Optimal Operating Policies for a Stochastic Clearing System with Bounded Waiting Times

By

Hisashi MINE* and Viliam MAKIŠ**

(Received May 9, 1984)

Abstract

This paper is a continuation of a previous paper where we investigated the steady-state behaviour of a stochastic clearing system with Poisson input operated under the following clearing policy: all the quantity is instantaneously removed from the system whenever there are at least M items in the queue, or every t time units since the first arrival after the last clearing, whichever occurs first. This type of policy was termed a bounded M-policy. The objective of this paper is to examine the behaviour of the expected average cost per unit time in the class of bounded M-policies and in the class of T-policies that clear the system every T time units. We find optimal policies in both classes by comparing the associated expected average costs, and present some computational results.

1. Introduction

We consider a stochastic clearing system with Poisson input operated under one of the following service policies:

(i) a T-policy consisting in removing all the quantity currently present in the system every T time units,

(ii) a bounded i-policy consisting in clearing whenever there are at least i items in the queue or every T time units since the first arrival after the last clearing, whichever occurs first.

In the previous paper[8], we derived the steady state characteristics for the stochastic clearing system operated under a bounded i-policy, and discussed some limiting cases. In this paper, we examine the behaviour of the expected average cost per unit time over an infinite time horizon under each of the above-mentioned service policies. We find optimal policies in both classes, and present some computational results. We now consider the following cost structure. The cost of serving i customers is $K+ci$, where $K>0$ and c is any constant. The waiting cost of i customers in the queue per unit time is $h(i)$ where $h(\cdot)$ is a non-negative function,

* Dept. of Applied Marhematics and Physics

** Dept. of Mathematial Methods, Technical University of Transport, Czechoslovakia

such that

$$h(i+1)-h(i)\geqslant\gamma>0 \tag{1}$$

for any non-negative integer i. We assume that in both cases the waiting times are bounded and cannot exceed a given constant $t>0$. To ensure the finitness of the cost incurred in a cycle, we assume that

$$\mathrm{E}\left(\int_0^t h(X_u+1)du\right)<+\infty, \tag{2}$$

where $\{X_u,\ u\geqslant 0\}$ is the input Poisson process with a given intensity $\lambda>0$. Note that (2) holds for any polynomial or exponential function $h(\cdot)$. Let $h(x)=\mathrm{e}^{ax}$, where a is any real constant. Then

$$\mathrm{E}\left(\int_0^t h(X_u+1)du\right)=\sum_{k=0}^{+\infty}h(k+1)\int_0^t q_k(u)du=\sum_{k=1}^{+\infty}\frac{h(k)}{\lambda}R_k(t)$$
$$=\sum_{k=1}^{+\infty}q_k(t)\frac{\mathrm{e}^a(1-\mathrm{e}^{ak})}{1-\mathrm{e}^a}<+\infty,$$

where

$$q_k(t)=\frac{(\lambda t)^k}{k!}\mathrm{e}^{-\lambda t},\ \ R_k(t)=\int_0^t \lambda q_{k-1}(u)du,\ \ R_0(t)=1. \tag{3}$$

We proceed to find optimal operating policies in both classes.

2. Optimal Operating Policies

In this section, we examine the behavior of the expected average cost per unit time under the service rules defined in Section 1, and find the optimal policy in each class. First, we investigate case (i).

2.1. The optimal T-policy

Consider a clearing policy that clears the system every $T\leqslant t$ time units. The long-run expected average cost can be obtained by applying a renewal argument. (See e. g. Ross[4].) We have

$$g(T)=\left(\mathrm{E}\left(\int_0^T h(X_u)du\right)+K+c\mathrm{E}(X_T)\right)/T$$
$$=\left(\int_0^T H(u)du+K\right)/T+\lambda c \tag{4}$$

for any $T>0$, where $H(u)=\mathrm{E}(h(X_u))$ and $\{X_u,\ u\geqslant 0\}$ is the input Poisson process. In the next lemma, we examine the behaviour of the expected average cost $g(\cdot)$.

Lemma 1. Let (1) hold. Then, the function $-g(\cdot)$ is unimodal with mode s_0, determined by

$$s_0\doteq\inf\{T>0:\ d(T)=0\}, \tag{5}$$

where

$$d(T)=\int_0^T udH(u)-K. \tag{6}$$

Proof. To prove that $-g(\cdot)$ is unimodal with mode s_0, it suffices to show that the derivative of $g(s)$ is negative for $s<s_0$ and positive for $s>s_0$. We have for any $s>0$

$$\frac{dg(s)}{ds}=\frac{sH(s)-\int_0^s H(u)du-K}{s^2}=\frac{d(s)}{s^2}, \tag{7}$$

and from (1), we get

$$\frac{dH(s)}{ds}=\lambda\Big(\sum_{j=1}^{+\infty}h(j)q_{j-1}(s)-\sum_{j=0}^{+\infty}h(j)q_j(s)\Big)$$
$$=\lambda\sum_{j=0}^{+\infty}(h(j+1)-h(j))q_j(s)\geqslant\lambda\gamma>0, \tag{8}$$

so that $d(s)$ is increasing in s and from (7), we conclude that $-g(\cdot)$ is unimodal with mode s_0 determined by (5). This completes the proof.

The optimal operating policy is given by the following theorem.

Theorem 1. Let (1) hold. The optimal clearing time is given by

$$s_{opt}=\min\{s_0,\ t\}, \tag{9}$$

where s_0 is determined by (5). Moreover,

$$s_{opt}\leqslant s_0\leqslant\sqrt{2K/(\lambda\gamma)}. \tag{10}$$

Proof. Formula (9) follows immediately from Lemma 1. Next, (6) and (8) yield for any $s>0$

$$d(s)\geqslant\lambda\gamma\int_0^s udu-K=\frac{\lambda\gamma s^2}{2}-K,$$

and the result follows, since $d(s)$ is increasing and $d(s)<0$ for $s<s_0$.

In the following section, we examine the behaviour of the average cost per unit time in the class of bounded i-policies and again find the optimal policy.

2.2. The optimal bounded i-policy

We assume that the system is operated under a bounded i-policy for some positive integer i, i.e., the system is cleared whenever the queue length reaches level i, or every t time units since the first arrival after the last clearing, whichever occurs first. It was proved elsewhere[2)] that this type of operating doctrine is the average cost optimal policy for a controlled M/G/1 queue with bounded waiting times. This is a generalization of the model considered here. We proceed to find the expected average cost per unit time. Under any bounded i-policy, the related decision process is a possibly delayed renewal-reward process. The long-run expected average cost per unit time is equal to the expected cost incurred in a cycle divided by the expected length of the cycle. The length of a cycle is the length of time between two successive clearings. The expected length of a cycle was derived in the previous paper[3)], and is given by

$$\mu=\frac{1}{\lambda}\sum_{n=0}^{i-1}R_n(t), \tag{11}$$

where $R_n(t)$ is defined by (3). The expected average cost is given by the following theorem.

Theorem 2. Under any bounded i-policy, the expected average cost per unit time is of the form

$$g_i=\left(\lambda K+\sum_{j=0}^{i-1}h(j)R_j(t)\right)/\sum_{n=0}^{i-1}R_n(t)+\lambda c. \tag{12}$$

Proof. To obtain (12), it suffices to evaluate the expected waiting cost $E_i(W)$ incurred in a cycle. (E_i denotes the expectation under a bounded i-policy.) Conditioning on the number of arrivals Y_t in the time interval $(\eta_1, \eta_1+t]$, we get

$$E_i(W)=h(0)/\lambda+\sum_{k=0}^{i-2}q_k(t)E\left(\int_0^t h(Y_u+1)du \mid Y_t=k\right)$$
$$+\sum_{k=i-1}^{+\infty}q_k(t)E\left(\int_0^{\bar{\eta}_{i-1}}h(Y_u+1)du \mid Y_t=k\right), \tag{13}$$

where $\bar{\eta}_{i-1}=\eta_i-\eta_1$ and η_k is the k-th arrival time in a given cycle. We have for $k\geqslant i-1$

$$E\left(\int_0^{\bar{\eta}_{i-1}}h(Y_u+1)du \mid Y_t=k\right)=\sum_{j=0}^{i-2}h(j+1)E(\bar{\eta}_{j+1}-\bar{\eta}_j \mid Y_t=k)$$
$$=\frac{t}{k+1}\sum_{j=1}^{i-1}h(j), \tag{14}$$

where the second equality follows from Theorem 2. 3[4] and from formula (2. 1. 6)[1]. Similarly,

$$E\left(\int_0^t h(Y_u+1)du \mid Y_t=k\right)=\frac{t}{k+1}\sum_{j=1}^{k+1}h(j), \tag{15}$$

and (13)—(15) yield, after some algebraic manipulations,

$$E_i(W)=\frac{1}{\lambda}\sum_{j=0}^{i-1}h(j)R_j(t), \tag{16}$$

and (12) follows from (11) and (16). This completes the proof.

Now we examine the behaviour of the average cost g_i given by (12). We denote for any positive integer i

$$r(i)=\sum_{n=0}^{i-1}(h(i)-h(n))R_n(t)-\lambda K. \tag{17}$$

We have

$$r(i+1)-r(i)=(h(i+1)-h(i))\sum_{n=0}^{i}R_n(t),$$

so that $r(\cdot)$ is increasing. Next, from (12)

$$g_{i+1}-g_i=R_i(t)r(i)/\left(\sum_{n=0}^{i}R_n(t)\sum_{j=0}^{i-1}R_j(t)\right),$$

and we get the following theorem.

Theorem 3. Let (1) hold. Then g_i is unimodal in i with mode

$$i^*=\inf\left\{i:\ \sum_{n=0}^{i-1}(h(i)-h(n))R_n(t)\geqslant\lambda K\right\}, \tag{18}$$

and the bounded i^*-policy is the optimal operating policy. In the final section, we compare both optimal policies and present some computational results.

3. Comparison of Operating Policies and Concluding Remarks

In this section, we assume that $h(i)=hi$ for some $h>0$ and all non-negative integers i. Then

$$H(u)\equiv \mathrm{E}(h(X_u))=\lambda hu,$$

$$d(s)=\lambda h\int_0^s udu-K=\frac{\lambda hs^2}{2}-K,$$

and the optimal clearing time is

$$s_{opt}=\min\{\sqrt{2K/(\lambda h)},\ t\}. \tag{19}$$

Note that for $t\to+\infty$, $s_{opt}=\sqrt{2K/(\lambda h)}$, which is also the optimal length of time between orders for the inventory system in Sivazlian[5], and the optimal dispaching time for a two-terminal shuttle system[6].

In the class of bounded i-policies, the expected average cost per unit time determined by (12) is in the linear case of the form

$$g_i=h\Big(u+\sum_{j=0}^{i-1}jR_j(t)\Big)\Big/\sum_{n=0}^{i-1}R_n(t)+\lambda c,$$

Table 1. i^* and $(g_{i^*}-\lambda c)/h$ as functions of $\lambda K/h$ and λ for $t=1$.

$\lambda K/h$	λ : 0.2	0.6	1	2	5	10
0.5	1, 0.5000	1, 0.5000	1, 0.5000	1, 0.5000	1, 0.5000	1, 0.5000
1	1, 1.0000	1, 1.0000	1, 1.0000	1, 1.0000	1, 1.0000	1, 1.0000
1.5	2, 1.4233	2, 1.3445	2, 1.3063	2, 1.2681	2, 1.2508	2, 1.2500
2	2, 1.8465	2, 1.6891	2, 1.6127	2, 1.5363	2, 1.5017	2, 1.5000
2.5	3, 2.2659	3, 2.0310	2, 1.9190	2, 1.8044	2, 1.7525	2, 1.7500
3	3, 2.6830	3, 2.3489	3, 2.1940	3, 2.0550	3, 2.0023	3, 2.0000
5	5, 4.3500	4, 3.6113	4, 3.2385	3, 2.8685	3, 2.6796	3, 2.6668
10	9, 8.5167	7, 6,7375	6, 5.7499	5, 4.6478	5, 4.0495	5, 4.0008
20	17, 16.8500	13, 12.9875	11, 10.7500	8, 7.9999	7, 6.0980	6, 5.8423
30	26, 25.1833	20, 19.2375	16, 15.7500	12, 11.3333	8, 7.8762	8, 7.2797

Table 2. $(g(s_{opt})-g_{i^*})/h$ as function of $\lambda K/h$ and λ for $t=1$.

$\lambda K/h$	λ : 0.2	0.6	1	2	5	10
0.5	2.1000	0.6333	0.5000	0.5000	0.5000	0.5000
1	4.1000	0.9667	0.5000	0.4142	0.4142	0.4142
1.5	6.1767	1.4555	0.6937	0.4639	0.4812	0.4820
2	8.2535	1.9442	0.8873	0.4637	0.4983	0.5000
2.5	10.3341	2.4356	1.0810	0.4456	0.4835	0.4861
3	12.4170	2.9511	1.3060	0.4450	0.4472	0.4495
5	20.7500	5.0221	2.2615	0.6315	0.4827	0.4955
10	41.5833	10.2292	4.7501	1.3522	0.4227	0.4714
20	83.2500	20.6458	9.7500	3.0001	0.4020	0.4823
30	124.9167	31.0625	14.7500	4.6667	0.6238	0.4663

and the optimal clearing level i^* is

$$i^* = \inf\left\{i:\ \sum_{j=0}^{i-1} (i-j)R_j(t) \geqslant u\right\},$$

where $u = \lambda K/h$. In Table 1, we present pairs i^* and $(g_{i^*} - \lambda c)/h$ and in Table 2 the differences $(g(s_{opt}) - g_{i^*})/h$ for $t=1$ and for several values of $u = \lambda K/h$ and λ. The results in Table 2 reveal that the optimal bounded i-policy is much better than the optimal T-policy. Since bounded i-policies are very simple, they are ideally suited for practical applications and can be used instead of the more traditional policy of a scheduled periodic service. The resulting saving will be substantial especially for the systems with low input intensities.

References

1) David, H. A., "Order Statistics", John Wiley and Sons, New York, 1970.
2) Makiš, V., "Optimal Control of a Batch Service Queueing System with Bounded Waiting Time", (submitted to Journal of Appl. Prob.).
3) Mine, H., and Makiš, V., "The Steady-State Behaviour of a Stochastic Clearing System with Bounded Waiting Times", Memoirs of the Faculty of Engineering, Kyoto University, Vol. 46, Part 3, 1984, pp. 30–36.
4) Ross, S. M., "Applied Probability Models with Optimization Applications", Holden Day, San Francisco, Calif., 1970.
5) Sivazlian, B. D., "A Continuous Review (s, S) Inventory Systems with Arbitrary Interarrival Distribution between Unit Demand", Opns. Res., Vol. 22, 1974, pp. 65–71.
6) Weiss, H. J., "Further Results on an Infinite Capacity Shuttle with Control at a Single Terminal", Opns. Res., Vol. 29, 1981, pp. 1212–1217.