Some Classes of Convergent Interval Matrices

By

Takehiro MORI* and Hideki KOKAME**

(Received June 19, 1987)

Abstract

Some classes of interval matrices for which a necessary and sufficient condition can be obtained in a simple form are indicated. A few sufficient conditions are also derived to assure convergence of interval matrices. Some of these results are discussed in association with their polynomial counterparts.

1. Introduction

In analyzing or implementing a physical system with perturbed parameters, it may be assumed that the upper and lower bounds for the parameters can be somehow estimated. A matrix(polynomial) is called an interval matrix(interval polynomial), if these two-sided bounds are specified for each element (coefficient) of the matrix (polynomial). In relation to robust stability, in these days considerable attention is being given to the stability property of interval matrices or interval polynomials.[1-9] However, the stability property here means mainly the Hurwitz property for continuous systems, and the results for discrete systems seem to be few.

In this paper, we will indicate several classes of convergent interval matrices and also give a few sufficient conditions assuring the convergence of interval matrices. Some of these results have corresponding interval polynomial counterparts. To begin with, the symbol conventions used throughout the paper are summarized below. Let (′), I be the transpose and the unit matrix, respectively. Interval matrices are defined as a set form as follows:

$$M \triangleq \{A \in R^{n\times n} : C \leq_e A \leq_e B,\ B,\ C \in R^{n\times n}\} \qquad (1)$$

where the inequality ($\leq_e$) between matrices is meant to hold elementwise, namely,

* Automation Research Laboratory, Kyoto University, Gokasho, Uji, Kyoto 611, Japan
** Osaka Institute of Technology, Ohmiya, Asahi-ku, Osaka 535, Japan

$$M \triangleq \{A = \{a_{ij}\} \in R^{n\times n}: c_{ij} \leq a_{ij} \leq b_{ij}, i, j = 1, \cdots, n\} \tag{2}$$

where $B = \{b_{ij}\}$ and $C = \{c_{ij}\}$. The usual inequality between two symmetric matrices X and Y, $X \geq Y$, implies $X - Y \geq 0$ (positive semidefinite). Strict inequalities are understood to be defined in the same way. For a matrix $X \in R^{n\times n}$, $\lambda_i(X)$ denote the eigenvalues of X and $\lambda_{\max}(X)$ the maximum of these when they are real. $\sigma(X)$ is the spectral norm of X defined by $\sigma(X) \triangleq \{\lambda_{\max}(X'X)\}^{1/2}$. $|X|$ is used to denote the matrix X where each element is replaced by its absolute value. For a nonnegative matrix X, i.e., $X \geq_e 0$, $p(X)$ denotes the Perron root of X.

2. Main Results

It is pointed out in 8) that for convergence of any matrix in M, the same property of $2^m (m \triangleq n^2)$ matrices corresponding to all the combinations of the end points of each interval is not sufficient. Then, a question arises: What condition is sufficient for the convergence property of interval matrices? In answer to this question partly, we will present some sufficient conditions and a few classes of interval matrices whose exact covergence condition can be obtained in a simple form.

Let us first define a non-negative matrix D in the following way.

$$D \triangleq \{\{d_{ij}\} \in R^{n\times n}: d_{ij} = \max(|c_{ij}|, |b_{ij}|), i, j = 1, \cdots, n\} \tag{3}$$

Then we have:

[Theorem 1]

If D is a convergent matrix, so is any matrix in M.

This theorem is a direct consequence of Lemma 1 below.

[Lemma 1][11)]

Let $F \in R^{n\times n}$ be an irreducible non-negative matrix and assume that

$$|X| \leq_e F \tag{4}$$

holds for some matrix $X \in R^{n\times n}$. Then we have

$$|\lambda_i(X)| \leq p(F), \qquad i = 1, \cdots, n. \tag{5}$$

Note that the assumption on the irreducibility of the above lemma can be substantially dispensed with by considering simultaneous appropriate permutation of the corresponding rows and columns in X and F. Theorem 1 would be apparent from the facts that the Perron root is the largest eigenvalue in absolute value and that $|A| \leq_e D$ is satisfied for any A in M.

Lemma 1 also gives some classes of interval matrices whose necessary and sufficient condition for the convergence is extremely simple.

[Theorem 2]

If $B \geq_e |C| \geq_e 0 (C \leq_e -|B| \leq_e 0)$ are satisfied in (1), the necessary and sufficient condition for the convergence of interval matrices is that the matrix $B(C)$ is convergent.

A special case of Theorem 2 is summarized in a corollary.

[Corollary 1]

If $c_{ij} = -b_{ij} \leq 0$, $i, j = 1, \cdots, n$ in (2), the necessary and sufficient condition for the convergence of interval matrices is that the non-negative matrix B is convergent, i.e., $p(B) < 1$.

We consider now the polynomial counterpart of the above results. Consider an interval plynomial given by

$$f(s) \triangleq s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \tag{6}$$

where

$$c_i \leq a_i \leq b_i, \qquad i = 1, \cdots, n. \tag{7}$$

Polynomial (6) can be linked with matrices through its companion form,

$$\begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \cdot & \cdot & 0 \cdots & 0 \\ 0 & 0 & 0 \cdots & 1 \\ -a_n & -a_{n-1} & \cdots\cdots & -a_1 \end{pmatrix} \tag{8}$$

and polynomial (6) is convergent if and only if the matrix (8) is convergent. Therefore we can immediately obtain the polynomial counterparts of Theorems 1, 2 and Corollary 1.

[Theorem 3]

Let d_i be given by $d_i \triangleq \max(|c_i|, |b_i|)$. Then the interval polynomials (6) and (7) are convergent, if polynomial (6) with $a_i = -d_i$ is convergent.

[Theorem 4]

If $c_i \leq -|b_i| \leq 0 (0 \leq |c_i| \leq b_i)$, $i = 1, \cdots, n$ in (7), the necessary and sufficient condition (sufficient condition) for the convergence of interval polynomials (6) and (7) is that the polynomial (6) with $a_i = c_i (a_i = -b_i)$ is convergent.

[Corollary 2]

If $c_i = -b_i \leq 0$, $i = 1, \cdots, n$ in (7), the convergence property of the interval matrices (6) and (7) is equivalent to that of (6) with $a_i = c_i$.

This last result has been already derived via another approach[7] and it has been shown that the condition for that is simply given by

$$\sum_{i=1}^{n} b_i < 1 \,.$$

Also, the assumption of Theorem 3 can be expressed succinctly by the above inequality where b_i is replaced by d_i. Note that when $|c_i| \leq b_i$ we can only obtain a sufficient condition in Theorem 4. This shows asymmetry in the polynomial case in contrast to the matrix case with respect to the sign of the coefficients or the elements. This is caused by 1's appearing in every row of (8) except the n-th.

We now turn back to interval matrices. Theorem 1 is very concise and useful for checking the convergence of interval matrices. However, one of its drawbacks is that it abandons information on the sign of the elements. We will devise a method which covers this point and makes the most of the information given to the matrices.

Let us define matrices N and E_1 by

$$N \triangleq (B+C)/2 \tag{9}$$

and

$$E_1 \triangleq (B-C)/2 \,. \tag{10}$$

It is obvious that M can be restated as

$$M \triangleq \{A\colon N-E_1 \leq_e A \leq_e N+E_1\} \tag{11}$$

and that A can be written as

$$A = N+E \tag{12}$$

where

$$|E| \leq_e E_1 \,. \tag{13}$$

Since the convergence of N is necessary for that of interval matrices, we can assume that the Lyapunov matrix equation,

$$P-N'PN = I/2 \,, \tag{14}$$

has a positive definite solution $P=P'$. We are now in position to state a sufficient condition for the convergence of interval matrices in terms of P and E_1.

[Theorem 5]

If the following inequality,

$$\sigma(P)+2\sigma(E_1'|P|E_1)>1 \,, \quad \text{holds,} \tag{15}$$

then any matrix in M is convergent.
Proof: It is enough to show that

$$P-(N+E)'P(N+E)>0 . \tag{16}$$

To do this, we make use of the relation,

$$(X+Y)'L(X+Y)\leq 2X'LX+2Y'LY , \tag{17}$$

which is valid for any X, Y and $L=L'\geq 0$. This comes from the fact that subtracting the left hand side from the right yields

$$(X-Y)'L(X-Y)\geq 0 .$$

By (14) and (17), we get

$$P-(N+E)'P(N+E)\geq P-2N'PN-2E'PE = I-P-2E'PE . \tag{18}$$

The necessary and sufficient condition for the right hand side of (18) to be positive definite is that

$$\sigma(P+2E'PE)>1 . \tag{19}$$

A sufficient condition for the above is

$$\sigma(P)+2\sigma(E'PE)<1 . \tag{20}$$

Using in (20) the following inequality which is due to Lemma 1,

$$\sigma(E'PE)\leq p(E_1'|P|E_1) = \sigma(E_1'|P|E_1) , \tag{21}$$

we arrive at the conclusion. This completes the proof. Q.E.D.

In some cases, we can get another sufficient condition without resorting to the Lyapunov matrix equation.
[Theorem 6]
If

$$\sigma(E_1'E_1)<1/2-\sigma^2(N) \tag{22}$$

is satisfied, then the interval matrices are convergent.
Proof: For the Lyapunov equation (14), the following bound has been known[10].

$$\sigma(P)\leq\frac{1}{2(1-\sigma^2(N))} , \tag{23}$$

if $\sigma(N)<1$.

We note that a sufficient condition for (20) is that

$$\sigma(P)\{1+2\sigma(E'E)\}<1\,. \tag{24}$$

Using $\sigma(E'E)\leq p(E_1'E_1)=\sigma(E_1'E_1)$ and (23) in (24) leads to the result. Q.E.D.

Theorems 5 and 6 assert that for the convergence of interval matrices the size of the perturbation should be sufficiently small in comparison with the stability margin of the matrix N.

3. Example

Let us consider the 2nd order interval matrix (1) where

$$C=\begin{bmatrix} 0 & 0 \\ -1/4 & 1/4 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 0 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}. \tag{25}$$

In this case we have

$$N=\begin{bmatrix} 0 & 1/4 \\ 0 & 1/2 \end{bmatrix},\; E_1=\begin{bmatrix} 0 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} \quad \text{and} \quad D=\begin{bmatrix} 0 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}.$$

The solution to (14) is given by

$$P=\begin{bmatrix} 1/2 & 0 \\ 0 & 17/24 \end{bmatrix}.$$

From these matrices, we obtain

$$\begin{aligned} \sigma(P) &= 17/24 = 0.7083\cdots \\ \sigma(E_1'|P|E_1) &= 0.1068\cdots \\ \sigma(N) &= 0.5590\cdots \\ \sigma(E_1'E_1) &= 0.1636\cdots\,. \end{aligned}$$

We can readily see that both Theorems 5 and 6 assure the convergence of interval matrix (25). It can also be confirmed that Theorem 1 leads to the same conclusion.

4. Concluding Remarks

Several classes of interval matrices for which the necessary and sufficient condition for the convergence is found to be a simple form are presented. Some sufficient conditions are also derived to assure their convergence property. The results are related to those for interval polynomials. It may be true that for stability of interval matrices or interval polynomials, information of all the extreme points of the intervals is not required as in the case of the Hurwitz property for interval polynomials[1]. Indeed, as shown in this paper, in several specific situations, only limited information is enough to ensure the convergence of interval matrices. The question: "When does this redundancy occur?," and also the improvement of the sufficient conditions are topics for further research.

Acknowledgement

A part of this study was supported by Grant-in-Aid C for General Scientific Research of the Ministry of Education, Science and Culture.

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