_{cc} = 15V$. For the amplifier, we assume that the input impedance is sufficiently large and the output impedance is sufficiently small. The saturation voltage *Vsa* is 14V, A is 31 and β is 0.4943.

In Table. 3 we show the values of the harmonic components calculated by the Volterra series expansion method ($R = 17$), compared to the results calculated by R.K.G. method.

Table. 3 Harmonic components of steady state solution

Harmonics n	Volterra $f = 3898.0$Hz x_n(V)	R.K.G x_n(V)
1	0.4178836	0.4178833
3	0.0006537	0.0006537
5	0.0000007	0.0000007
7	0.0000000	0.0000000
9	0.0000000	0.0000000

Since this circuit has a low pass characteristic, the harmonic components decrease rapidly, and both results show good agreement.

5.2 Transistor *RC* phase shift oscillator

Here, we will consider a *RC* high pass phase shift oscillator using a transistor as an amplifier, as shown in Fig. 10. The equivalent circuit of transistor is represented by the following simplified Ebers-Moll model, as shown in Fig. 11.

An emitter current is given by the following equation

$$I_E = I_{SO}(e^{\lambda V_{BE}} - 1) \tag{65}$$

where I_{SO} is the saturation current for the inverse direction, and $\lambda = q/kT$ (q: charge

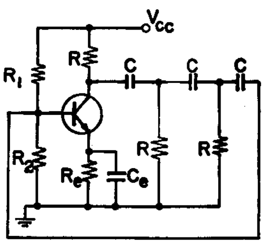

Fig. 10 Transistor *CR* phase shift oscillator

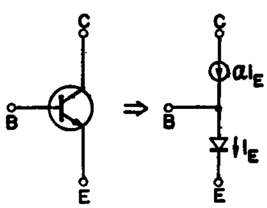

Fig. 11 Equivalent model of transistor

of electron, k: Boltzmann constant, T: absolute temperature).

Let us separate every current and voltage as the following form:

$$\left.\begin{array}{l} I_E = I_{EO} \text{ (DC bias component)} + i_e \text{ (signal component)} \\ I_C = I_{CO} \text{ (〃)} + i_c \text{ (〃)} \\ V_{BE} = V_{BEO} \text{ (〃)} + v_{be} \text{ (〃)} \end{array}\right\} \tag{66}$$

First, we determine DC components. From Fig. 11, we have

$$I_{CO} = \alpha I_{EO} = \alpha I_{SO}(e^{\lambda V_{BEO}} - 1) \tag{67}$$

and the DC load line is given as follows:

$$\left.\begin{array}{l} I_{CO} = \dfrac{1}{R_e + R_x/\beta}(V_{BEO} - V_x) \\ \beta = \dfrac{\alpha}{1-\alpha} \quad R_x = R_1 /\!/ R_2 \quad V_x = \dfrac{R_x}{R_1} V_{cc} \end{array}\right\} \tag{68}$$

Using the DC bias points determined by Eqs. (67), (68), we have the following relation for the signal components.

$$i_c = \alpha I_{SO} e^{\lambda V_{BE}} (e^{\lambda v_{be}} - 1) \tag{69}$$

For the linear subsystem show in Fig. 12, we have

$$\left.\begin{array}{l} i_1 = C(in\omega) v_2 + D(in\omega) i_2 \\ C(in\omega) = \dfrac{1}{R}\left\{3 + \dfrac{4}{in\omega CR} - \dfrac{1}{(n\omega CR)^2}\right\} \\ D(in\omega) = 1 - \dfrac{5}{(n\omega CR)^2} + \dfrac{1}{in\omega CR}\left\{6 - \dfrac{1}{(n\omega CR)^2}\right\} \end{array}\right\}. \tag{70}$$

Substituting the following relation

$$\left.\begin{array}{l} i_1 = -i_c \\ v_2 = v_{be} \\ i_2 = i_b + v_{be}/R_x \\ i_b = (1-\alpha) i_e \end{array}\right\} \tag{71}$$

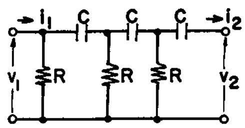

Fig. 12 Linear subsystem of oscillator

Eq. (70) becomes as follows:

$$\left.\begin{array}{l} -\alpha I_v (e^{\lambda v_{be}} - 1) \\ \quad = \left\{C(in\omega) + \dfrac{D(in\omega)}{R_x}\right\} v_{be} + D(in\omega)(1-\alpha) I_v (e^{\lambda v_{be}} - 1) \\ I_v = I_{SO} e^{\lambda v_{BE}} \end{array}\right\} \tag{72}$$

For the open loop block diagram of this oscillator we have

$$f(v_{be}) = I_v (e^{\lambda v_{be}} - 1) \tag{73}$$

$$G(in\omega) = \frac{(1-\alpha)D(in\omega)+\alpha}{-C(in\omega) - \frac{D(in\omega)}{R_x}} \tag{74}$$

To analyse this system by the Volterra series expansion, f must be described by a polynomial, so we will approximate Eq. (73) by the following best approximation polymomial.

$$f(v_{be}) = \sum_{i=1}^{7} a_i v_{be}^i \tag{75}$$

Volterra kernels of the total system are given as follows.

$$\left. \begin{array}{l} H_i(i\omega_1, \cdots, i\omega_i) = a_i G\{i(w_1 + \cdots + \omega_i)\} \\ i=1, \cdots, 7 \end{array} \right\} \tag{76}$$

Since this system has all the kernels from the lst to the 7th, the steady state oscillation wave contains the DC component and all the even and odd harmonics.

Here, we calculated the steady state oscillation wave when $T=300°$K, $R_1=35k\Omega$, $R_2=10k\Omega$, $R_e=4k\Omega$, $I_{SO}=10^{-7}$A, $V_{cc}=12$V, $C_e=50\mu$F, $R=10$kΩ, $C=0.01\mu$F. In Table 4, we show the values of the harmonic components calculated by the Volterra series expansion method.

Since this circuit has a high pass property, and the property of the transistor is expressed by the exponential function, the effects of the harmonic components become large as β increases.

In previous examples, we regarded the solutions calculated by the R. K. G. method as exact solutions, and compared them to the solutions calculated by the Volterra series expansion method. In this section we did not do so, but from previous results it may be well to conclude that the solutions of this example are sufficiently accurate.

Table. 4 Harmonic components of v_{be}

α	0.9755	0.977
β	39.8	42.5
V_{BE}(V)	0.224761	0.224796
ω(rad/s)	3898.69	3861.62
Harmonics		
x_0(mV)	0.12040	3.1251
x_1(mV)	3.91116	20.1691
x_2(mV)	0.15220	4.2373
x_3(mV)	0.00783	1.2134
x_4(mV)	0.00046	0.3177
R	7	7

6. Conclusion

In this paper, we showed a method to analyse an oscillator which generates a nearly sinusoidal oscillation wave by the Volterra series expansion method. By this method, we do not use the traditional method to derive and solve nonlinear differential equations, but solve algebraic equations derived from the block diagram of the oscillator.

An amplitude and a period of a fundamental component of the oscillation wave and harmonic components which concern the determining of the above quantities can be easily calculated.

We showed a numerical procedure to determine the above quantities. The method has been applied to the analysis of a van der Pole oscillator and a phase shift oscillator and we obtained useful results.

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