An Optimal Replacement Problem of A Semi-Markovian Deteriorating System

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Abstract

This paper discusses an optimal replacement problem of a multi-state system when the deterioration of the system state is described by a semi-Markov process. It is assumed that the system has operating costs and replacement costs depending on its states. The problem is to derive a replacement policy which minimizes the expected average cost per unit time over the infinite horizon. Moreover, under some reasonable conditions reflecting the physical and economical meaning of the deterioration, we show that an optimal replacement policy has a monotone structure.

1. Introduction

Up to now, several researchers have discussed many types of problems on optimal maintenance policies for various binary state systems i. e. systems which are assumed to have only two posible states, mainly because of theoretical simplicity. That is, they have considered the systems which are normally in good states as like new, and directly fall into failed states after some random duration.

From the practical point of view, however, it is reasonable to deal with multi-state systems i. e. systems which have many states corresponding to the degrees of the deterioration of the system performance, and to consider operating costs and maintenance costs which increase with the deterioration of the system state.

For optimal maintenance problems of multi-state systems, Derman [2] and others studied optimal replacement problems of systems, assuming that the deteriorating processes are discrete-time and finite state Markov chains. They discussed the structural

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properties of optimal replacement policies. Kao [5] formulated optimal replacement problems as semi-Markov decision processes, when the changes of system states are described by discrete-time and finite state semi-Markov processes.

This paper discusses an optimal replacement problem for a multi-state system with a finite number of deteriorated states, whose deteriorating process is described by a continuous-time semi-Markov process. The problems are to obtain an optimal replacement policy which minimizes the expected average cost per unit time over the infinite horizon, and to derive a sufficient condition under which an optimal replacement policy has a monotone structure.

2. Model Description

First, we describe the semi-Markovian deteriorating system. The system is assumed to be in any one of the states denoted by $0, 1, \ldots, N, N+1$. Number 0 means a good state where the system perfectly operates as a new one. Numbers $1, \ldots, N$ denote deteriorated states, and $i<j$ means that state j is more deteriorated than state i. Number $N+1$ denotes a failed state, and the failed system is immediately replaced with a new one.

We assume that the system deteriorates according to the following semi-Markov process.

$F_i(x)$ $(0\leqq i\leqq N)$: the cumulative distribution function of the sojourn time in state i. It is assumed that $F_i(x)$ has a continuous density function $f_i(x)$ and a finite positive mean.

p_{ij} $(0\leqq i<j\leqq N+1)$: the probability that the next visited state of the system in state i is j, where

$$p_{ij}=0 \text{ for } i\geqq j.$$

For the purpose of notational convenience, we use the following notations:

$$\bar{F}_i(x):=1-F_i(x),$$
$$r_i(x):=f_i(x)/\bar{F}_i(x),$$
$$\lambda_i:=\int_0^\infty uf_i(u)\,du=\int_0^\infty \bar{F}_i(u)\,du.$$

As maintenance activities for the system, we consider only replacement, and let $(0<\mu<+\infty)$ be the expected time for the replacement of the system.

Morever, we consider the following cost structure:

a_i $(\geqq 0,\ 0\leqq i\leqq N)$: the operating cost rate per unit time in state i,

c_i $(>0,\ 0\leqq i\leqq N+1)$: the replacement cost of the system in state i.

3. Optimal Replacement Problem

In this section, we discuss an optimal replacement problem for the semi-Markovian deteriorating system described in the preceding section. Since the condition of the system at any time instant is completely specified by the pair of the current system state and the sojourn time in the said system state, the problem is to determine, for each state i, the time length t_i from the entrance time instant into state i, after which the system should be preventively replaced. This means that if the sojourn time in state i reaches t_i, then the system is preventively replaced at this time and that if the system state makes a transition to another state (say state j) at a certain time instant before the said event occurs, then the system obeys the preventive replacement rule t_j specified for the new system state j.

Thus, if we define

$$\boldsymbol{t} := (t_0, t_1, \ldots, t_N, t_{N+1})$$
$$0 \leqq t_i \leqq +\infty \quad \text{for } 0 \leqq i \leqq N+1,$$

then t specifies a replacement policy. It is noted that $t_i=0$ implies the system is immediately replaced when the system state enters into state i. Also, $t_i=+\infty$ means the system is never preventively replaced in state i until the system state makes a transition to another state.

It is reasonable that in order to exclude practically meaningless replacement policies, we confine ourselves to the following class of policies:

$$R := \{\boldsymbol{t}; 0 \leqq t_i \leqq +\infty \text{ for } 0 \leqq i \leqq N, \text{ and } t_{N+1}=0\},$$

that is, under any policy $\boldsymbol{t}(\in R)$, the system is replaced immediately after a system failure.

Now, we consider the problem to obtain an optimal policy minimizing the expected average cost per unit time over the infinite horizon among the policies of class R. For this purpose, we define

$x_i(\boldsymbol{t})$: the expected time length from the entrance into state i till the completion of the replacement of the system under the replacement policy $\boldsymbol{t}$,

$y_i(\boldsymbol{t})$: the expected total cost incurred from the entrance into state i till the completion of the replacement of the system under the replacement policy $\boldsymbol{t}$.

Then, the following relations hold:

$$x_i(\boldsymbol{t}) = \mu\bar{F}_i(t_i) + \int_0^{t_i} \bar{F}_i(u)\,du + F_i(t_i) \sum_{j=i+1}^{N+1} p_{ij} x_j(\boldsymbol{t}) \quad \text{for } 0 \leqq i \leqq N \tag{3.1}$$

$$x_{N+1}(\boldsymbol{t}) = \mu, \tag{3.2}$$

and

$$y_i(\boldsymbol{t}) = C_i\bar{F}_i(t_i) + a_i\int_0^{t_i}\bar{F}_i(u)du + F_i(t_i)\sum_{j=i+1}^{N+1}p_{ij}y_j(\boldsymbol{t}) \quad \text{for } 0\leq i\leq N \tag{3.3}$$

$$y_{N+1}(\boldsymbol{t}) = C_{N+1}. \tag{3.4}$$

It is easy to show the inequalities

$$0<\mu\leq x_i(\boldsymbol{t})\leq \sum_{j=i}^{N}\lambda_j+\mu \quad \text{for } 0\leq i\leq N+1 \tag{3.5}$$

hold.

According to the standard theoretical result of the renewal theory, the problem is reduced to that of obtaining a replacement policy $\boldsymbol{t}^*=(t_0^*, t_1^*, \ldots, t_N^*, t_{N+1}^*)$ $(\in R)$, which attains

$$g^* := \inf_{\boldsymbol{t}\in R} y_0(\boldsymbol{t})/x_0(\boldsymbol{t}). \tag{3.6}$$

4. Structure of Optimal Replacement Policy

First, we define the following functions defined on R which include a real parameter g:

$$v_i(\boldsymbol{t};g) := y_i(\boldsymbol{t}) - gx_i(\boldsymbol{t}) \quad \text{for } 0\leq i\leq N+1. \tag{4.1}$$

Then, the following lemma holds from the well known result of the theory of the fractional programming.

[LEMMA 4-1]

A replacement policy $\boldsymbol{t}^*(\in R)$ satisfies

$$v_0(\boldsymbol{t}^*;g^*) = \inf_{\boldsymbol{t}\in R}v_0(\boldsymbol{t};g^*) = 0, \tag{4.2}$$

if and only if $\boldsymbol{t}^*$ is optimal, that is,

$$y_0(\boldsymbol{t}^*)/x_0(\boldsymbol{t}^*) = g^*. \tag{4.3}$$

It is noted that the following relations hold from (3.1)-(3.4).

$$v_i(\boldsymbol{t};g^*) = (C_i-g^*\mu)\bar{F}_i(t_i) + (a_i-g^*)\int_0^{t_i}\bar{F}_i(u)du$$

$$+F_i(t_i)\sum_{j=i+1}^{N+1}p_{ij}v_j(\boldsymbol{t};g^*) \quad \text{for } 0\leq i\leq N, \tag{4.4}$$

$$v_{N+1}(\boldsymbol{t};g^*) = C_{N+1}-g^*\mu. \tag{4.5}$$

Moreover, defining

$$v_i := \inf_{\boldsymbol{t}\in R}v_i(\boldsymbol{t};g^*), \quad \text{for } 0\leq i\leq N \tag{4.6}$$

we obtain the following optimality equations from (4.4) and (4.5)

$$v_i = \inf_{0 \leq t \leq +\infty} [(C_i - g^* \mu)\bar{F}_i(t) + (a_i - g^*)\int_0^t \bar{F}_i(u)du + F_i(t) \sum_{j=i+1}^{N+1} p_{ij} v_j] \text{ for } 0 \leq i \leq N, \tag{4.7}$$

$$v_{N+1} = C_{N+1} - g^* \mu. \tag{4.8}$$

An optimal replacement policy

$$\boldsymbol{t}^* = (t_0^*, t_1^*, \ldots, t_N^*, t_{N+1}^*)$$

can be obtained by choosing as t_i^*, for each i, a time length t which attains the minimum of the right hand side of (4.7). Let us define

$$G_i(t) := (C_i - g^* \mu)\bar{F}_i(t) + (a_i - g^*)\int_0^t \bar{F}_i(u)du + F_i(t) \sum_{j=i+1}^{N+1} p_{ij} v_j, \tag{4.9}$$

and

$$g_i(t) := \frac{d}{dt} G_i(t) = \bar{F}(t)[a_i - r_i(t) C_i + \{\sum_{j=i+1}^{N+1} p_{ij} v_j + g^* \mu\} r_i(t) - g^*] \text{ for } 0 \leq i \leq N. \tag{4.10}$$

Then, (4.8) can be rewritten as

$$v_i = \inf_{0 \leq t \leq +\infty} G_i(t). \tag{4.11}$$

In the sequel, we assume that the following five conditions are satisfied.

(C-1) $r_i(x)$ is non-decreasing in both x and i.

(C-2) $\sum_{j=k}^{N+1} p_{ij}$ is non-decreasing in i for an arbitrarily fixed $k (=0, 1, \ldots, N, N+1)$.

(C-3) C_i is non-decreasing in i.

(C-4) $a_i - r_i(x) C_i$ is non-decreasing in i for an arbitrarily fixed $x (\in [0, +\infty))$.

(C-5) $a_i \lambda_i$ is non-decreasing in i.

The conditions (C-1) and (C-2) state that as the system becomes deteriorated, it is more likely to make a transition to the more deteriorated state. (C-3) and (C-5) show that as the system deteriorates it becomes more costly to operate or replace. Finally, (C-4) implies that a replacement improves in comparison to an operation with deterioration. These conditions are reasonable to describe real systems in practical respects.

Under these conditions, we investigate the structure of an optimal replacement policy $\boldsymbol{t}^*$ which is obtained from the optimality equations (4.7) (or (4.11)).

We prepare the following lemmas.

[LEMMA 4-2]

For $i = 0, 1, \ldots, N-1$

$$\bar{F}_i(x) \geqq \bar{F}_{i+1}(x) \text{ for all } x(\in[0, +\infty)) \tag{4.12}$$

holds, and hence,

$$\lambda_i \geqq \lambda_{i+1}. \tag{4.13}$$

(PROOF)

The proof is easy from the condition (C-1) and the following relations:

$$\bar{F}_i(x) = \exp\{-\int_0^x r_i(u)du\} \tag{4.14}$$

and

$$\lambda_i = \int_0^\infty \bar{F}_i(u)du. \quad \square$$

[LEMMA 4-3]

The following inequalities hold:

$$0 < C_0/\{\sum_{i=0}^{N} \lambda_i + \mu\} \leqq g^* \leqq C_0/\mu. \tag{4.15}$$

(PROOF)

The first inequality is obvious. The second inequality follows from the fact that, for all $\boldsymbol{t}(\in R)$, from (3.5)

$$x_0(\boldsymbol{t}) \leqq \sum_{i=0}^{N} \lambda_i + \mu,$$

and

$$y_0(\boldsymbol{t}) \geqq C_0.$$

The final inequality is shown from

$$g^* = \inf_{\boldsymbol{t}\in R} \frac{y_0(\boldsymbol{t})}{x_0(\boldsymbol{t})} \leqq \frac{y_0(\boldsymbol{t}^0)}{x_0(\boldsymbol{t}^0)} = \frac{C_0}{\mu}, \tag{4.16}$$

where

$$\boldsymbol{t}^0 = (\underset{0}{0}, \underset{1}{0}, \ldots, \underset{N}{0}, \underset{N+1}{0}) \ (\in R). \quad \square$$

Now we obtain the following theorem.

[THEOREM 4-1]

If

$$v_k = C_k - g^* \mu \tag{4.17}$$

holds for some $k(0\leqq k\leqq N+1)$, then

$$v_i=C_i-g^*\mu \text{ for all } i(\geqq k). \tag{4.18}$$

(PROOF)

It suffices to show that if, for some m and $n(0\leqq m\leqq n\leqq N+1)$,

$$v_m=C_m-g^*\mu \tag{4.19}$$

and

$$v_i=C_i-g^*\mu \text{ for all } i(\geqq n+1) \tag{4.20}$$

hold, then

$$v_n=C_n-g^*\mu. \tag{4.21}$$

From (4.19), it holds that

$$v_m=C_m-g^*\mu\leqq G_m(t) \text{ for all } t(\in[0, +\infty)). \tag{4.22}$$

Noting that from (4.20)

$$G_n(t)=\{\sum_{j=n+1}^{N+1}p_{nj}(C_j-g^*\mu)-(C_n-g^*\mu)\}F_n(t)$$
$$+(a_n-g^*)\int_0^t \bar{F}_n(u)du+C_n-g^*\mu,$$

we have only to show

$$G_n(t)\geqq G_n(0)=C_n-g^*\mu.$$

Defining, for $0\leqq i\leqq N$,

$$A_i(t):=G_i(t)-(C_i-g^*\mu)$$
$$=F_i(t)\{\sum_{j=i+1}^{N+1}p_{ij}v_j-(C_i-g^*\mu)\}+(a_i-g^*)\int_0^t \bar{F}_i(u)du$$

and

$$B_i(t):=F_i(t)\{\sum_{j=i+1}^{N+1}p_{ij}C_j-C_i\}+(a_i-g^*)\int_0^t \bar{F}_i(u)du,$$

we have from (4.20)

$$A_i(t)=B_i(t) \text{ for } t\in[0, +\infty),\ n\leqq i\leqq N.$$

On the other hand,

$$A_i(t)\leqq B_i(t) \text{ for } t\in[0, +\infty),\ 0\leqq i\leqq n-1 \tag{4.23}$$

holds from the following obvious inequalities:

$$v_i\leqq C_i-g^*\mu \text{ for } 0\leqq i\leqq N+1.$$

Since, from (4. 22) and (4. 23)

$$0 \leqq G_m(t) - C_m + g^* \mu = A_m(t) \leqq B_m(t) \text{ for all } t(\in[0, +\infty)),$$

it is sufficient to show that

$$B_m(t) \geqq 0 \text{ for all } t(\in[0, +\infty))$$

implies

$$B_i(t) \geqq 0 \text{ for all } t(\in[0, +\infty)) \text{ and } i(\geqq m).$$

For this means

$$G_n(t) - C_n + g^* \mu = A_n(t) = B_n(t) \geqq 0 \text{ for all } t(\in[0, +\infty)),$$

that is,

$$v_n = \inf_{0 \leqq t \leqq +\infty} G_n(t) \geqq C_n - g^* \mu,$$

and hence (4. 21) i. e.

$$v_n = C_n - g^* \mu$$

holds.

Now we have

$$\begin{aligned} \frac{d}{dt} B_i(t) &= f_i(t) \{ \sum_{j=i+1}^{N+1} p_{ij} C_j - C_i \} + (a_i - g^*) \bar{F}_i(t) \\ &= \bar{F}_i(t) b_i(t) \end{aligned}$$

where

$$\begin{aligned} b_i(t) :&= \{ \sum_{j=i+1}^{N+1} p_{ij} C_j - C_i \} r_i(t) + a_i - g^* \\ &= \{ \sum_{j=i+1}^{N+1} p_{ij} C_j \} r_i(t) + a_i - r_i(t) C_i - g^*. \end{aligned}$$

Since from (C-3)

$$\sum_{j=i+1}^{N+1} p_{ij} C_j - C_i \geqq 0$$

and (C-1) holds, for each i, $b_i(t)$ is non-decreasing in t. On the other hand, because (C-2) and (C-3) imply

$$\sum_{j=i+1}^{N+1} p_{ij} C_j (>0)$$

is non-decreasing in i, and (C-4) holds, we conclude that $b_i(t)$ is non-decreasing in i for an arbitrarily fixed $t(\in[0, +\infty))$. Moreover,

$$B_m(t) \geqq 0 \text{ for all } t(\in[0, +\infty))$$

implies

$$0 \leqq \frac{d}{dt} B_m(t) \,|_{t=0} = \bar{F}_m(0) b_m(0) = b_m(0).$$

Therefore, we obtain

$$\frac{d}{dt} B_i(t) = \bar{F}_i(t) b_i(t) \geqq \bar{F}_i(t) b_i(0) \geqq \bar{F}_i(t) b_m(0)$$
$$\geqq 0 \text{ for all } t(\in [0, +\infty)),$$

that is, $B_i(t)$ is non-decreasing in t for each $i(\geqq m)$. Thus,

$$0 = B_i(0) \leqq B_i(t) \text{ for all } t(\in [0, +\infty)) \text{ and } i(\geqq m).$$

The proof is completed. □

It is noted that

$$v_i = C_i - g^* \mu$$

implies that we can choose

$$t_i^* := 0.$$

Hence, THEOREM 4-1 states that a Control Limit Rule (CLR) holds for replacements under an optimal replacement policy. That is, under an optimal replacement policy, there exists a threshold state such that if the system state makes a transition into a more deteriorated state than that state, then it is optimal to replace immediately at that time.

[LEMMA 4-4]

v_i is non-decreasing in i, that is,

$$(0=) v_0 \leqq v_1 \leqq \cdots \leqq v_N \leqq v_{N+1} \tag{4.24}$$

holds.

(PROOF)

We prove by mathematical induction on i that

$$v_i \leqq v_{i+1} \text{ for } i = N, N-1, \ldots, 1, 0.$$

First, for $i=N$,

$$v_N \leqq C_N - g^* \mu \leqq C_{N+1} - g^* \mu = v_{N+1}$$

holds. Now. assuming that

$$v_{k+1} \leqq \cdots \leqq v_N \leqq v_{N+1}$$

holds for some $k(0\leqq k\leqq N)$, we prove

$$v_k\leqq v_{k+1}.$$

We consider the following two cases.

i) Case 1. $v_k=G_k(0)=C_k-g^*\mu$:

In this case, from THEOREM 4-1, we obtain that

$$v_i=C_i-g^*\mu \text{ for all } i(\geqq k),$$

and hence

$$v_k=C_k-g^*\mu\leqq C_{k+1}-g^*\mu=v_{k+1}$$

holds.

ii) Case 2. $v_k<C_k-g^*\mu$:

Let us define

$$\begin{aligned} h_i(t):&=a_i-r_i(t)C_i+\{\sum_{j=i+1}^{N+1}p_{ij}v_j+g^*\mu\}r_i(t)-g^* \\ &=\{\sum_{j=i+1}^{N+1}p_{ij}v_j-(C_i-g^*\mu)\}r_i(t)+a_i-g^*, \end{aligned}$$

then we have

$$G_i(t)=\int_0^t\bar{F}_i(u)h_i(u)du+C_i-g^*\mu.$$

Now we define, for each $i(0\leqq i\leqq N)$,

$$\begin{aligned} D_i^-:&=\{t\in[0,+\infty);h_i(t)<0\}, \\ D_i^+:&=\{t\in[0,+\infty);h_i(t)\geqq 0\}. \end{aligned}$$

It is noted that (C-1) implies that each $h_i(t)$ is a monotone function of t, and hence D_i^- and D_i^+ are respectively some intervals of the half line $[0,+\infty)$.

ii-1) When

$$\sum_{j=k+1}^{N+1}p_{kj}v_j-(C_k-g^*\mu)\geqq 0,$$

$h_k(t)$ is non-decreasing in t. Moreover, from (C-2) and the hypothesis of the induction, it follows that

$$0<C_k\leqq\sum_{j=k+1}^{N+1}p_{kj}v_j+g^*\mu\leqq\sum_{j=k+2}^{N+1}p_{k+1,j}v_j+g^*\mu.$$

Thus, we obtain from (*C*-1) and (*C*-4)

$$h_k(t)\leqq h_{k+1}(t) \text{ for all } t(\in[0,+\infty)). \tag{4.25}$$

Since $h_{k+1}(t)<0$ implies $h_k(t)<0$,

$$D_{k+1}^{-} \subseteq D_k^{-}$$

and

$$D_{k+1}^{+} \supseteq D_k^{+}$$

hold. Therefore, we conclude that

$$\begin{aligned} v_k &= \inf_{0 \leq t \leq +\infty} G_k(t) \\ &= \int_{D_k^-} \bar{F}_k(u) h_k(u) du + C_k - g^* \mu \\ &= \int_{D_{k+1}^-} \bar{F}_k(u) du + \int_{D_k^- / D_{k+1}^-} \bar{F}_k(u) h_k(u) du + C_k - g^* \mu \\ &\leq \int_{D_{k+1}^-} \bar{F}_k(u) h_k(u) du + C_k - g^* \mu \\ &\leq \int_{D_{k+1}^-} \bar{F}_{k+1}(u) h_{k+1}(u) du + C_{k+1} - g^* \mu = v_{k+1} \end{aligned}$$

where the last inequality follows that as from (4. 25)

$$h_k(t) \leq h_{k+1}(t) < 0 \text{ for all } t(\in D_{k+1}^-)$$

holds and

$$0 \leq \bar{F}_{k+1}(t) \leq \bar{F}_k(t) \text{ for all } t(\in [0, +\infty))$$

follows LEMMA 4-2, we have

$$\bar{F}_k(t) h_k(t) \leq \bar{F}_{k+1}(t) h_{k+1}(t) \leq 0 \text{ for all } t(\in D_{k+1}^-).$$

ii-2) When

$$\sum_{j=k+1}^{N+1} p_{kj} v_j - (C_k - g^* \mu) < 0,$$

$h_k(t)$ is non-increasing in t. Moreover, the assumption

$$v_k < C_k - g^* \mu$$

implies

$$C_k - g^* \mu > v_k = G_k(+\infty) = (a_k - g^*) \lambda_k + \sum_{j=k+1}^{N+1} p_{kj} v_j.$$

ii-2-a) When

$$\sum_{j=k+2}^{N+1} p_{k+1,j} v_j - (C_{k+1} - g^* \mu) \geq 0,$$

$h_{k+1}(t)$ is non-decreasing in t and

$$\sum_{j=k+1}^{N+1} p_{kj} v_j + g^* \mu < C_k \leq C_{k+1} \leq \sum_{j=k+2}^{N+1} p_{k+2,j} v_j + g^* \mu$$

holds. Since

$$0 \leqq \sum_{j=k+1}^{N+1} p_{kj} v_j + g^* \mu$$

implies

$$0 \leqq \{ \sum_{j=k+1}^{N+1} p_{kj} v_j + g^* \mu \} r_k(0) \leqq \{ \sum_{j=k+2}^{N+1} p_{k+1,j} v_j + g^* \mu \} r_{k+1}(0),$$

and

$$\sum_{j=k+1}^{N+1} p_{kj} v_j + g^* \mu < 0 (< C_k)$$

implies

$$\{ \sum_{j=k+1}^{N+1} p_{kj} v_j + g^* \mu \} r_k(0) \leqq 0 \leqq \{ \sum_{j=k+2}^{N+1} p_{k+1,j} v_j + g^* \mu \} r_{k+1}(0),$$

in both cases, we have

$$h_k(s) \leqq h_k(0) \leqq h_{k+1}(0) \leqq h_{k+1}(t) \text{ for all } s,\ t (\in [0,\ +\infty)).$$

If $h_{k+1}(0) \geqq 0$ then $D_{k+1}^- = \phi$, and we obtain

$$v_{k+1} = G_{k+1}(0) = C_{k+1} - g^* \mu \geqq C_k - g^* \mu > v_k.$$

On the other hand, if $h_{k+1}(0) < 0$ then $D_{k+1}^- \neq \phi$, and we obtain

$$\begin{aligned} v_{k+1} &= \inf_{0 \leqq t \leqq +\infty} G_{k+1}(t) \\ &= \int_{D_{k+1}^-} \bar{F}_{k+1}(u) h_{k+1}(u) du + C_{k+1} - g^* \mu \\ &\geqq \int_{D_{k+1}^-} \bar{F}_k(u) h_k(u) du + C_k - g^* \mu \\ &\geqq \int_0^\infty \bar{F}_k(u) h_k(u) du + C_k - g^* \mu = G_{k+1}(+\infty) = v_k. \end{aligned}$$

ii-2-b) When

$$\sum_{j=k+2}^{N+1} p_{k+1,j} v_j - (C_{k+1} - g^* \mu) < 0,$$

$h_{k+1}(t)$ is non-increasing in t, and we have

$$v_{k+1} = \min \{ G_{k+1}(0),\ G_{k+1}(+\infty) \}.$$

If $v_{k+1} = G_{k+1}(0)$, then we obtain

$$v_k < C_k - g^* \mu \leqq C_{k+1} - g^* \mu = G_{k+1}(0) = v_{k+1}.$$

On the other hand, if $v_{k+1} = G_{k+1}(+\infty)$, then we obtain

$$\begin{aligned} v_k = G_k(+\infty) &= (a_k - g^*)\lambda_k + \sum_{j=k+1}^{N+1} p_{kj} v_j \\ &\leqq a_k\lambda_k - g^*\lambda_k + \sum_{j=k+2}^{N+1} p_{k+1,j} v_j \\ &\leqq a_{k+1}\lambda_{k+1} - g^*\lambda_{k+1} + \sum_{j=k+2}^{N+1} p_{k+1,j} v_j \\ &= G_{k+1}(+\infty) = v_{k+1}, \end{aligned}$$

where the first inequality holds from (C-2) and the hypothesis of the induction, the second inequality from (C-5) and LEMMA 4-2. The proof is completed. □

From these results, we obtain the following final theorem, which precisely describes the structural properties of an optimal replacement policy.

[THEOREM 4-2]

An optimal replacement policy

$$t^* = (t_0^*, t_1^*, \ldots, t_N^*, t_{N+1}^*)$$

has the following structural properties: there exist, in general, a state h and a state $k(0 \leqq h \leqq k \leqq N+1)$ such that

$$\begin{aligned} t_0^* = t_1^* &= \ldots = t_{h-1}^* = +\infty, \\ +\infty > t_h^* \geqq t_{h+1}^* \geqq &\ldots \geqq t_{k-1}^* > 0, \\ t_k^* = t_{k+1}^* &= \ldots = t_{N+1}^* = 0. \end{aligned} \tag{4.26}$$

(PROOF)

From THEOREM 4-1, there exists some $k(0 \leqq k \leqq N+1)$ such that

$$v_i = C_i - g^*\mu \text{ for } k \leqq i \leqq N+1,$$

and

$$v_i < C_i - g^*\mu \text{ for } 0 \leqq i < k.$$

Thus, we can choose

$$t_i^* := 0 \text{ for } k \leqq i \leqq N+1,$$

and

$$0 < t_i^* \leqq +\infty \text{ for } 0 \leqq i < k.$$

Therefore, we have only to consider the range $0 \leqq i < k$, that is, the i's satisfying $v_i < C_i - g^*\mu$. Since $D_i^- \neq \phi$ for $0 \leqq i < k$, we can choose as

$$t_i^* := \sup D_i^- \text{ for } 0 \leqq i < k.$$

Moreover,

$$v_i=G_i(t_i^*) \text{ for } 0\leqq i<k.$$

i) Case 1. $0<t_i^*<+\infty$:

If $t_{i-1}^*=+\infty$, then

$$+\infty=t_{i-1}^*>t_i^*,$$

and we have nothing to prove. Let us assume $0<t_{i-1}^*<+\infty$. Then, it necessarily follows that both $h_{i-1}(t)$ and $h_i(t)$ are non-decreasing in t. Moreover, (C-2) and LEMMA 4-4 imply

$$0<C_{i-1}\leqq \sum_{j=i}^{N+1} p_{i-1,j}v_j+g^*\mu\leqq \sum_{j=i+1}^{N+1} p_{ij}v_j+g^*\mu.$$

Hence, we obtain

$$h_{i-1}(t)\leqq h_i(t) \text{ for all } t(\in[0, +\infty)),$$

that is,

$$t_{i-1}^*\geqq t_i^*.$$

ii) Case 2. $t_i^*=+\infty$:

If $t_{i-1}^*=+\infty$, then

$$t_{i-1}^*=t_i^*=+\infty.$$

Assuming that $0<t_{i-1}^*<+\infty$, we conclude that $h_{i-1}(t)$ is non-decreasing in t. Moreover, from (C-2) and LEMMA 4-4, it follows that

$$0<C_{i-1}\leqq \sum_{j=i}^{N+1} p_{i-1,j}v_j+g^*\mu\leqq \sum_{j=i+1}^{N+1} p_{ij}v_j+g^*\mu,$$

and this implies

$$h_{i-1}(t)\leqq h_i(t) \text{ for all } t(\in[0, +\infty)). \tag{4.27}$$

On the other hand, if we choose some t satisfying

$$t_{i-1}^*<t<t_i^*=+\infty,$$

then

$$h_i(t)<0\leqq h_{i-1}(t),$$

and this contradicts (4.27). Thus, it necessarily holds that

$$t_{i-1}^*=+\infty. \quad \square$$

This theorem states that under an optimal replacement policy there exist, in general, a state h and a state $k(0\leqq h\leqq k\leqq N+1)$ such that

i) In states 0, 1, ..., $h-1$, the system is never replaced preventively, and is kept till the next transition time instant of the system state.

ii) In states h, $h+1$, $\dots$, $k-1$, the system has optimal finite and positive preventive replacement time lengths respectively. Moreover, these time lengths become shorter with the deterioration of the system state.

iii) If a transition into any state of k, $k+1$, $\dots$, $N+1$ occurs, the system is replaced immediately.

5. Concluding Remarks

In this paper, we have discussed an optimal replacement problem when the deterioration of the system state is expressed by a semi-Markov process. We showed that under some reasonable conditions reflecting practical situations, an optimal replacement policy has a monotone structure. That is, the preventive replacement time lengths become shorter as the system state deteriorates.

Though we assumed that the system state can be completely identified at any time instant, cases are not rare where we must consider the incompleteness of the state observation of the systems. Moreover, we assumed that the underlying probabilistic law of the deterioration of the system is completely known. However, it is more practical to assume that the probabilistic law includes certain unknown elements. Further studies should be done on the maintenance problems in the cases of incomplete state observation and/or lack of knowledge of the underlying probabilistic law.

References

1) Barlow, R. E. and Proschan, F., *Mathematical Theory of Reliability*, Wiley, New York, 1965

2) Derman, C., "On Optimal Replacement Rules when Changes of States are Markovian", in: *Mathematical Optimization Techniques* (Bellman, R. ed.), Chapter 9, pp. 201-210, University of California Press, Berkeley, California, 1963

3) Feldman, R. M., "Optimal Replacement with semi-Markov Shock Models", *Journal of Applied Probability*, Vol. 13, pp. 108-117, 1976

4) Feldman, R. M., "Optimal Replacement with semi-Markov Shock Models Using Discounted Costs", *Mathematics of Operations Research*, Vol. 2, pp. 78-90, 1977

5) Kao, E. P. C., "Optimal Replacement Rules when Changes of States are semi-Markovian", *Operations Research*, Vol. 21, pp. 1231-1249, 1973

6) Nguen, D. G. and Murthy, N. P., "Optimal Preventive Maintenance Policies for Repairable Systems", *Operations Research*, Vol. 29, pp. 1181-1194, 1981

7) Taylor, H. M., "Optimal Replacement under Additive Damage and Other Failure Models", *Naval Research Logistics Quarterly*, Vol. 22, pp. 1-18, 1975

8) Zuckerman, D., "Optimal Stopping in a semi-Markov Shock Model", *Journal of Applied Probability*, Vol. 15, pp. 629-634, 1978