General Orientation Problem of Photographs with Planar Objects**

By

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Abstract

The orientation problem of photographs taken of planar objects can be treated as a special case for that of a three-dimensional object space. Based on this concept, this paper derived an orientation theory for a two-dimensional object space where the geometry of a picture can, in general, be determined by eight independent central projective parameters. First, the relationship between an object plane and its picture was explored in different ways. Then, the characteristics of the orientation problem were clarified for a stereopair of photographs and for multiple pictures overlapped. The orientation calculation was formulated algebraically with the DLT approach. The orientation methods proposed here are quite general and applicable to the rectification of non-metric photographs with planar objects. In particular, using distances as object space controls, we can form a model congruent to the object by means of a single photograph having non-linear distortions. Also, elements describing the non-linear errors can be provided from the potential of a stereopair of photographs. Furthermore, by employing three pictures overlapped, a united model similar to the object can be constructed without object space information, when we can assume that the interior orientation is unchanged in the three photographs.

1. Introduction

From the time since general and rigorous considerations on the rectification problem from one plane to another were given by Thompson (1965) and others, photogrammetric mensulation of planar objects has been widely employed in many fields such as architecture, the conservation of monuments, archaeology, etc.. Also, to these ends, differential rectification methods have been developed by Kraus (1976), Vozikis (1979, 1984), and Vozikis and Loitsch (1980). However, the rectification theory has not been fully extended to the case of overlapped pictures taken of planar objects, although this is an interesting and instructive problem for photogrammetrists. Further, it is useful for many purposes such as precision measurement of

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planar displacements by means of "before" and "after" photographs (Wong and Vonderohe (1981)), rectification without object space information (Kager, et al. (1985)), and for photogrammetric cadastral surveying of almost flat terrains.

In this paper, the DLT approach (Abdel-Aziz and Karara (1971)) is applied to the orientation problem of photographs with planar objects. Firstly, it is because parameters defining linear systematic errors included in measured image coordinates are absorbed by the coefficients of the collinearity condition, and they can thus be corrected automatically in the orientation calculations. Secondly, it is because its treatment is remarkably simple in comparison with the conventional geometric approach of using photogrammetric orientation parameters. On the other hand, the coplanarity condition of corresponding rays has the potential to provide parameters describing non-linear errors of photo coordinates such as lens distortion and the effect of a lack of film flatness. Hence, based on the general orientation theory of overlapped pictures for a three-dimensional object space (Okamoto (1981a, 1981b)), the characteristics of the orientation problem for a stereopair of photographs with a planar object are clarified when the geometry of a picture can be determined by eight independent orientation elements. Furthermore, the orientation theory is extended to the case of multiple overlapped photographs, which is important for the rectification of non-metric photographs having planar objects without object space information, and for the self calibrating block adjustment for almost flat terrains.

2. Characteristics of General Central Projective One-To-One Correspondence Between Image and Object Planes

In this section, the characteristics of the general central projective transformation relating an object plane and its picture plane will be explored when the object space coordinate system (X, Y, Z) is taken arbitrarily. The collinearity condition between an object point P (X, Y, Z) and its measured image point p_c (x_c, y_c) is described in the following form (Abdel-Aziz and Karara (1971), and Thompson (1971))

$$\begin{aligned} x_c&=\frac{A_1X+A_2Y+A_3Z+A_4}{A_9X+A_{10}Y+A_{11}Z+1}\\ y_c&=\frac{A_5X+A_6Y+A_7Z+A_8}{A_9X+A_{10}Y+A_{11}Z+1}\end{aligned} \tag{1}$$

which is equivalent to the equations of the object ray at the exposure instant. However, Equation 1 cannot simply be solved with respect to the coefficients A_i ($i=1$, $\cdots$, 11), because the 11 parameters are not independent for the case of a two-dimensional object space. By selecting a new object space coordinate system ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) with its $\bar{X}\bar{Y}$-plane on the object plane (See Fig. 1), Equation 1 can be simplified as

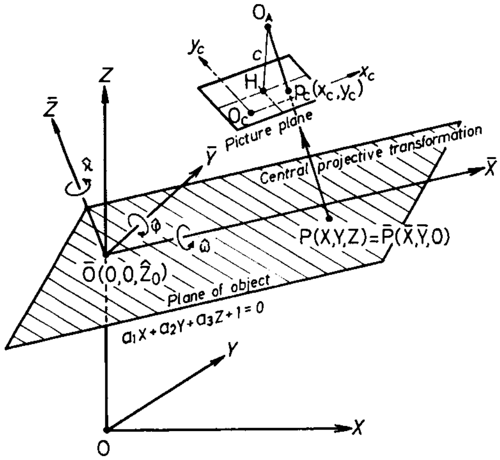

Fig. 1. Central projective relationship between an object plane and its picture plane for the object space coordinate system ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) having the $\bar{X}\bar{Y}$-plane on the object plane.

$$
\begin{aligned}
x_c &= \frac{\bar{A}_1\bar{X}+\bar{A}_2\bar{Y}+\bar{A}_3}{\bar{A}_7\bar{X}+\bar{A}_8\bar{Y}+1} \\
y_c &= \frac{\bar{A}_4\bar{X}+\bar{A}_5\bar{Y}+\bar{A}_6}{\bar{A}_7\bar{X}+\bar{A}_8\bar{Y}+1}
\end{aligned}
\qquad (2)
$$

in which $\bar{A}_i$ ($i=1, \cdots, 8$) denotes independent coefficients. The solution of Equation 2 can solve for $\bar{A}_i$ ($i=1, \cdots, 8$), if four points are given on the object plane. Then, we can reconstruct all object rays at the exposure instant, which means that Equation 2 can be considered in a central perspective way.

The general central projective relationship between an object plane and its picture plane can be considered in still another way of employing the concept of fictitious image and fictitious object planes. The actual object plane is expressed in the reference coordinate system (X, Y, Z) in the form

$$a_1X+a_2Y+a_3Z+1=0 \qquad (3)$$

in which the three coefficients a_i ($i=1, 2, 3$) are independent. By substituting Equation 3 into Equation 1 and further adding Equation 3, we have

$$
\begin{aligned}
x_c &= \frac{A'_1X+A'_2Y+A'_3}{A'_7X+A'_8Y+1} \\
y_c &= \frac{A'_4X+A'_5Y+A'_6}{A'_7X+A'_8Y+1} \\
0 &= a_1X+a_2Y+a_3Z+1
\end{aligned}
\qquad (4)
$$

Equation 4 is equivalent to the general central projective one-to-one correspondence relating two planes, whose coordinates are, however, measured three-dimensionally (See Okamoto (1981c)). Also, the inverse transformation of Equation 4 yields:

$$
\begin{aligned}
X&=\frac{B'_1x_c+B'_2y_c+B'_3}{B'_{10}x_c+B'_{11}y_c+1}\\
Y&=\frac{B'_4x_c+B'_5y_c+B'_6}{B'_{10}x_c+B'_{11}y_c+1}\\
Z&=\frac{B'_7x_c+B'_8y_c+B'_9}{B'_{10}x_c+B'_{11}y_c+1}
\end{aligned}
\tag{5}
$$

Equations 4 and 5 have the following geometrical characteristics:

(i) The central projective relationship between the object plane and its measured image plane can be expressed in terms of 11 independent parameters, if the coordinates of object points are given three-dimensionally.

(ii) The unique determination of this transformation requires four control points, namely, three points with the space coordinates given and one point with the planimetric coordinates known.

(iii) This one-to-one correspondence is very difficult to consider central-perspectively in a three-dimensional space.

The third property of Equation 4 can further be explained as follows. The first and second parts of Equation 4 (or Equation 5) mean geometrically a projective transformation of the measured image plane into the XY-plane of the object space coordinate system (X, Y, Z) arbitrarily selected. Also, only by means of these two equations we can determine the planimetric coordinates (X, Y) of all photographed object points, if four control points in planimetry are available. Furthermore, Z-coordinates of the object points can be obtained by substituting the computed planimetric coordinates (X, Y) into the third part of Equation 4 (or Fquation 5), if we have three control points in height. The latter procedure corresponds to an orthogonal transformation of the object plane into the XY-plane of the reference coordinate system. From these facts we can see that the general central projective one-to-one correspondence (Equation 4) relating the object and measured image planes can be divided into two transformations: a projective transformation relating two planes whose coordinates are given two-dimensionally and an orthogonal transformation.

The orientation problem of photographs taken of planar objects can also be analyzed, based on Equation 4. In particular, this orientation technique is very effective when only planimetric coordinates of objects points are asked for in the object space coordinate system arbitrarily taken, because the first and second parts of Equation 4

$$
\begin{aligned}
x_c&=\frac{A'_1X+A'_2Y+A'_3}{A'_7X+A'_8Y+1}\\
y_c&=\frac{A'_4X+A'_5Y+A'_6}{A'_7X+A'_8Y+1}
\end{aligned}
\tag{6}
$$

can be solved separately from the third part of Equation 4. However, it will be

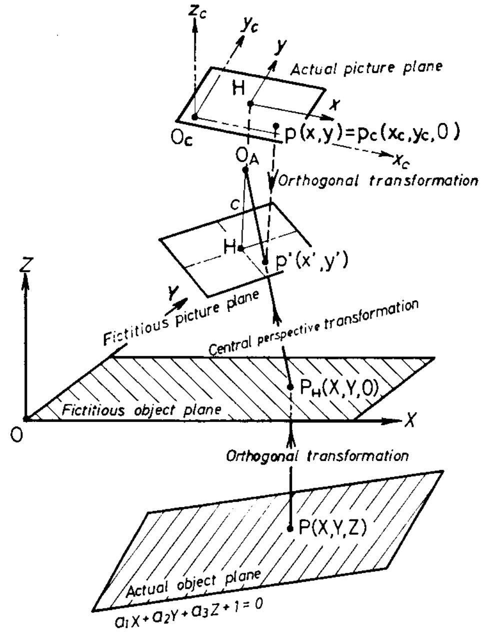

Fig. 2. Central projective relationship between an object plane and its piture plane for the reference coordinate system (X, Y, Z) arbitrarily selected.

noted that Equation 6 has no central perspective characteristics in a three-dimensional space, as is demonstrated in Fig. 2. First, we will introduce the concept of a fictitious picture which is assumed to have been taken of a fictitious object (an image obtainen by an orthogonal (affine) transformation of the actual object into the XY-plane of the reference coordinate system (X, Y, Z)). In addition, the interior and exterior orientation parameters of the fictitious picture are assumed to have been the same as those of the actual picture. This fictitious photograph can be obtained by an affine transformation of the actual picture. Thus, the relationship (Equation 6) between the actual picture and the fictitious object can be considered to be divided into two projections in a three dimensional space: a central perspective projection and an orthogonal projection. It follows that the transformation (Equation 6) relating the actual picture plane and the fictitious object plane is not perspective but projective (See Thompson (1965)).

3. Orientation Problem of a Single Photograph with a Planar Obiect

The orientation problem of a single photograph taken of a planar object can readily be solved by using Equation 2 or Equation 4, if the picture does not have non-linear distortions such as lens distortion and the effect of a lack of film flatness.

However, for the analysis of pictures taken with non-metric cameras, the non-linear distortions are not usually negligibly small. Also the parameters defining the non-linear systematic errors (Δx, Δy) are not absorbed by the coefficients of Equation 2 (or Equation 4). Thus, the fundamental equations must be modified in the form

$$\begin{aligned} x_c-\Delta x&=\frac{\bar{A}_1\bar{X}+\bar{A}_2\bar{Y}+\bar{A}_3}{\bar{A}_7\bar{X}+\bar{A}_8\bar{Y}+1} \\ y_c-\Delta y&=\frac{\bar{A}_4\bar{X}+\bar{A}_5\bar{Y}+\bar{A}_6}{\bar{A}_7\bar{X}+\bar{A}_8\bar{Y}+1} \end{aligned} \tag{7}$$

for the case where the object space coordinate system ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) is selected with its $\bar{X}\bar{Y}$-plane on the object plane, and as

$$\begin{aligned} x_c-\Delta x&=\frac{A'_1X+A'_2Y+A'_3}{A'_7X+A'_8Y+1} \\ y_c-\Delta y&=\frac{A'_4X+A'_5Y+A'_6}{A'_7X+A'_8Y+1} \\ 0&=a_1X+a_2Y+a_3Z+1 \end{aligned} \tag{8}$$

if the reference coordinate system (X, Y, Z) is taken arbitrarily. In the orientation problem of a single picture, the elements modeling the non-linear distortions must be determined from object space information. On the other hand, the coplanarity condition of corresponding rays can provide such parameters, since it does not require object space controls. Therefore, in the following sections, the orientation problem of overlapped pictures taken of planar objects will be discussed in detail.

4. Orientation Problem of a Stereopair of Photographs

4.1. Basic consideration

We will begin with the discussion on the general orientation problem of a stereopair of photographs taken of a three-dimensional object, because that for a two-dimensional object space can be treated as a special case where a photograph has in general only eight independent central projective parameters. The general collinearity equations relating an object point P (X, Y, Z) and its measured image point p_c (x_c, y_c) are described as

$$\begin{aligned} x_{c1}&=\frac{{}_1A_1X+{}_1A_2Y+{}_1A_3Z+{}_1A_4}{{}_1A_9X+{}_1A_{10}Y+{}_1A_{11}Z+1} \\ y_{c1}&=\frac{{}_1A_5X+{}_1A_6Y+{}_1A_7Z+{}_1A_8}{{}_1A_9X+{}_1A_{10}Y+{}_1A_{11}Z+1} \end{aligned} \tag{9}$$

for the left picture, and in the form

$$\begin{aligned} x_{c2}&=\frac{{}_2A_1X+{}_2A_2Y+{}_2A_3Z+{}_2A_4}{{}_2A_9X+{}_2A_{10}Y+{}_2A_{11}Z+1} \\ y_{c2}&=\frac{{}_2A_5X+{}_2A_6Y+{}_2A_7Z+{}_2A_8}{{}_2A_9X+{}_2A_{10}Y+{}_2A_{11}Z+1} \end{aligned} \tag{10}$$

for the right photograph, respectively. The condition that Equations 9 and 10 are valid for all object points photographed in common on the left and right pictures can be formulated as

$$\begin{vmatrix} x_{c1}\ {}_1A_9-{}_1A_1 & x_{c1}\ {}_1A_{10}-{}_1A_2 & x_{c1}\ {}_1A_{11}-{}_1A_3 & x_{c1}-{}_1A_4 \\ y_{c1}\ {}_1A_9-{}_1A_5 & y_{c1}\ {}_1A_{10}-{}_1A_6 & y_{c1}\ {}_1A_{11}-{}_1A_7 & y_{c1}-{}_1A_8 \\ x_{c2}\ {}_2A_9-{}_2A_1 & x_{c2}\ {}_2A_{10}-{}_2A_2 & x_{c2}\ {}_2A_{11}-{}_2A_3 & x_{c2}-{}_2A_4 \\ y_{c2}\ {}_2A_9-{}_2A_5 & y_{c2}\ {}_2A_{10}-{}_2A_6 & y_{c2}\ {}_2A_{11}-{}_2A_7 & y_{c2}-{}_2A_8 \end{vmatrix}=0 \tag{11}$$

which is equivalent to the coplanarity condition of corresponding rays. Under the condition of Equation 11, we can define one space (X_M, Y_M, Z_M) which can be transformed into the object space (X, Y, Z) by the general projective transformation having 15 independent elements, i. e.,

$$\begin{aligned} X_M&=\frac{B_1X+B_2Y+B_3Z+B_4}{B_{13}X+B_{14}Y+B_{15}Z+1} \\ Y_M&=\frac{B_5X+B_6Y+B_7Z+B_8}{B_{13}X+B_{14}Y+B_{15}Z+1} \\ Z_M&=\frac{B_9X+B_{10}Y+B_{11}Z+B_{12}}{B_{13}X+B_{14}Y+B_{15}Z+1} \end{aligned} \tag{12}$$

or inversely as

$$\begin{aligned} X&=\frac{C_1X_M+C_2Y_M+C_3Z_M+C_4}{C_{13}X_M+C_{14}Y_M+C_{15}Z_M+1} \\ Y&=\frac{C_5X_M+C_6Y_M+C_7Z_M+C_8}{C_{13}X_M+C_{14}Y_M+C_{15}Z_M+1} \\ Z&=\frac{C_9X_M+C_{10}Y_M+C_{11}Z_M+C_{12}}{C_{13}X_M+C_{14}Y_M+C_{15}Z_M+1} \end{aligned} \tag{13}$$

Equation 12 (or Equation 13) is equivalent to the general central projective one-to-one correspondence between the model and object spaces.

Based on the orientation theory for a three-dimensional object space, we can explore the properties of the orientation problem of a stereopair of photographs with a planar object (See Fig. 3). For this purpose, we can, without loss of generality, select the object space coordinate system ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) having the $\bar{X}\bar{Y}$-plane on the object plane. Then, Equation 13 becomes

$$\begin{aligned} \bar{X}&=\frac{\bar{C}_1X_M+\bar{C}_2Y_M+\bar{C}_3Z_M+\bar{C}_4}{\bar{C}_{13}X_M+\bar{C}_{14}Y_M+\bar{C}_{15}Z_M+1} \\ \bar{Y}&=\frac{\bar{C}_5X_M+\bar{C}_6Y_M+\bar{C}_7Z_M+\bar{C}_8}{\bar{C}_{13}X_M+\bar{C}_{14}Y_M+\bar{C}_{15}Z_M+1} \\ 0&=\frac{\bar{C}_9X_M+\bar{C}_{10}Y_M+\bar{C}_{11}Z_M+\bar{C}_{12}}{\bar{C}_{13}X_M+\bar{C}_{14}Y_M+\bar{C}_{15}Z_M+1} \end{aligned} \tag{14}$$

The third part of Equation 14 can be reduced to:

$$\bar{D}_9X_M+\bar{D}_{10}Y_M+\bar{D}_{11}Z_M+1=0 \tag{15}$$

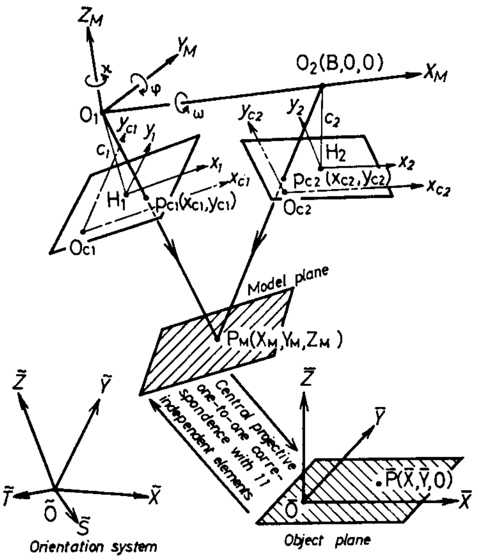

Fig. 3. Orientation problem of a stereopair of photographs taken of a planar object.

We substitute Equation 15 into Equation 14 to obtain

$$\begin{aligned}\bar{X}&=\frac{\bar{D}_1X_M+\bar{D}_2Y_M+\bar{D}_3}{\bar{D}_7X_M+\bar{D}_8Y_M+1}\\ \bar{Y}&=\frac{\bar{D}_4X_M+\bar{D}_5Y_M+\bar{D}_6}{\bar{D}_7X_M+\bar{D}_8Y_M+1}\\ 0&=\bar{D}_9X_M+\bar{D}_{10}Y_M+\bar{D}_{11}Z_M+1\end{aligned}\tag{16}$$

The inverse transformation of Equation 16 yields:

$$\begin{aligned}X_M&=\frac{\bar{B}_1\bar{X}+\bar{B}_2\bar{Y}+\bar{B}_3}{\bar{B}_{10}\bar{X}+\bar{B}_{11}\bar{Y}+1}\\ Y_M&=\frac{\bar{B}_4\bar{X}+\bar{B}_5\bar{Y}+\bar{B}_6}{\bar{B}_{10}\bar{X}+\bar{B}_{11}\bar{Y}+1}\\ Z_M&=\frac{\bar{B}_7\bar{X}+\bar{B}_8\bar{Y}+\bar{B}_9}{\bar{B}_{10}\bar{X}+\bar{B}_{11}\bar{Y}+1}\end{aligned}\tag{17}$$

From Equation 16 we can find the following characteristics of the central projective one-to-one correspondence between the model and object spaces for the special case where the object is a plane:

(i) This transformation can be expressed in terms of 11 independent central projective parameters.

(ii) Four control points are necessary for the unique determination of this one-to-one correspondence.

(iii) The model is constructed in a two-dimensional space, because Equation 15 indicates a plane.

(iv) The model is not similar to the object.

Using the results obtained above, we can further investigate the important properties of the orientation problem of a stereopair of photographs for a two-dimensional object space. The collinearity equations (Equation 2) relating a photographed object point $\bar{P}$ ($\bar{X}$, $\bar{Y}$, 0) and its measured image point p_c (x_c, y_c) have eight independent orientation parameters. Thus, the geometry of a stereopair of photographs can be perfectly determined by 16 independent elements. As 11 elements among these 16 ones can be provided during the phase of the central projective transformation between the model and object planes, the coplanarity condition of corresponding rays must provide only five independent orientation parameters (exterior). However, a stereo model constructed only by means of the coplanarity condition is not similar to the object. In order to make the stereo model similar to the object, we must apply the similarity condition (Okamoto (1981a)) for five corresponding line segments on the model and object planes. This is because the central projective transformation between both planes can be determined uniquely with four points, and the degrees of freedom of four points on a plane are five. Also, from the similarity condition we can determine four interior orientation parameters. If we treat the model scale as an unknown in this model construction process, a stereo model congruent to the object can be formed.

If a metric camera is employed, a stereo model similar to the object can be constructed only by means of the coplanarity condition of corresponding rays. This is due to the fact that all interior orientation parameters can be considered to be given in metric photogrammetry. Kager, et al. proposed in 1985 an orientation method of a stereopair of metric pictures, in which one coplanarity equation was replaced by a condition equation that the object space is two-dimensional. This procedure is mathematically possible, as will be explained later.

4.2. Characteristics of the Orientation Calculation

In this paragraph, the orientation calculation techniques for a stereopair of photographs having a planar object will be discussed by means of the collinearity equations. Thus, all the orientation unknowns are determined simultaneously and together with the unknown coordinates of orientation points, exluding the necessary control points.

(A) For the object space coordinate system ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) with its $\bar{X}\bar{Y}$-plane on the object plane

The collinearity condition relating an object point $\bar{P}$ ($\bar{X}$, $\bar{Y}$, 0) and its measured image point p_c (x_c, y_c) is written together for a stereopair of photographs as

$$
\left.
\begin{aligned}
x_{c1} &= \frac{{}_1\bar{A}_1\bar{X}+{}_1\bar{A}_2\bar{Y}+{}_1\bar{A}_3}{{}_1\bar{A}_7\bar{X}+{}_1\bar{A}_8\bar{Y}+1}\\
y_{c1} &= \frac{{}_1\bar{A}_4\bar{X}+{}_1\bar{A}_5\bar{Y}+{}_1\bar{A}_6}{{}_1\bar{A}_7\bar{X}+{}_1\bar{A}_8\bar{Y}+1}\\
x_{c2} &= \frac{{}_2\bar{A}_1\bar{X}+{}_2\bar{A}_2\bar{Y}+{}_2\bar{A}_3}{{}_2\bar{A}_7\bar{X}+{}_2\bar{A}_8\bar{Y}+1}\\
y_{c2} &= \frac{{}_2\bar{A}_4\bar{X}+{}_2\bar{A}_5\bar{Y}+{}_2\bar{A}_6}{{}_2\bar{A}_7\bar{X}+{}_2\bar{A}_8\bar{Y}+1}
\end{aligned}
\right\}
\tag{18}
$$

Mathematically, Equation 18 includes one equation equivalent to the coplanarity condition of corresponding rays. Thus, we must set up Equation 18 for five points in order to use five coplanarity equations mathematically required. However, Equation 18 can solve for the 16 independent coefficients ${}_i\bar{A}_j$ $(i=1,\ 2;\ j=1,\ \cdots,\ 8)$, if Equation 18 is set up for the four control points mathematically necessary. This means that the orientation problem of the stereopair can be solved by using only four coplanarity equations, which seems to contradict the orientation theory in the preceding paragraph. This apparent contradiction can, however, be explained as follows. In order to determine a plane in a reference coordinate system, we need only three points with the space coordinates known. However, in the orientation calculation described above, four points with the space coordinates given are considered to have been used, because the $\bar{Z}$-coordinates of the four points are known as $\bar{Z}=0$. From this fact it can be understood that the $\bar{Z}$-coordinate of one control point is mathematically redundant, and it plays a roll of one coplanarity equation. It follows that the condition of an object plane can be used as a part of the coplanarity equations mathematically required (See Kager, et al. (1985)).

(B) For the reference coordinate system $(X,\ Y,\ Z)$ arbitrarily taken

The orientation problem of a stereopair of photographs with planar objects can also be analyzed, based on Equation 4. Writing down the first and second parts of Equation 4 together for the stereopair, we have

$$
\left.
\begin{aligned}
x_{c1} &= \frac{{}_1A'_1X+{}_1A'_2Y+{}_1A'_3}{{}_1A'_7X+{}_1A'_8Y+1}\\
y_{c1} &= \frac{{}_1A'_4X+{}_1A'_5Y+{}_1A'_6}{{}_1A'_7X+{}_1A'_8Y+1}\\
x_{c2} &= \frac{{}_2A'_1X+{}_2A'_2Y+{}_2A'_3}{{}_2A'_7X+{}_2A'_8Y+1}\\
y_{c2} &= \frac{{}_2A'_4X+{}_2A'_5Y+{}_2A'_6}{{}_2A'_7X+{}_2A'_8Y+1}
\end{aligned}
\right\}
\tag{19}
$$

In principle, the orientation calculation based on Equation 19 includes the following two main processes: a fictitious model construction and a central projective transformation between the fictitious model and a fictitious object (an image obtainen by an orthogonal transformation of the actual object into the XY-plane of the reference

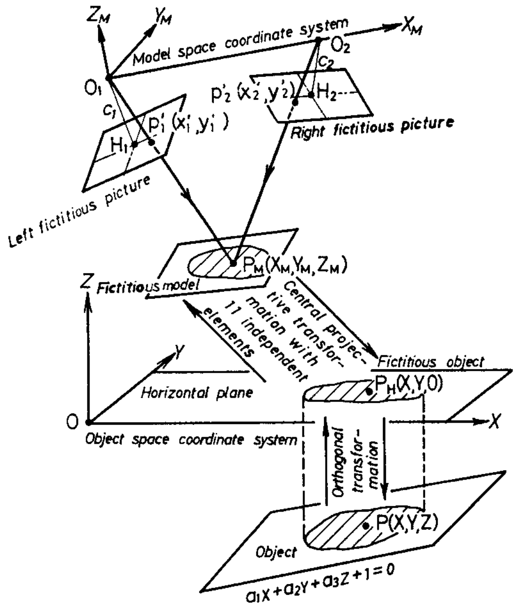

Fig. 4. Geometrical characteristics of Equation 19.

coordinate system) (See Fig. 4). Also, the orientation calculation may be performed by using planimetric coordinates of four control points. Setting up Equation 19 for four points on the fictitious object plane, corresponding to the four control points, we get 16 equations. Solving these 16 equations with respect to the 16 unknown coefficients ${}_iA'_j$ ($i=1, 2$; $j=1, \cdots, 8$), we can determine planimetric coordinates of all photographed object points. This is because the 16 equations, in principle, contain four coplanarity equations and one condition equation that the object space is two-dimensional.

The transformation of the fictitious object into the actual object can be carried out by means of the third part of Equation 4, namely,

$$a_1X+a_2Y+a_3Z+1=0$$

which requires mathematically three control points in height.

(C) A method of using the relationship between the left and right photographs

First, we will investigate the relationship between a stereopair of pictures having a planar object (See Fig. 5). The object space coordinate system ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) is selected with its $\bar{X}\bar{Y}$-plane on the object plane. The central projective transformation between the object plane and the left picture can be described in the form of Equation 2. The same relationship is valid between the object plane and the right photograph. Substituting the inverse transformation of Equation 2 for the right picture into Equation 2 for the left photograph, we obtain

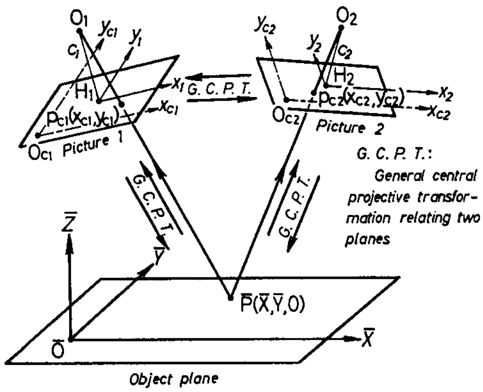

Fig. 5. Central projective one-to-one correspondence between two pictures taken of the same object plane.

$$
\begin{aligned}
x_{c1} &= \frac{\bar{E}_1 x_{c2} + \bar{E}_2 y_{c2} + \bar{E}_3}{\bar{E}_7 x_{c2} + \bar{E}_8 y_{c2} + 1} \\
y_{c1} &= \frac{\bar{E}_4 x_{c2} + \bar{E}_5 y_{c2} + \bar{E}_6}{\bar{E}_7 x_{c2} + \bar{E}_8 y_{c2} + 1}
\end{aligned}
\tag{20}
$$

Equation 20 shows that the general central projective one-to-one correspondence with eight independent parameters is also satisfied between the left and right photographs. Using this property, measured image coordinates (x_{c2}, y_{c2}) for the right picture can be transformed to the comparator coordinate system for the left picture. Also, the orientation of the left picture with respect to the control points can be determined by means of Equation 2. Thus, this method may be effective for measuring planar displacements by using a pair of "before" and "after" photographs when the non-linear distortions are negligibly small or when they do not affect significantly the precision of the measurements (See Wong and Vonderohe (1981)).

However, corrections for the non-linear distortions are required in such cases where absolute positioning with respect to the reference coordinate system is a matter of major importance and where, at least, the shape of the planar object must be accurately measured. The orientation method based on Equation 20 can readily be extended to the analysis of such photographs, because Equation 20 includes the concept of both the coplanarity condition and the condition of a planar object.

(D) Orientation calculation with non-linear distortions

If a stereopair of photographs with a planar object have non-linear systematic errors, both the coplanarity condition of the corresponding rays and the condition due to the planar object can mathematically provide parameters defining the non-linear distortions (Δx, Δy). Taking the object space coordinate system ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) with its $\bar{X}\bar{Y}$-plane on the object plane, the orientation calculation will be, briefly outlined, as follows.

The collinearity equations are described together for the left and right photographs as

$$
\begin{aligned}
x_{c1}-\Delta x&=\frac{{}_1\bar{A}_1\bar{X}+{}_1\bar{A}_2\bar{Y}+{}_1\bar{A}_3}{{}_1\bar{A}_7\bar{X}+{}_1\bar{A}_8\bar{Y}+1}\\
y_{c1}-\Delta y&=\frac{{}_1\bar{A}_4\bar{X}+{}_1\bar{A}_5\bar{Y}+{}_1\bar{A}_6}{{}_1\bar{A}_7\bar{X}+{}_1\bar{A}_8\bar{Y}+1}\\
x_{c2}-\Delta x&=\frac{{}_2\bar{A}_1\bar{X}+{}_2\bar{A}_2\bar{Y}+{}_2\bar{A}_3}{{}_2\bar{A}_7\bar{X}+{}_2\bar{A}_8\bar{Y}+1}\\
y_{c2}-\Delta x&=\frac{{}_2\bar{A}_4\bar{X}+{}_2\bar{A}_5\bar{Y}+{}_2\bar{A}_6}{{}_2\bar{A}_7\bar{X}+{}_2\bar{A}_8\bar{Y}+1}
\end{aligned}
\qquad (21)
$$

Assuming the number of parameters modeling the non-linear distortions to be n for one picture, we have (16+2n) orientation unknowns for the stereopair. Setting up Equation 21 for four control points mathematically necessary and n unknown object points, we get (16+4n) equations with respect to (16+4n) unknowns (the 16 coefficients of Equation 21, the 2n elements for the non-linear systematic errors, and 2n unknown coordinates of the n orientation points). Solving these (16+4n) equations, all object points can be determined in the reference coordinate system.

5. Orientation Problem of Multiple Photographs Overlapped

5.1. Basic Consideration

We will assume that a planar object was photographed by using four different non-metric cameras. The first stereo model is constructed with the first and second photographs and the second stereo model with the third and fourth pictures. What

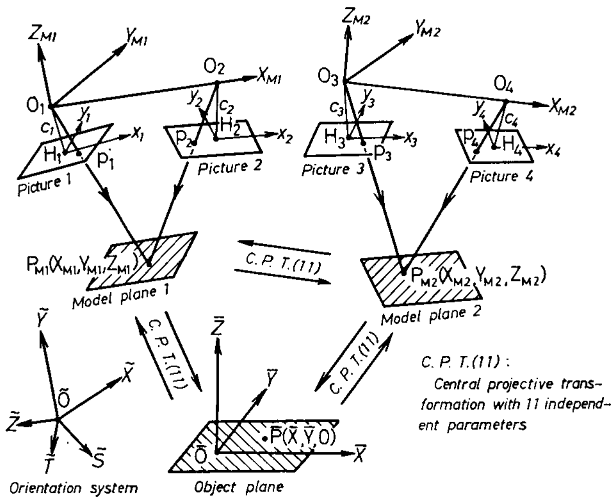

Fig. 6. Central projective relationship between the first and second model planes in case of four non-metric pictures taken of the same planar object.

relationship is valid between these two model planes? This problem will be explored, based on the united model construction theory for a three-dimensional object space (See Okamoto (1981b)). The object space coordinate system ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) is selected with its $\bar{X}\bar{Y}$-plane on the object plane (See Fig. 6). The central projective transformation between the first model space (X_{M1}, Y_{M1}, Z_{M1}) and the object space ($\bar{X}$, $\bar{Y}$, 0) is given in the form of Equation 16. Also, the inverse transformation is of the form of Equation 17. The same relationships must be valid between the second model space (X_{M2}, Y_{M2}, Z_{M2}) and the object space ($\bar{X}$, $\bar{Y}$, 0). Substituting Equation 16 for the second model space into Equation 17 for the first model space, we can find the relationship between both model spaces, i. e.,

$$\begin{aligned} X_{M1} &= \frac{\bar{F}_1 X_{M2} + \bar{F}_2 Y_{M2} + \bar{F}_3}{\bar{F}_{10} X_{M2} + \bar{F}_{11} Y_{M2} + 1} \\ Y_{M1} &= \frac{\bar{F}_4 X_{M2} + \bar{F}_5 Y_{M2} + \bar{F}_6}{\bar{F}_{10} X_{M2} + \bar{F}_{11} Y_{M2} + 1} \\ Z_{M1} &= \frac{\bar{F}_7 X_{M2} + \bar{F}_8 Y_{M2} + \bar{F}_9}{\bar{F}_{10} X_{M2} + \bar{F}_{11} Y_{M2} + 1} \end{aligned} \tag{22}$$

Equation 22 means that the central projective one-to-one correspondence with 11 independent parameters is valid between the first and second model planes. In addition, these 11 elements can be divided into seven exterior and four interior orientation parameters. This is because the three-dimensional similarity transformation with seven exterior ones must be satisfied between both model planes, if we employ metric cameras in this measurement. It follows that four interior orientation parameters among the eight independent ones of the four non-metric photographs can be provided from the model connection condition (Equation 22).

The case of three photographs taken of the same planar object can be treated as a special case for the previous one, where the second and third photographs have the same orientation parameters. In other words, the case of three pictures is equivalent to the case of four photographs with eight constraints, because eight independent orientation elements can determine perfectly the geometry of a picture of a planar object. Thus, considering that the first model is formed with the first and second pictures and the second model by means of the second and third photographs, we can see that the relationship between the two model planes can be expressed in terms of three independent parameters (one exterior and two interior orientation elements).

5.2. Orientation Calculation of Three Photographs Overlapped

Figure 7 illustrates the arrangement of two kinds of points required for the orientation calculation of three overlapped photographs with a planar object. For object points which lie in the overlapped part of the three pictures, we can set up the following six equations, i. e.,

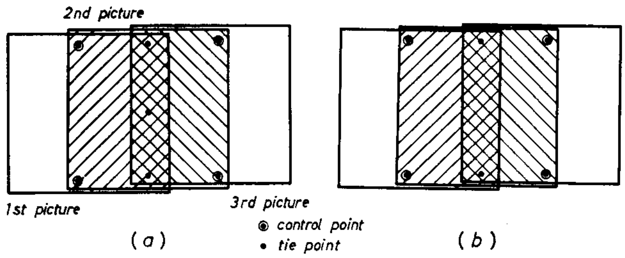

Fig. 7. Arrangement of control and tie points.

$$
\begin{aligned}
x_{c1}&=\frac{{}_1\bar{A}_1\bar{X}+{}_1\bar{A}_2\bar{Y}+{}_1\bar{A}_3}{{}_1\bar{A}_7\bar{X}+{}_1\bar{A}_8\bar{Y}+1}\\
y_{c1}&=\frac{{}_1\bar{A}_4\bar{X}+{}_1\bar{A}_5\bar{Y}+{}_1\bar{A}_6}{{}_1\bar{A}_7\bar{X}+{}_1\bar{A}_8\bar{Y}+1}\\
x_{c2}&=\frac{{}_2\bar{A}_1\bar{X}+{}_2\bar{A}_2\bar{Y}+{}_2\bar{A}_3}{{}_2\bar{A}_7\bar{X}+{}_2\bar{A}_8\bar{Y}+1}\\
y_{c2}&=\frac{{}_2\bar{A}_4\bar{X}+{}_2\bar{A}_5\bar{Y}+{}_2\bar{A}_6}{{}_2\bar{A}_7\bar{X}+{}_2\bar{A}_8\bar{Y}+1}\\
x_{c3}&=\frac{{}_3\bar{A}_1\bar{X}+{}_3\bar{A}_2\bar{Y}+{}_3\bar{A}_3}{{}_3\bar{A}_7\bar{X}+{}_3\bar{A}_8\bar{Y}+1}\\
y_{c3}&=\frac{{}_3\bar{A}_4\bar{X}+{}_3\bar{A}_5\bar{Y}+{}_3\bar{A}_6}{{}_3\bar{A}_7\bar{X}+{}_3\bar{A}_8\bar{Y}+1}
\end{aligned}
\tag{23}
$$

in which ${}_i\bar{A}_j$ ($i=1$, 2, 3; j$=1$, ⋯, 8) are independent coefficients and (x_{ci}, y_{ci}) ($i=1$, 2, 3) denote measured plate coordinates, respectively, for the three photographs. Equation 23 contains one equation corresponding to the coplanarity condition for the first and second pictures, one equation equivalent to that for the second and third photographs, one equation for the united model construction, and one condition equation that the object space is two-dimensional. On the other hand, the collinearity equations in the form of Equation 18 are valid for object points which exist in the overlapped parts of two adjacent pictures. From the following facts which were described earlier, namely:

(i) The coplanarity condition provides mathematically five independent orientation elements (exterior),

(ii) The transformation of the second model plane into the first one can be determined by three independent orientation parameters (one exterior and two interior), and

(iii) The one-to-one correspondence between the united model plane and the object plane is expressed in terms of 11 independent orientation parameters (seven exterior and four interior), which can be uniquely determined with four control points,

the orientation calculation of the three non-metric photographs seems to require such an arrangement of four control points and three tie points as is demonstrated in Fig. 7a. Then, we have 34 equations for the determination of 30 unknowns (the 24 unknown coefficients ${}_i\bar{A}_j$ ($i=1, 2, 3$; $j=1, \cdots, 8$) of Equation 23 plus six unknown coordinates ($\bar{X}$, $\bar{Y}$) of the three tie points). This is an overdetemined system due to the fact that we used seven condition equations for the planar object, though only three are necessary for the determination of the object plane in the reference coordinate system. In order to avoid the overdetermined system, we may configurate four control points and two tie points, as is shown in Fig. 7b. Then, 28 Equations are obtained for the unique determination of 28 unknowns (the 24 coefficients ${}_i\bar{A}_j$ ($i=1, 2, 3$; $j=1, \cdots, 8$) plus four planimetric coordinates of the two tie points). In the latter orientation calculation, three redundant condition equations for the planar object play the roll of two coplanarity equations and one transformation equation of the second model plane into the first one.

The orientation problem of multiple photographs taken of the same planar object can also be analyzed by applying the orientation method based on Equation 4. In particular, this method may be effectively used for cadastral surveying of almost flat terrains. In addition, rectification of non-metric photographs can be performed without object space information, if we employ three pictures taken of the same planar object. Under the assumption that the interior orientation is unchanged in the three pictures, we have only two interior orientation parameters to be determined. These two elements can be provided from the model connection condition, as was explained earlier. Then, a united stereo model similar to the planar object can be formed, which is the end product in this case.

6. Concluding Discussions

The general orientation problem of non-metric photographs taken of planar objects has been discussed fundamentally in this paper. First, the central projective relationship relating a planar object and its picture has been investigated and the next properties have been clarified, namely:

(i) The transformation between the picture and object planes can be expressed in terms of 11 independent parameters, if the object plane is inclined with respect to the reference coordinate system, and

(ii) This one-to-one correspondence can be diveded into two transformations: a projective transformation with eight independent elements and an orthogonal transformation having three independent parameters.

Using the characteristics cited above, a direct (or differential) rectification method of a single photograph can be developed, when the object plane is not paralled to the XY-plane of the reference coordinate system (X, Y, Z).

Next, the rectification theory has been extended to the case of multiple photographs taken of the same planar object and the following interesting characteristics have been revealed that:

(i) The coplanarity condition of corresponding rays can provide five independent orientation parameters (exterior) and the stereo model is constructed in a two-dimensional space.

(ii) The stereo model is not similar to the object. However, if we employ metric cameras, a stereo model similar to the object can be formed only by means of the coplanarity condition (See Kager, et al. (1985)).

(iii) The central projective one-to-one correspondence with 11 independent parameters is valid between the model and object planes, which can be uniquely determined with four control points.

(iv) The relationship between two photographs taken of the same planar object can be given by the general central projective transformation with eight independent parameters.

(v) In the analysis of three overlapped photographs with a planar object, the model connection condition can determine three independent orientation elements (one exterior and two interior). Thus, we can form a united stereo model similar to the object under the assumption that the interior orientation is unchanged in the three pictures,

(vi) Parameters defining non-linear distortions of photographs can be provided from not only the coplanarity condition but also the model connection condition and the condition for a planar object.

The orientation calculations have been formulated mainly algebraically with the DLT method, because the treatment is remarkably simple in comparison with the conventional geometric approach.

Based on the characteristics clarified here, we can develop different orientation and rectification methods for photographs taken of planar objects. These problems will be discussed in a later report.

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