A Note on Canonical Transforms Representing $SL_2(2, \boldsymbol{R})$, a Two-Fold Covering of $SL(2, \boldsymbol{R})$, in $L^2(\boldsymbol{R})$

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Abstract

Canonical transforms representing unitarily $SL_2(2, \boldsymbol{R})$, a two-fold covering of $SL(2, \boldsymbol{R})$, are explicitly constructed in the form of integral transforms in $L^2(\boldsymbol{R})$ with the main stress laid on the composition of canonical transforms.

1. Introduction

In his book[1], K. B. Wolf treated in detail canonical transformations associated with $SL(2, \boldsymbol{R})$ in the form of integral transforms. His method, due to M. Moshinsky and C. Quesne[2], is a quantization of the symplectic action of $SL(2, \boldsymbol{R})$ on $\boldsymbol{R}^2$, the phase space for a classical system of one degree of freedom. In the case of several degrees of freedom, the integral transforms are discussed in reference 3).

It was D. Shale[4] who observed that this quantization determines a double-valued unitary representation of the symplectic group. N. R. Wallach showed in his book[5] that this quantization determines a unitary representation of the metaplectic group, a two-fold covering of the symplectic group, which becomes known as the Segal-Shale-Weil representation[6].

The two-fold covering causes a sign problem in the integral kernel. Though the problem has been solved in an abstract manner, it seems to require further investigation in an explicit manner. The present article gives integral transforms for $SL_2(2, \boldsymbol{R})$ with the phase factors assigned strictly, and shows that the integral transforms actually constitute a unitary representation of $SL_2(2, \boldsymbol{R})$, by composing the integral transforms in an explicit form.

2. Canonical Transforms

In this section, we review quantum mechanical canonical transforms associated with $SL(2, \boldsymbol{R})$. Assume that a unitary operator U on $L^2(\boldsymbol{R})$ induces a linear trans-

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formation of x and $y=-id/dx$;

$$U\begin{bmatrix} x \\ y \end{bmatrix}U^{-1}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \qquad ad-bc=1. \tag{2.1}$$

This transformation corresponds to the canonical transformation on $\boldsymbol{R}^2$ by $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for the classical system. It is known[1] that U is expressed in the integral transform with the integral kernel, for $b\neq 0$,

$$K(x, \xi)=\frac{\alpha}{(2\pi|b|)^{1/2}}\exp\left[\frac{i}{2b}(dx^2-2x\xi+a\xi^2)\right] \tag{2.2}$$

where α is a complex constant with $|\alpha|=1$. To determine α, one has to pass on to the two-fold covering $SL_2(2, \boldsymbol{R})$ of $SL(2, \boldsymbol{R})$. We will discuss this in the next section.

On supposing that α is determined in some way or other, we calculate compositions of the integral transforms. Let

$$M_{21}=M_2M_1, \qquad M_k=\begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}, \qquad k=1, 2, 21. \tag{2.3}$$

By U_k, $k=1, 2, 21$, we mean the unitary operator determined by (2.1) with coefficient matrices M_k^{-1}. For the unitary operators U_k, the integral kernels and the phase factors are denoted by K_k and α_k, respectively. Performing an iterated integral of the kernels yields

$$U_2U_1=\begin{cases} \dfrac{\alpha_1\alpha_2}{\alpha_{21}}e^{i\pi/4}U_{21} & \text{for} \dfrac{b_{21}}{b_1b_2}>0, \\ \dfrac{\alpha_1\alpha_2}{\alpha_{21}}e^{-i\pi/4}U_{21} & \text{for} \dfrac{b_{21}}{b_1b_2}<0. \end{cases} \tag{2.4}$$

Our purpose is to show that for suitably chosen phase factors α_k, Eq. (2.4) becomes exactly $U_2U_1=U_{21}$, which means that U is a unitary representation of $SL_2(2, \boldsymbol{R})$.

3. One-Parameter Subgroups

To realize how the two-fold covering SL_2(a, $\boldsymbol{R}$) emerges in the integral transforms, we consider one-parameter groups of unitary transformations corresponding to an Iwasawa decomposition of $SL(2, \boldsymbol{R})$;

$$SL(2, \boldsymbol{R})\simeq KAN,$$

or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix}\begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} e^{\tau}\cos\theta & e^{\tau}\zeta\cos\theta+e^{-\tau}\sin\theta \\ -e^{\tau}\sin\theta & -e^{\tau}\zeta\sin\theta+e^{-\tau}\cos\theta \end{bmatrix}. \tag{3.1}$$

The parameters (θ, τ, ζ) are given by

$$e^{i\theta}=\frac{a-ic}{(a^2+c^2)^{1/2}},\quad e^{\tau}=(a^2+c^2)^{1/2},\quad \zeta=\frac{ab+cd}{a^2+c^2}. \tag{3.2}$$

The one-parameter subgroup $K(\theta)$ is associated with the harmonic oscillator, $A(\tau)$ with the dilatation, and $N(\zeta)$ with a free particle. The $K(\theta)$, diffeomorphic with S^1, causes the problem of covering, but $A(\tau)$ and $N(\zeta)$ do not. Let $e^{-i\theta H}$, $e^{-i\tau D}$ and $e^{-i\zeta F}$ denote the one-parameter groups of unitary operators corresponding to $K(\theta)$, $A(\tau)$, and $N(\zeta)$, respectively, where H, D, and F, are given, as is well known, by

$$\begin{aligned} H&=\frac{1}{2}\left(-\frac{d^2}{dx^2}+x^2\right),\\ D&=\frac{1}{i}\left(x\frac{d}{dx}+1/2\right),\\ F&=-\frac{1}{2}\frac{d^2}{dx^2}. \end{aligned} \tag{3.3}$$

Then $e^{-i\theta H}$ should be a representation of a two-fold covering of $K(\theta)$, while $e^{-i\tau D}$ and $e^{-i\zeta F}$ are representation of $A(\tau)$ and $N(\zeta)$, respectively.

To fix an idea of covering, we look into $e^{-i\theta H}$. The integral kernel of $e^{-i\theta H}$ is given from (2.2) as

$$K_\theta(x,\ \xi)=\frac{\alpha(\theta)}{(2\pi|\sin\theta|)^{1/2}}\exp\left[\frac{i}{2\sin\theta}((x^2+\xi^2)\cos\theta-2x\xi)\right]. \tag{3.4}$$

The phase factor $\alpha(\theta)$ should be determined under the condition that U_θ should be a one-parameter group of unitary operators with the initial value $U_0=id$. Put another way, $\alpha(\theta)$ must be determined in each interval $n\pi<\theta<(n+1)\pi$, $n=0, \pm1, \pm2, \ldots$, so that U_θ may be continuous at $n\pi$. As a consequence we obtain

$$\alpha(\theta)=e^{-i\pi/4}\beta(\theta),\quad \alpha(\theta+4n\pi)=\alpha(\theta),$$

with

$$\beta(\theta)=\begin{cases} -1 & \text{for} \quad -2\pi<\theta<-\pi,\\ i & \text{for} \quad -\pi<\theta<0,\\ 1 & \text{for} \quad 0<\theta<\pi,\\ -i & \text{for} \quad \pi<\theta<2\pi \end{cases} \tag{3.5}$$

and

$$K_\theta(x,\xi)\to\begin{cases} -\delta(x-\xi) & \text{as} \quad \theta\to-2\pi,\\ i\delta(x+\xi) & \text{as} \quad \theta\to-\pi,\\ \delta(x-\xi) & \text{as} \quad \theta\to0,\\ -i\delta(x+\xi) & \text{as} \quad \theta\to\pi. \end{cases} \tag{3.6}$$

We notice here that $\alpha(\theta)$ is periodic in θ with period 4π, while $K(\theta)$ has period 2π. For the U_θ, we can prove that $U_sU_t=U_{s+t}$ by using (2.4).

The operators $e^{-i\tau D}$ and $e^{-i\zeta F}$ cause no problem of covering, and are known to have respective integral kernels

$$K_\tau(x, \xi)=e^{-\tau/2}\delta(e^{-\tau}x-\xi) \tag{3.7}$$

and

$$K_\zeta(x, \xi)=\frac{\alpha(\zeta)}{(2\pi|\zeta|)^{1/2}}\exp\left[\frac{i}{2\zeta}(x-\xi)^2\right] \tag{3.8}$$

with $\alpha(\zeta)=e^{-i\pi(sgn\zeta)/4}$.

In view of the Iwasawa decomposition (3.1), we infer that compositions $e^{-i\zeta F}$ $e^{-i\tau D}$ $e^{-i\theta H}$ will provide a unitary representation of $SL_2(2, \boldsymbol{R})$, where we consider (θ, τ, ζ) as the parameters of $SL_2(2, \boldsymbol{R})$ with θ ranging over the interval of length 4π.

4. A Representation of $SL_2(2, \boldsymbol{R})$

We are going to determine the factor α in the expression (2.2) for the parameters (θ, τ, ζ) of $SL_2(2, \boldsymbol{R})$.

Let

$$\alpha=e^{-i\pi/4}\beta. \tag{4.1}$$

For $e^{-i\theta H}$, β takes values ± 1, $\pm i$, and for $e^{-i\zeta F}$, $\beta=1$ or i, depending on the sign of b.

Let $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix assigned by the parameters (3,1). For the M, there are two elements in $SL_2(2, \boldsymbol{R})$ assigned by the parameters (θ, τ, ζ) and $(\theta\pm 2\pi, \tau, \zeta)$, the sign $\pm$ depending on the value of θ. To determine β, we consider the sign of

$$b=\zeta e^\tau \cos\theta+e^{-\tau}\sin\theta \tag{4.2}$$

for τ and ζ fixed arbitrarily. On setting

$$\tan\chi_0=-\zeta e^{2\tau} \quad \text{with } |\chi_0|<\pi/2, \tag{4.3}$$

the sign of b is alternating, as θ goes into intervals $\chi_0+n\pi<\theta<\chi_0+(n+1)\pi$ in such a manner that, for example, $b>0$ for $\chi_0<\theta<\chi_0+\pi$. In view of the values of β for $e^{-i\theta H}$, we define β to take the values such that

$$\beta=\begin{cases} -1 & \text{for } \chi_0-2\pi<\theta<\chi_0-\pi, \\ i & \text{for } \chi_0-\pi<\theta<\chi_0, \\ 1 & \text{for } \chi_0<\theta<\chi_0+\pi, \\ -i & \text{for } \chi_0+\pi<\theta<\chi_0+2\pi, \end{cases} \tag{4.4}$$

If $\tau=\zeta=0$, then $\chi_0=0$, so that Def. (4.4) reduces to (3.5). We turn to the case of $\theta=\tau=0$, whence $\tan\chi_0=-\zeta$ with $|\chi_0|<\pi/2$. If $\zeta>0$, then $\chi_0<0$, so that $\chi_0<0<\chi_0+\pi$. Hence, from (4.4) we have $\beta=1$. If $\zeta<0$, then $\chi_0>0$, so that $\chi_0-\pi<0<\chi_0$. Hence, $\beta=1$. Thus, Def. (4.4) covers the values of β for $e^{-i\zeta F}$. We wish to show that the integral transforms, whose kernels (2.2) have the factors $\alpha=e^{-i\pi/4}\beta$ with β defined by (4.4), indeed give a unitary representation of $SL_2(2, \boldsymbol{R})$. Let

M_k, $k=1, 2, 21$, be matrices in $SL(2, \boldsymbol{R})$ with parameters $(\theta_k, \tau_k, \zeta_k)$ such that $M_{21}=M_2M_1$. Then from (3.1) and (3.2) applied for $M_{21}=M_2M_1$, we have

$$\theta_{21}=\theta_2+\arg[e^{\tau_1+\tau_2}(\cos\theta_1-\zeta_2\sin\theta_1)+ie^{\tau_1-\tau_2}\sin\theta_1]. \tag{4.5}$$

When the argument and the equality in Eq. (4.5) are taken modulo 2π, Eq. (4.5) gives the parameter θ_{21} for M_{21}. However, if we take Eq. (4.5) modulo 4π, the θ_{21} should be considered as a parameter of $SL_2(2, \boldsymbol{R})$. Moreover, Eq. (4.5) modulo 4π gives the multiplication law for $SL_2(2, \boldsymbol{R})$ together with similar equations obtained from the second and third equations in Eq. (3.2). See also reference 7) for parameters of the universal convering group of $SL(2, \boldsymbol{R})$.

We are now looking into Eq. (4.5). Let $Z(\theta_1)$ denote the quantity in (4.5) (enclosed by the square brackets), the values of τ_1, τ_2, and ζ_2 being fixed. Let $\cot\kappa_1=\zeta_2$ with $|\kappa_1|<\pi/2$ for ζ_2 given. Then for $\zeta_2>0$, arg $Z(\theta_1)$ ranges from -2π to 2π, when θ_1 goes from -2π to 2π, with arg $Z(\kappa_1+n\pi)=n\pi+\pi/2$ and arg $Z(n\pi)=n\pi$, $n=-2, -1, 0, 1$. For $\zeta_2<0$, one has the same range of arg $Z(\theta_1)$ as that for $\zeta_2>0$, with the difference arg $Z(\kappa_1+n\pi)=(n-1)\pi+\pi/2$ and arg $Z((n-1)\pi)=(n-1)\pi$, $n=-1, 0, 1, 2$. Thus we know from Eq. (4.5) how θ_{21} depends on θ_1 and θ_2.

We are now in a position to know the sign of b_{21}/b_1b_2 in the (θ_1, θ_2)-plane, and thereby are able to show by virtue of Eq. (2.4) that the integral transforms under consideration give a representation of $SL_2(2, \boldsymbol{R})$. Let

$$\tan\chi_k=-\zeta_k e^{2\tau_k}, \quad k=1, 2, 21, \tag{4.6}$$

where $|\chi_k|<\pi/2$ for $k=1, 2$. For $k=21$, χ_k will be soon determined uniquely. For $(\chi_k, \tau_k, \zeta_k)$, b_k vanish, $k=1, 2, 21$. Since one has $b_{21}=0$ when $b_1=b_2=0$, Eq. (4.5) gives

$$\chi_{21}=\chi_2+\arg Z(\chi_1), \ \text{mod.}\ 4\pi, \tag{4.7}$$

by which χ_{21} is determined uniquely. To identify the sign of b_{21}/b_1b_2, we draw orthogonal lines $\theta_1=\chi_1+n\pi$ and $\theta_2=\chi_2+m\pi$ in the (θ_1, θ_2)-plane, n and m being integers. We next draw the curves defined by $\theta_{21}=\chi_{21}+k\pi$, k being integers, which pass the intersection points of the already drawn orthogonal lines, because at those points $b_1=b_2=b_{21}=0$. For example, the curve $\theta_{21}=\chi_{21}$ passes the points $(\chi_1+l\pi, \chi_2-l\pi)$, l being integers, and so on. Thus, the (θ_1, θ_2)-plane is broken up into curved triangles. The sign of b_{21}/b_1b_2 can now be definitely settled in each curved triangle.

We are ready to describe the composition of the integral transforms. Let U_k, $k=1, 2, 21$, be the integral transforms with certain values of β, which are determined by (4.4) with χ_k replaced for χ_0. The operators $\pm U_k$ are in two-to-one correspondence with M_k. Now we can prove $U_{21}=U_2U_1$ by the use of (2.4). For example, in the region in which $\chi_1-\pi<\theta_1<\chi_1$, $\chi_2-\pi<\theta_2<\chi_2$, and $\chi_{21}-2\pi<\theta_{21}<\chi_{21}$

$-\pi$, b_{21}/b_1b_2 is positive. Moreover, from (4.1) and (4.4) with χ_k replaced for χ_0, $\alpha_1=\alpha_2=ie^{-i\pi/4}$ and $\alpha_{21}=-e^{-i\pi/4}$. Then we have from (2.4)

$$U_1U_2=\frac{\alpha_1\alpha_2}{\alpha_{21}}e^{i\pi/4}U_{21}=U_{21}. \tag{4.8}$$

In other regions the same reasoning holds. Thus we have proved $U_2U_1=U_{21}$ for $b_1\neq 0$, $b_2\neq 0$.

We are in the final stage of studying canonical transforms. So far we have defined integral transforms for $\theta\neq\chi_0+n\pi$, n being integers. The last task is to define the canonical transforms for $\theta=\chi_0+n\pi$. A definition can be made by finding the limits of the integral kernels as θ tends to $\chi_0+n\pi$. We take the integral kernel (2.2) again to consider its limit as $b\to 0$. Since $ad\text{-}bc=1$, we may understand that $a\neq 0$ when b is near zero. Hence the integral kernel can be writen, by replacing d with $(1+bc)/a$, in the form

$$K(x,\ \xi)=\frac{\alpha}{(2\pi|b|)^{1/2}}\exp\left(\frac{ic}{2a}x^2\right)\exp\left(\frac{ia}{2b}\left(\xi-\frac{x}{a}\right)^2\right) \tag{4.9}$$

It follows from (4.9) that, as $b\to 0$,

$$\int K(x,\ \xi)d\xi\to\frac{\alpha_0}{|a_0|^{1/2}}e^{\pm i\pi/4}\exp\left(\frac{ic_0}{2a_0}x^2\right), \tag{4.10}$$

where the $+$sign or $-$sign is taken, according as $a/b>0$ or $a/b<0$, and the subscript 0 indicates the limit as $b\to 0$.

We are going to the limit of the kernel (4.9) as $\theta\to\chi_0$. The condition $|\chi_0|<\pi/2$ means that $a_0=e^{\tau}\cos\chi_0>0$, so that $a/b>0$ for $\theta\to\chi_0+0$, and $a/b<0$ for $\theta\to\chi_0-0$. Thus, from (4.1) and (4.4), α tends to $\alpha_0=e^{-i\pi/4}$ or $\alpha_0=ie^{-i\pi/4}$ according as $\theta\to\chi_0+0$ or $\theta\to\chi_0-0$. Hence, the limit (4.10) has a unique value as $\theta\to\chi_0\pm 0$, so that it follows from (4.9) that, as $\theta\to\chi_0$,

$$K(x,\ \xi)\to\frac{\exp\left(-\frac{i}{2}x^2\tan\chi_0\right)}{|e^{\tau}\cos\chi_0|^{1/2}}\delta\left(\frac{x}{e^{\tau}\cos\chi_0}-\xi\right) \tag{4.11}$$

When $\tau=\zeta=0$, one has $\chi_0=0$, so that Expression (4.11) reduces to $K(x,\ \xi)\to\delta(x-\xi)$.

In the same manner we obtain the limits of $K(x,\ \xi)$ as $\theta\to\chi_0+n\pi$, n being integers;

$$K(x,\ \xi)\to\frac{i^k\exp\left(-\frac{i}{2}x^2\tan\chi_0\right)}{|e^{\tau}\cos\chi_0|^{1/2}}\delta\left(\frac{(-1)^kx}{e^{\tau}\cos\chi_0}-\xi\right) \tag{4.12}$$

as $\theta\to\chi_0-k\pi$, $k=-2,\ -1,\ 0,\ 1$. Expression (4.12) reduces to (3.6) when $\tau=\zeta=0$. When $\tau=0$, one has $\chi_0=0$. Then, Expression (4.12) reads

$$K(x,\ \xi)\to i^ke^{-\tau/2}\,\delta((-1)^ke^{-\tau}-\xi). \tag{4.13}$$

If $\theta\to 0$, Expression (4.13) with $k=0$ reduces to (3.7).

Thus, we have defined the unitary operators U for all parameters of $SL_2(2,\ \boldsymbol{R})$.

The composition of operators $U_2 U_1 = U_{21}$ is true for all parameters of $SL_2(2, \boldsymbol{R})$ because of the continuity of U.

In conclusion, we point out that the canonical transformations we have constructed have two invariant closed subspaces, which are the space of even functions and the space of odd functions.

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