Asymptotic Theory of Slightly Rarefied Gas Flow and Force on a Closed Body

By

Yoshio SONE* and Kazuo AOKI*

(Received March 31, 1987)

Abstract

Steady gas flows at small Knudsen numbers around arbitrary bodies (asymptotic behavior for small Knudsen numbers of the solution of time-independent boundary value problems of the Boltzmann equation over a general domain) are considered when the Reynolds number of the system is of the order of unity. The generalized slip flow theory developed for the Boltzmann-Krook-Welander equation is extended for the standard Boltzmann equation. From the result, the effect of gas rarefaction on the flow (the relation between Boltzmann and hydrodynamic systems) is clarified, and several features of the force on a closed body in the gas are derived.

I. Introduction

The relation between the hydrodynamic equation and the Boltzmann equation has been discussed by various authors.[1)-9)] In this connection the Hilbert and the Chapman-Enskog expansions are often mentioned. The expansions, however, are not derived in the framework of the boundary-value problem, and the hydrodynamic equations derived have some awkward properties in considering the boundary-value problem.[2),3)]

In this paper, taking the time-independent boundary value problem of the Boltzmann equation over a general domain, we investigate the asymptotic behavior of the solution for small Knudsen numbers to derive a set of hydrodynamic equations and their boundary conditions that covers some effects of the Knudsen number (gas rarefaction). From the result, the effect of gas rarefaction on velocity and temperature fields is discussed, and several features of the force acting on a closed body in a slightly rarefied gas are derived.

* Department of Aeronautical Engineering

II. Asymptotic Solution for Small Knudsen Numbers

II-1. Analysis and Hydrodynamic Systems

Mach number Ma, Reynolds number Re, and Knudsen number Kn, important parameters in characterizing slightly rarefied gas flows, are related as[3]

$$Ma \sim Re\ Kn.$$

This relation is important in considering the asymptotic analysis for small Knudsen numbers ($Kn \ll 1$). The linear theory[4),5)], where the quantities of $O(Ma^2)$ are neglected, is applicable only for very small Re ($Re \ll Kn$). The standard Hilbert expansion[1] corresponds to the case with $Re \to \infty$. When Re is of the order of unity (the case of our interest), we must take into account that Ma is of the same order of smallness as Kn. In the present paper, noting that Ma is a measure of deviation from an equilibrium state at rest, we investigate the asymptotic behavior for $Kn \ll 1$ of the system where the deviation from a uniform equilibrium state at rest is of the order of the Knudsen number of the system. Owing to limited space, we give only the outline of the analysis.

We introduce the notations: T_0, p_0, f_0, and l_0 are the temperature, the pressure, the velocity distribution of gas molecules, and the mean free path of our reference equilibrium state at rest; R is the (specific) gas constant; L is the characteristic length of our system; Lx_i is the rectangular space coordinates; $(2RT_0)^{1/2}\zeta_i$ is the molecular velocity; $f_0(1+\phi)$ is the velocity distribution of gas molecules.

The behavior of the gas ϕ is described by the Boltzmann equation:

$$\zeta_i \frac{\partial \phi}{\partial x_i} = \frac{1}{k}\left[L(\phi) + J(\phi, \phi)\right], \tag{1}$$

$$k = \frac{\sqrt{\pi}}{2}\frac{l_0}{L} = \frac{\sqrt{\pi}}{2} Kn, \tag{2}$$

where the standard collision integral $J(1+\phi, 1+\phi)$ is split into two parts: the linearized operator $L(\phi)$ and the remainder $J(\phi, \phi)$. The complete definition of the collision operators is not given here, but no confusion will take place.

On the boundary a condition for the reflected molecules is imposed[1),2)]:

$$\phi = \phi_w, \qquad (\zeta_i n_i > 0), \tag{3}$$

where n_i is the unit normal to the boundary, pointed to the gas, and ϕ_w is a given function or is related with ϕ ($\zeta_i n_i < 0$).

The asymptotic solution of the boundary-value problem is obtained in the form:

$$\phi=\phi_H+\phi_K, \tag{4}$$
$$\phi_H=\phi_{H1}k+\phi_{H2}k^2+\cdots, \tag{5}$$
$$\phi_K=\phi_{K1}k+\phi_{K2}k^2+\cdots, \tag{6}$$

where ϕ_H, called hydrodynamic part, represents the overall solution, and ϕ_K, Knudsen-layer part, the correction near the boundary. Since we are considering the case where the perturbed distribution ϕ is of the order of k, ϕ_{Hm} and ϕ_{Km} are of the order of unity.

First we determine ϕ_H as a solution of the Boltzmann equation whose length scale of variation is of the order of the characteristic length L of the system $[\partial\phi/\partial x_i=O(\phi)]$. Substituting Eq. (5) in the Boltzmann equation (1) and arranging the same order terms of k, we obtain a sequence of integral equations for ϕ_{Hm}:

$$L(\phi_{Hm}) = \text{Inhomogeneous term } (\phi_{Hm-1},\cdots,\phi_{H1}), \tag{7}$$

which can in principle be solved from the lowest order. From the solvability condition* of Eq. (7) with $m \geqq 2$, we get a sequence of partial differential equations, called hydrodynamic equations, that govern the component functions of the expansions corresponding to Eq. (5) of hydrodynamic quantities (velocity, temperature, etc.).

Since the hydrodynamic part ϕ_H, obtained without paying attention to the boundary condition, cannot in general be made to satisfy the boundary condition (3) [the differential operator is multiplied by the small parameter k in Eq. (1)], we introduce the Knudsen-layer correction ϕ_K, which is assumed to have the length scale of variation normal to the boundary of the order of l_0 $[kn_i\partial\phi/\partial x_i=O(\phi)]$ and to be appreciable only near the boundary. Substituting Eq. (6) with ϕ_H previously obtained in Eq. (1) and arranging the terms with the properties of ϕ_K and ϕ_H in mind, we obtain a sequence of (inhomogeneous) one-dimensional linearized Boltzmann equations.

* Homogeneous integral equation $L(\phi)=0$ has the five independent solutions 1, ζ_i, and ζ_i^2.

$$\zeta_i n_i \frac{\partial \phi_{K1}}{\partial \eta} = L(\phi_{K1}), \tag{8}$$

$$\zeta_i n_i \frac{\partial \phi_{K2}}{\partial \eta} = L(\phi_{K2}) + 2J((\phi_{H1})_0,\ \phi_{K1}) + J(\phi_{K1}, \phi_{K1}) - \zeta_i \left[\left(\frac{\partial s_1}{\partial x_i}\right)_0 \frac{\partial \phi_{K1}}{\partial s_1} + \left(\frac{\partial s_2}{\partial x_i}\right)_0 \frac{\partial \phi_{K1}}{\partial s_2}\right], \tag{9}$$

$\cdots$

$$x_i = n_i k\eta + x_{wi}\ (s_1,\ s_2), \tag{10}$$

where x_{wi} is the boundary surface, η is a stretched coordinate normal to the boundary, s_1 and s_2 are (unstretched) coordinates within a parallel surface $\eta=$ const., and $(\ )_0$ denotes that the quantity in $(\ )$ is evaluated at $\eta=0$. The boundary condition for ϕ_{Km} at $\eta=0$ is

$$\phi_{Km} = \phi_{wm} - \phi_{Hm}, \qquad (\zeta_i n_i > 0), \tag{11}$$

where ϕ_{wm} is defined by

$$\phi_w = \phi_{w1} k + \phi_{w2} k^2 + \cdots. \tag{12}$$

The boundary value of ϕ_{Hm}, which is undetermined, is involved in the boundary condition (11). The analysis of the equations under the condition that ϕ_K vanishes rapidly away from the boundary gives conditions among the boundary values of hydrodynamic parts of hydrodynamic quantities and their derivatives[6), 10)-12)] as well as the Knudsen-layer correction ϕ_K. These conditions serve as boundary conditions for the hydrodynamic equations and are hereafter called slip boundary condition for convenience.

Here we list the hydrodynamic equations. The non-dimensional hydrodynamic quantities u_i, p, τ, and ω are introduced: $(2RT_0)^{1/2} u_i$ is the gas velocity, $p_0(1+p)$ the pressure, $T_0(1+\tau)$ the temperature, $p_0(RT_0)^{-1}(1+\omega)$ the density. The hydrodynamic quantities are split into H and K parts and expanded in power series of k as in Eqs. (4), (5), and (6). [u_{iH}, p_H, τ_H, and ω_H are defined in ϕ_H by the same formulae as u_i etc. in ϕ (Appendix 2), and u_{iK} etc. are defined as the remainders.]

$$\frac{\partial p_{H1}}{\partial x_i} = 0, \tag{13}$$

$$\frac{\partial u_{iH1}}{\partial x_i} = 0, \tag{14 a}$$

$$u_{jH1}\frac{\partial u_{iH1}}{\partial x_j}=-\frac{1}{2}\frac{\partial p_{H2}}{\partial x_i}+\frac{1}{2}\gamma_1\frac{\partial^2 u_{iH1}}{\partial x_j^2}, \tag{14 b}$$

$$u_{jH1}\frac{\partial \tau_{H1}}{\partial x_j}=\frac{1}{2}\gamma_2\frac{\partial^2 \tau_{H1}}{\partial x_j^2}, \tag{14 c}$$

$$\frac{\partial u_{jH2}}{\partial x_j}=-u_{jH1}\frac{\partial \omega_{H1}}{\partial x_j}, \tag{15 a}$$

$$u_{jH1}\frac{\partial u_{iH2}}{\partial x_j}+(\omega_{H1}u_{jH1}+u_{jH2})\frac{\partial u_{iH1}}{\partial x_j}$$
$$=-\frac{1}{2}\frac{\partial}{\partial x_i}\left[p_{H3}-\frac{1}{6}(\gamma_1\gamma_2-4\gamma_3)\frac{\partial^2 \tau_{H1}}{\partial x_j^2}\right]+\frac{1}{2}\gamma_1\frac{\partial^2 u_{iH2}}{\partial x_j^2}$$
$$+\frac{1}{2}\gamma_4\frac{\partial}{\partial x_j}\left[\tau_{H1}\left(\frac{\partial u_{iH1}}{\partial x_j}+\frac{\partial u_{jH1}}{\partial x_i}\right)\right], \tag{15 b}$$

$$u_{jH1}\frac{\partial \tau_{H2}}{\partial x_j}+(\omega_{H1}u_{jH1}+u_{jH2})\frac{\partial \tau_{H1}}{\partial x_j}-\frac{2}{5}u_{jH1}\frac{\partial p_{H2}}{\partial x_j}$$
$$=\frac{1}{5}\gamma_1\left(\frac{\partial u_{iH1}}{\partial x_j}+\frac{\partial u_{jH1}}{\partial x_i}\right)^2+\frac{1}{2}\frac{\partial^2}{\partial x_j^2}\left(\gamma_2\tau_{H2}+\frac{1}{2}\gamma_5\tau_{H1}^2\right), \tag{15 c}$$

. . . . . .

$$p_{H1}=\omega_{H1}+\tau_{H1},\qquad p_{H2}=\omega_{H2}+\tau_{H2}+\omega_{H1}\tau_{H1},\cdots\cdots \tag{16}$$

where γ_i are numerical constants related with the collision operators L and J (App. 1).

Equations for (u_{iH1}, τ_{H1}, p_{H2}) [Eqs. (14 a~c)] are the Navier-Stokes equations for an incompressible fluid, and the successive equations for (u_{iHm}, τ_{Hm}, p_{Hm+1}, $m\geqq 2$) are the same order differential equations as Eqs. (14 a~c). These sets of equations are derived by a systematic small parameter (k) expansion, where no assumption is made on the form of the velocity distribution function but special attention is paid to the estimate of physical variables so that the analysis may cover physically interesting cases with finite Reynolds numbers. Incidentally, in the standard Hilbert expansion, sets of the first-order differential equations, starting with the Euler equations for an ideal gas, are derived; in the Chapman-Enskog expansion, the order of the differential equations, starting also with the Euler equations, increases with the progress of approximation.

The slip boundary conditions for the hydrodynamic equations (13 ~ 15 c) take the same form as those for the Boltzmann-Krook-Welander equation except for numerical constants. The latter results are given in Ref. 5 for solid boundary where neither evaporation nor condensation takes place and in Ref. 13 for interface between gas and its condensed phase where evaporation or condensation is taking place. (For brevity, the former boundary is hereafter called solid

boundary and the latter interface.) The slip boundary conditions are as follows:
(i) On the solid boundary

$$u_{iH1}-u_{wi1}=0, \tag{17 a}$$
$$\tau_{H1}-\tau_{w1}=0, \tag{17 b}$$

$$(u_{iH2}-u_{wi2})t_i=-k_0\left(\frac{\partial u_{iH1}}{\partial x_j}+\frac{\partial u_{jH1}}{\partial x_i}\right)n_i t_j-K_1\frac{\partial \tau_{H1}}{\partial x_i}t_i, \tag{18 a}$$

$$u_{iH2}n_i=0, \tag{18 b}$$

$$\tau_{H2}-\tau_{w2}=d_1\frac{\partial \tau_{H1}}{\partial x_i}n_i, \tag{18 c}$$

(ii) On the interface

$$(u_{iH1}-u_{wi1})t_i=0, \tag{19 a}$$

$$\begin{bmatrix} p_{H1}-p_{w1} \\ \tau_{H1}-\tau_{w1} \end{bmatrix}=u_{iH1}n_i\begin{bmatrix} C_4^* \\ d_4^* \end{bmatrix}, \tag{19 b, 19 c}$$

$$(u_{iH2}-u_{wi2})t_i=-k_0\left(\frac{\partial u_{iH1}}{\partial x_j}+\frac{\partial u_{jH1}}{\partial x_i}\right)n_i t_j$$
$$-K_1\frac{\partial \tau_{H1}}{\partial x_i}t_i+K_2 t_j\frac{\partial}{\partial x_j}(u_{iH1}n_i), \tag{20 a}$$

$$\begin{bmatrix} p_{H2}-p_{w2} \\ \tau_{H2}-\tau_{w2} \end{bmatrix}=u_{iH2}n_i\begin{bmatrix} C_4^* \\ d_4^* \end{bmatrix}+n_i\frac{\partial \tau_{H1}}{\partial x_i}\begin{bmatrix} C_1 \\ d_1 \end{bmatrix}$$
$$+\left(\frac{\partial u_{iH1}}{\partial x_j}+\frac{\partial u_{jH1}}{\partial x_i}\right)n_i n_j\begin{bmatrix} C_6 \\ d_6 \end{bmatrix}-2\bar{\kappa}u_{iH1}n_i\begin{bmatrix} C_7 \\ d_7 \end{bmatrix}$$
$$+(u_{iH1}n_i)^2\begin{bmatrix} C_8 \\ d_8 \end{bmatrix}+\tau_{w1}u_{iH1}n_i\begin{bmatrix} C_9 \\ d_9 \end{bmatrix}+p_{w1}u_{iH1}n_i\begin{bmatrix} C_{10} \\ d_{10} \end{bmatrix}, \tag{20 b, 20 c}$$

where t_i is the direction cosine of a tangential vector to the boundary; $\bar{\kappa}/L$ is the mean curvature of the boundary where the sign of each principal curvature is taken negative when the corresponding center of curvature is on the gas side; u_{wim}, τ_{wm}, and p_{wm} are the terms of the expansions of the velocity $(2RT_0)^{1/2}u_{wi}$ (with $u_{wi}n_i=0$), the temperature $T_0(1+\tau_w)$ of the boundary, and the saturation gas pressure $p_0(1+p_w)$ at temperature $T_0(1+\tau_w)$:

$$u_{wi}=u_{wi1}k+u_{wi2}k^2+\cdots, \tag{21 a}$$
$$\tau_w=\tau_{w1}k+\tau_{w2}k^2+\cdots, \tag{21 b}$$
$$p_w=p_{w1}k+p_{w2}k^2+\cdots, \tag{21 c}$$

[u_{wi}, τ_w, and p_w correspond to the deviation from our reference equilibrium state and thus are of the order of k. The higher order terms of k are retained for the convenience of treating the problems where the boundary values are not known beforehand.]; k_0, K_1, d_1, K_2, C_1, C_4^*, C_6, C_7, C_8, C_9, C_{10}, d_4^*, d_6, d_7, d_8, d_9, d_{10} are numerical constants. For *B-K-W* equation,

$$
\begin{array}{lll}
C_1 = 0.558437, & C_4^* = -2.132039, & C_6 = 0.820853, \\
C_7 = -0.380569, & C_8 = 2.320074, & C_9 = 1.066019, \\
C_{10} = C_4^*, & d_1 = 1.302716, & d_4^* = -0.446749, \\
d_6 = 0.330345, & d_7 = -0.131574, & d_8 = -0.0028315, \\
d_9 = -0.223375, & d_{10} = 0, & k_0 = -1.016191, \\
K_1 = -0.383161, & K_2 = -0.795186. &
\end{array}
$$

Finally, we list the hydrodynamic parts of the stress tensor[1),2)] $p_0(\delta_{ij}+P_{ij})$ and heat flow vector[1),2)] $p_0(2RT_0)^{1/2}Q_i$ (App. 2). The component functions of their expansions corresponding to Eq. (5) are:

$$
\left.
\begin{aligned}
&P_{ijH1}=p_{H1}\delta_{ij}, \qquad P_{ijH2}=p_{H2}\delta_{ij}-\gamma_1\left(\frac{\partial u_{iH1}}{\partial x_j}+\frac{\partial u_{jH1}}{\partial x_i}\right), \\
&P_{ijH3}=p_{H3}\delta_{ij}-\gamma_1\left(\frac{\partial u_{iH2}}{\partial x_j}+\frac{\partial u_{jH2}}{\partial x_i}-\frac{2}{3}\frac{\partial u_{kH2}}{\partial x_k}\delta_{ij}\right) \\
&\qquad -\gamma_4\tau_{H1}\left(\frac{\partial u_{iH1}}{\partial x_j}+\frac{\partial u_{jH1}}{\partial x_i}\right)+\gamma_3\left(\frac{\partial^2\tau_{H1}}{\partial x_i\partial x_j}-\frac{1}{3}\frac{\partial^2\tau_{H1}}{\partial x_k^2}\delta_{ij}\right),
\end{aligned}
\right\} \quad (22)
$$

$$
\left.
\begin{aligned}
&Q_{iH1}=0, \qquad Q_{iH2}=-\frac{5}{4}\gamma_2\frac{\partial\tau_{H1}}{\partial x_i}, \\
&Q_{iH3}=-\frac{5}{4}\gamma_2\frac{\partial\tau_{H2}}{\partial x_i}-\frac{5}{4}\gamma_5\tau_{H1}\frac{\partial\tau_{H1}}{\partial x_i}+\frac{1}{2}\gamma_3\frac{\partial^2 u_{iH1}}{\partial x_j^2}.
\end{aligned}
\right\} \quad (23)
$$

The last term of P_{ijH3} (Q_{iH3}) is non Navier-Stokes stress (heat flow) and is called thermal stress. The term before the last in P_{ijH3} (Q_{iH3}) shows the temperature dependence of viscosity (thermal conductivity).

The results of this subsection (Sec. II-1) are the generalization of the senior author's work (Ref. 5) developed for B-K-W equation.

II-2. Velocity and Temperature Fields

The first order hydrodynamic equations [Eqs. (14a~c)] are the Navier-Stokes equations for an incompressible fluid. The second order equations [Eqs. (15a~c)] combined with Eqs. (14a~c) differ a little from the Navier-Stokes equations of a slightly compressible gas. If γ_3 in the numerical coefficient of

$\partial^2\tau_{H1}/\partial x_j^2$ in the square brackets of the first term on the right hand side of Eq. (15 b) is zero, Eqs. (15 a~c) coincide with the second order equations of the Mach number expansion of the Navier-Stokes equations for a compressible gas. [Noting that the case $Ma=ak$ with $a=O(1)$ is under consideration, transform the k-expansion to Ma-exp.] The difference is due to the thermal stress in P_{ijH3}.

This difference, however, can be eliminated by the replacement:

$$p_{H3}^*=p_{H3}+\frac{2}{3}\gamma_3\frac{\partial^2\tau_{H1}}{\partial x_j^2}. \tag{24}$$

Further, the slip boundary condition (up to the second order of k) does not contain p_{H3} [cf. Eqs. (17 a) ~ (18 c), (19 a) ~ (20 c)]. Thus, we conclude:

Proposition 1 : *Except for the Knudsen-layer correction, the velocity and the temperature fields of a slightly rarefied gas can be calculated correctly up to the second order in the Knudsen number by the slightly compressible Navier-Stokes equations with the slip boundary conditions. The effect of gas rarefaction comes in through the boundary condition.*

(N. B. In an infinite-domain problem where the pressure is specified at infinity, the pressure modified by Eq. (24) should be used. In most physical problems, however, $\partial^2\tau_{H1}/\partial x_j^2$ vanishes at infinity and no correction is necessary.)

III. Force and Its Moment on a Closed Body

Take a closed body B_1 in a gas. The gas may or may not be bounded, and other bodies may lie in the gas. We will investigate the force and its moment on B_1. In the following analysis, ∂B_1 denotes the boundary of B_1; ∂B_0 a closed surface that encloses only B_1 in the gas; n_i the unit normal of the surface of integration under consideration pointed to the region including infinity; dS its surface element.

Theorem 1 : *The Knudsen-layer part of the momentum flux does not contribute to the force acting on a closed body.*

Proof: Let $p_0(\delta_{ij}+\Psi_{ij})$ be momentum flux tensor, where $\Psi_{ij}=P_{ij}+2(1+\omega)u_iu_j$, and F_i be the force, normalized by $-p_0L^2$, acting on a closed body B_1 in the gas. Then,

$$F_i=\int_{\partial B_1}\Psi_{ij}n_j\,dS. \tag{25}$$

Because $\partial\Psi_{ij}/\partial x_j=0$ in the gas (App. 3), the surface of integration can be deformed arbitrarily in the gas. Taking a surface of integration ∂B_0 outside the

Knudsen layer, we have

$$F_i = \int_{\partial B_0} \Psi_{ijH} n_j dS, \tag{26}$$

since Ψ_{ijK} vanishes there. Further, because $\partial \Psi_{ijH}/\partial x_j = 0$ (App. 3), we can deform ∂B_0 in Eq. (26) arbitralily in the gas. ∂B_0 may be in the Knudsen layer, especially on the body ∂B_1. (QED)

Theorem 2 : The Knudsen-layer part of the momentum flux does not contribute to the moment of force acting on a closed body.

Proof: The moment of force M_i around origin, normalized by $p_0 L^3$, is expressed by

$$M_i = \int_{\partial B_1} \varepsilon_{ihk} x_h \Psi_{kj} n_j dS, \tag{27}$$

where ε_{ijk} is Eddington's ε. The proof goes parallel to that of Theorem 1 if Ψ_{ij} is replaced by $\varepsilon_{ihk} x_h \Psi_{kj}$ because $\dfrac{\partial}{\partial x_j} \varepsilon_{ihk} x_h \Psi_{kj} = 0$ from $\partial \Psi_{ij}/\partial x_j = 0$ and $\Psi_{ij} = \Psi_{ji}$.

(*QED*)

Corollary: On a solid boundary, F_i and M_i are calculated correctly up to the k^3-order only by P_{ijH} on ∂B_1.

Proof: From the Knudsen-layer analysis, $u_{iH1} n_i = u_{iH2} n_i = 0$ on a solid boundary [cf. Eqs. (17 a) and (18 b)]. (Incidentally, $u_{iH3} n_i$ is not necessarily zero.) (QED)

As in Theorem 1, we can prove the following theorem with the aid of the formulae in Appendix 3. (Proof omitted)

Theorem 3 : The Knudsen-layer part of mass (energy) flux does not contribute to the mass (energy) flow to a closed body.

We prepare a lemma for Theorem 4 :

Lemma: Let $f(x_i)$ be a function three times continuously differentiable in a domain containing a closed surface (say ∂B_0). Then

$$\int_{\partial B_0} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_k^2} \delta_{ij} \right) n_j dS = 0, \tag{28 a}$$

$$\int_{\partial B_0} \varepsilon_{ihk} x_h \left(\frac{\partial^2 f}{\partial x_k \partial x_j} - \frac{\partial^2 f}{\partial x_m^2} \delta_{kj} \right) n_j dS = 0. \tag{28 b}$$

Proof: Extend $f(x_i)$ over the whole region inside ∂B_0 keeping its smoothness and apply Gauss theorem. (QED)

Theorem 4 : The non Navier-Stokes stress in p_{H3}^* system contributes neither to the force nor to the moment of force on a closed body.

The non N-S stress in p^*_{H3} syst. means the thermal stress in P_{ijH3} modified by the replacement (24).

Proof: Its contributions to F_i and M_i are, respectively, proportional to:

$$\int_{\partial B_1}\left(\frac{\partial^2\tau_{H1}}{\partial x_i\partial x_j}-\frac{\partial^2\tau_{H1}}{\partial x_k^2}\delta_{ij}\right)n_j dS, \tag{29 a}$$

and

$$\int_{\partial B_1}\varepsilon_{ihk}x_h\left(\frac{\partial^2\tau_{H1}}{\partial x_k\partial x_j}-\frac{\partial^2\tau_{H1}}{\partial x_m^2}\delta_{kj}\right)n_j dS. \tag{29 b}$$

After deforming ∂B_1 to ∂B_0, apply the lemma. (QED)

Combining Theorems 1, 2, 4 with Proposition 1 we find:

Proposition 2: Under the condition of Proposition 1, the force and the moment of force on a closed body can be computed correctly up to the k^3-order of F_i and M_i by the classical hydrodynamic procedure based on the Navier-Stokes solution and the N-S stress if the slip boundary condition is taken into account.

The results of this section are the generalization of the senior author's work (Ref. 14) developed for the linearized Boltzmann equation.

Appendix

1. Numerical constants γ_i

Let $A(\zeta^2)$ and $B(\zeta^2)$ be the solutions of the integral equations:

$$L[\zeta_i A(\zeta^2)]=-\zeta_i(\zeta^2-\frac{5}{2}),$$

$$L[(\zeta_i\zeta_j-\frac{1}{3}\zeta^2\delta_{ij})B(\zeta^2)]=-2(\zeta_i\zeta_j-\frac{1}{3}\zeta^2\delta_{ij}),$$

with the subsidiary condition:

$$\int_0^\infty \zeta^4 A(\zeta^2)\exp(-\zeta^2)\,d\zeta=0,$$

where $L[\cdots]$ is the linearized collision operator [cf. Eq. (1)] and $\zeta^2=\zeta_i^2$. $C(\zeta^2)$, $D(\zeta^2)$, and $G(\zeta^2)$ are introduced by

$$2J[\zeta^2-\frac{3}{2},\ \zeta_i\zeta_j B(\zeta^2)]=\zeta_i\zeta_j C(\zeta^2)+D(\zeta^2)\delta_{ij},$$

$$2J\left[\zeta^2-\frac{3}{2},\ \zeta_i A(\zeta^2)\right]=\zeta_i G(\zeta^2).$$

The γ_i are defined by the integrals of these functions:

$$\gamma_1=I_6(B),\qquad \gamma_2=2I_6(A),\qquad \gamma_3=I_6(AB),$$

$$\gamma_4=-\frac{5}{2}\gamma_1+I_8(B)+\frac{1}{2}I_6(BC),\qquad \gamma_5=-6\gamma_2+2I_8(A)+2I_4(AG),$$

where

$$I_n(F)=\frac{8}{15\sqrt{\pi}}\int_0^\infty \zeta^n F(\zeta^2)\exp(-\zeta^2)\,d\zeta,$$

with $F=A, B$, etc. For B–K–W equation $\gamma_i=1$, and for the hard sphere model γ_i are[15]:

$$\gamma_1=1.2700,\qquad \gamma_2=1.9223,\qquad \gamma_3=1.9479,\qquad \gamma_4=0.63489,\qquad \gamma_5=0.96070.$$

2. Relations between u_i, τ, etc. and ϕ

$$\omega=\int\phi E d\boldsymbol{\zeta},\qquad (1+\omega)u_i=\int\zeta_i\phi E d\boldsymbol{\zeta},$$

$$\frac{3}{2}(1+\omega)\tau=\int\left(\zeta_i^2-\frac{3}{2}\right)\phi E d\boldsymbol{\zeta}-(1+\omega)u_i^2,\qquad p=\omega+\tau+\omega\tau,$$

$$P_{ij}=2\int\zeta_i\zeta_j\phi E d\boldsymbol{\zeta}-2(1+\omega)u_i u_j,$$

$$Q_i=\int\zeta_i\zeta_j^2\phi E d\boldsymbol{\zeta}-\frac{5}{2}u_i-u_jP_{ij}-\frac{3}{2}pu_i-(1+\omega)u_iu_j^2,$$

where

$$E=\pi^{-3/2}\exp(-\zeta_i^2),\qquad d\boldsymbol{\zeta}=d\zeta_1 d\zeta_2 d\zeta_3,$$

and the integration is carried out over the whole space of ζ_i.

3. Conservation equations

Multiplying Eq. (1) by E, $\zeta_i E$, or $\zeta_i^2 E$ and integrating over the whole space of ζ_i, we have

$$\frac{\partial}{\partial x_j}\left[(1+\omega)u_j\right]=0,\qquad \frac{\partial}{\partial x_j}\left[2(1+\omega)u_iu_j+P_{ij}\right]=0,$$

$$\frac{\partial}{\partial x_j}\left[\frac{5}{2}u_j+u_iP_{ji}+\frac{3}{2}pu_j+(1+\omega)u_ju_i^2+Q_j\right]=0.$$

These relations also hold with subscript H since hydrodynamic part is a solution of Eq. (1).

References

1) H. Grad, in "Handbuch der Physik," edited by S. Flügge, Springer-Verlag, Berlin (1958), Vol. 12, p. 205.
2) C. Cercignani, "Theory and Application of the Boltzmann Equation," Scottish Academic, Edinburgh (1975).
3) Y. Sone, in "Rarefied Gas Dynamics," edited by H. Oguchi, University of Tokyo Press, Tokyo (1984), Vol. 1, p. 71.
4) Y. Sone, in "Rarefied Gas Dynamics," edited by L. Trilling and H. Y. Wachman, Academic, New York (1969), Vol. 1, p. 243.
5) Y. Sone, in "Rarefied Gas Dynamics," edited by D. Dini, Editrice Tecnico Scientifica, Pisa (1971), Vol. 2, p. 737.
6) H. Grad, in "Transport Theory," edited by R. Bellman, G. Birkhoff, and I. Abu-Shumays, American Mathematical Society, Providence (1969), p. 269.
7) H. Grad, Phys. Fluids, **6**, 147 (1963).
8) C. Cercignani, Phys. Fluids, **11**, 303 (1968).
9) J. S. Darrozes, in "Rarefied Gas Dynamics," edited by L. Trilling and H. Y. Wachman, Academic, New York (1969), Vol. 1, p. 111.
10) W. Greenberg and C. V. M. van der Mee, Z. Angew. Math. Phys., **35**, 156 (1984).
11) C. Bardos, R. E. Caflisch, and B. Nicolaenko, Comm. Pure Appl. Math., **39**, 323 (1986).
12) C. Cercignani, in "Trends in Applications of Pure Mathematics to Mechanics," edited by E. Kröner and K. Kirchgässner, Springer-Verlag, Berlin (1986), p. 35.
13) Y. Onishi and Y. Sone, J. Phys. Soc. Japan, **47**, 1676 (1979).
14) Y. Sone, in "Rarefied gas Dynamics," edited by H. Oguchi, University of Tokyo Press, Tokyo (1984), Vol. 1, p. 117.
15) T. Ohwada, private communication.