On the Behaviour of Elastic Potentials

By

N. NISHIMURA* and S. KOBAYASHI*

(Received June 26, 1987)

Abstract

Behaviour of various potentials in the linear anisotropic elastostatics is investigated. Explicit formulae for the derivatives of these potentials including their boundary values are obtained with the help of the theory of distribution.

1. Introduction

The linear theory of elastostatics is a branch of applied mechanics which investigates the behaviour of the solution of a boundary value problem of the following form:

To find a solution u_i (displacement in physical terms) of an equation

$$\operatorname{div} \boldsymbol{C}[\nabla \boldsymbol{u}]+\boldsymbol{F}=\boldsymbol{0} \quad \text{in} \quad D \quad (C_{ijkl}u_{k,lj}+F_i=0) \tag{1}$$

subject to the boundary conditions

$$\boldsymbol{u}=\boldsymbol{u}_0 \quad \text{on} \quad \partial D_u \quad (u_i=u_{0i})$$
$$\mathrm{T}\boldsymbol{u}:=\boldsymbol{C}[\nabla \boldsymbol{u}]\boldsymbol{n}=\boldsymbol{t}_0 \quad \text{on} \quad \partial D_s \quad (\mathrm{T}u_i=t_{0i}) \tag{2a, b}$$

where D is a domain in $R^N(N=2, 3)$, ∂D is its boundary with unit outward normal vector $\boldsymbol{n}$, ∂D_u and ∂D_s are the portions of ∂D such that $\overline{\partial D_u \cup \partial D_s}=\partial D$, and $\partial D_u \cap \partial D_s=\phi$ hold, $\boldsymbol{C}$ is the elasticity tensor, $\boldsymbol{F}$ is the body force (given), and $\boldsymbol{u}_0$ and $\boldsymbol{t}_0$ are given functions. As for $\boldsymbol{C}$, we require the usual symmetry $C_{ijkl}=C_{jikl}=C_{klij}$ and positive definiteness $\nabla\boldsymbol{u}\cdot\boldsymbol{C}[\nabla\boldsymbol{u}]\geqq C'|\nabla\boldsymbol{u}|^2$ where C' is a positive constant.

As is usually the case in applied mechanics, the exact solution to (1) is very difficult to construct analytically except in simple cases. Hence, one often has to resort to some numerical methods of analysis in order to obtain an approximate solution to (1). The boundary integral equation method (BIEM) is one of such numerical methods. BIEM usually seeks the solution in the form of combined

* Department of Civil Engineering.

potentials:

$$\boldsymbol{u} = \int_{\partial D} \boldsymbol{\Gamma t} \mathrm{d}S - \int_{\partial D} \boldsymbol{\Gamma}_{\mathrm{I}} \boldsymbol{u} \mathrm{d}S + \int_{D} \boldsymbol{\Gamma F} \mathrm{d}V , \tag{3}$$

where $\boldsymbol{\Gamma}$ stands for the fundamental solution of the problem, $\boldsymbol{\Gamma}_{\mathrm{I}}$ for the double layer kernel defined by

$$\Gamma_{\mathrm{I}ij}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\partial}{\partial y_l} \Gamma_{ik}(\boldsymbol{x}-\boldsymbol{y}) C_{klmj} n_m(\boldsymbol{y}) \tag{4}$$

and $\boldsymbol{t}$ for $\boldsymbol{C}[\nabla \boldsymbol{u}]\boldsymbol{n}$ on ∂D. The explicit form of $\boldsymbol{\Gamma}$ is given in terms of the Fourier inverse transform $\mathcal{F}^{-1}$ as

$$\boldsymbol{\Gamma} = \mathcal{F}_{\xi}^{-1}(\Delta^{*-1}(\boldsymbol{\xi})) , \tag{5}$$

where $\boldsymbol{\xi}$ is the parameter of the transform and

$$\Delta^{*}_{ij}(\boldsymbol{\xi}) = C_{ikjl} \xi_k \xi_l , \tag{6}$$

respectively. The three integrals in (3) are called the simple layer potential, double layer potential and volume potential, respectively. After solving a boundary integral equation obtained from (2) and (3), one computes the solution $\boldsymbol{u}$ by (3) and the stress $\boldsymbol{C}[\nabla \boldsymbol{u}]$ by differentiating (3).

A difficulty one encounters in this BIEM procedure is the singularity of the kernel functions $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}_{\mathrm{I}}$. Due to this singularity, (3) yields the so called non-integral terms as $\boldsymbol{x}$ approaches ∂D, or as one differentiates (3) in D twice. In addition, the resulting integrals may not be integrable in the classical sense. All such details have been investigated so far by using explicit forms of $\boldsymbol{\Gamma}$ and tedious limit calculations.[1] This explains why formulae for these limits and derivatives were not available in 3D anisotropic elastostatics, because $\boldsymbol{\Gamma}$ for this case has not been computed explicitly.

In this paper, we shall show an alternative for determining the behaviour of these potentials and their derivatives. Our method uses the theory of distribution and computes the various limits and derivatives of elastic potentials for the general anisotropic case. Specifically, we consider a somewhat more general case in which the potentials of the forms

$$\int_{\partial D} F \varphi \, dS \quad \text{or} \quad \int_{D} F \varphi \, dV$$

are considered, where φ is a certain smooth density and F is a kernel function which has a homogeneous Fourier transform $\hat{F}$ of order -2, -1, or 0. After determining the general forms of various limiting values and derivatives of these potentials, we consider the special case of elastostatics by using (3–6). This paper concludes with a few comments on the implication of the present investigation in elastodynamics.

2. Behaviour of Potentials

We introduce the notion of (m, n) homogeneity of a function $f(\boldsymbol{x})$ ($\boldsymbol{x}\in R^N$, $N=2, 3$): A function is (m, n) homogeneous if

$$f(\lambda \boldsymbol{x}) = \lambda^m |\lambda|^n f(\boldsymbol{x}) \qquad \text{for} \quad \lambda \in R. \tag{7}$$

We are particularly interested in functions having $(n, 0)$ homogeneous Fourier transforms ($n=-2, -1, 0$) because we have (5) and (6). Such functions are known to be either functions with logarithmic singularities at the origin for ($N=2$, $n=-2$), or $(-n, -N)$ homogeneous functions for $-N<n<0$, or linear combinations of $\delta(\boldsymbol{x})$ (Dirac's delta) and v.p. of $(0, -N)$ homogeneous functions for $n=0$.[3]

We now consider a domain D in R^N ($N=2$ or 3) whose boundary ∂D is very smooth near $\boldsymbol{x}_0\in\partial D$. Our purpose is to determine the behaviour of the potentials of the forms

$$\int_{\partial D} F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\,dS_y, \quad \int_D F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})dS_y, \tag{8a, b}$$

near $\boldsymbol{x}_0$, where $F(\cdot)$ is a kernel having an $(n, 0)$ homogeneous Fourier transform ($n=-2, -1, 0$), and φ is a smooth density function. Since the comments below (7) show that F is an ordinary function away from $\boldsymbol{x}_0$, we may introduce a ball $B_\rho(\boldsymbol{x}_0)$ which has a radius of $\rho>0$ and is centred at $\boldsymbol{x}_0$, and then concentrate our attention on the contribution to (8) from within $B_\rho(\boldsymbol{x}_0)$. The assumed smoothness of ∂D then enables us to approximate $\partial D\cap B_\rho(\boldsymbol{x}_0)$ and $D\cap B_\rho(\boldsymbol{x}_0)$ by a circular plane segment S and a half ball B shown in Fig. 1. Hence, we are lead to the

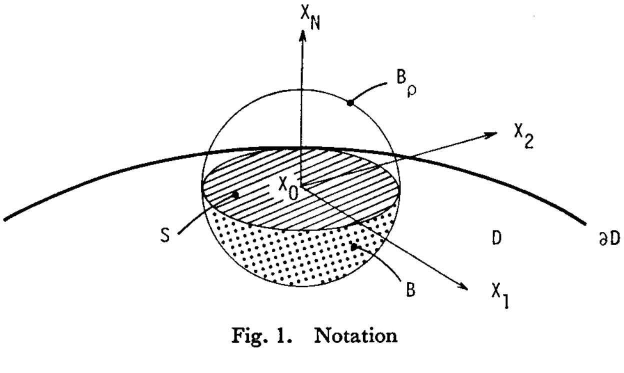

Fig. 1. Notation

investigation of the limits of the forms

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_S F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})dS_y\,,\quad \lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_B F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})dS_y\,,$$

where the approach of $\boldsymbol{x}$ to $\boldsymbol{x}_0$ is along the unit outward normal vector to S at $\boldsymbol{x}_0$, denoted by $\boldsymbol{n}$. In the sequel, we shall compute these limits by using a cartesian frame whose Nth axis is in the direction of $\boldsymbol{n}$ and whose origin is at $\boldsymbol{x}_0$.

2.1 Surface integrals

We consider the following integral

$$\int_S F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})dS_y\,. \tag{9}$$

Obviously, one may replace the domain of integration by R^{N-1} with the extension $\varphi(\boldsymbol{y})=0$ for $\boldsymbol{y}\in R^{N-1}\backslash S$. We have the following results for the different n:

i) $n=-2$. Since the kernels have estimates of the form

$$|F(x_\alpha, x_N)|\leqq\frac{C}{(|x_\alpha|^2+x_N^2)^{(N-2)/2}}\leqq\frac{C}{|x_\alpha|^{N-2}}\qquad (N=3)^*$$

or

$$|F(x_\alpha, x_N)|\leqq C_1+C_2|\log\sqrt{(|x_\alpha|^2+x_N^2)}|\leqq C_1+C_2|\log|x_\alpha||\qquad (N=2)$$

near the origin, (See the comments below (7).) we have

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_S F(x_\alpha, x_N)\varphi(\boldsymbol{y})dS_y=\int_S F(\boldsymbol{x}_0-\boldsymbol{y})\varphi(\boldsymbol{y})dS_y \tag{10}$$

for a sufficiently smooth $\varphi(\boldsymbol{y})$.

ii) $n=-1$. We use the notion of the partial Fourier (inverse) transform of $\hat{F}(\xi_\alpha, \xi_N)$ with respect to ξ_N, which is denoted by $\hat{F}(\xi_\alpha|x_N)$, and is defined by

$$\hat{F}(\xi_\alpha|x_N)=\frac{1}{2\pi}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}e^{i\xi_N x_N-\varepsilon\xi_N^2}\hat{F}(\xi_\alpha, \xi_N)d\xi_N\,. \tag{11}$$

The required limit is calculated in terms of $\hat{F}(\xi_\alpha|x_N)$ as

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_S F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})d\boldsymbol{y}=\frac{1}{(2\pi)^{N-1}}\lim_{x_N\to 0}\int_{R^{N-1}}\hat{F}(\xi_\alpha|x_N)\hat{\varphi}(\xi_\alpha)d\xi_1\cdots d\xi_{N-1}\,,$$

$$\text{(Note } \boldsymbol{x}_0=(0,\cdots,0) \text{ by definition.)} \tag{12}$$

* Greek indices run from 1 to $N-1$.

where $\phi(\xi_\alpha)$ is the Fourier transform of $\varphi(y_\alpha)$ on S. Into the right hand side of (12) we substitute an expansion

$$\hat{F}(\xi_\alpha, \xi_N) = \xi_N^n \hat{F}\left(\frac{\xi_\alpha}{\xi_N}, 1\right) = \xi_N^n \hat{F}(\mathbf{0}, 1) + \xi_N^{n-1}\xi_\alpha \frac{\partial}{\partial \xi_\alpha}\hat{F}(\mathbf{0}, 1) + \frac{1}{2}\xi_N^{n-2}\xi_\alpha\xi_\beta \frac{\partial^2}{\partial\xi_\alpha\partial\xi_\beta}\hat{F}(\mathbf{0}, 1) + \cdots$$

which is valid for a large ξ_N. Subsequent use of Lebesgue's theorem converts the limit in (11) into

$$\lim_{\varepsilon\downarrow 0}\left[\int_{|\xi_N|<\delta}\hat{F}(\boldsymbol{\xi})e^{i\xi_N x_N-\varepsilon\xi_N^2}d\xi_N + \int_{|\xi_N|>\delta}\left(\hat{F}(\boldsymbol{\xi}) - \frac{\hat{F}(\mathbf{0}, 1)}{\xi_N}\right)e^{i\xi_N x_N-\varepsilon\xi_N^2}d\xi_N + \hat{F}(\mathbf{0}, 1)\int_{|\xi_N|>\delta}\frac{e^{i\xi_N x_N-\varepsilon\xi_N^2}}{\xi_N}d\xi_N\right]$$

$$= \int_{|\xi_N|<\delta}\hat{F}(\boldsymbol{\xi})e^{i\xi_N x_N}d\xi_N + \int_{|\xi_N|>\delta}\left(\hat{F}(\boldsymbol{\xi}) - \frac{\hat{F}(\mathbf{0}, 1)}{\xi_N}\right)e^{i\xi_N x_N}d\xi_N + \hat{F}(\mathbf{0}, 1)\lim_{\varepsilon\downarrow 0}\int_{|\xi_N|>\delta}\frac{e^{i\xi_N-\varepsilon\xi_N^2}}{\xi_N}d\xi_N$$

for a $\delta>0$ and $|\xi_\alpha|\neq 0$. The last integral is equal to

$$\begin{bmatrix}\pi i\\ 0\\ -\pi i\end{bmatrix} - i\,\mathrm{sgn}\,x_N\int_{-\delta|x_N|}^{\delta|x_N|}\frac{\sin\xi}{\xi}d\xi \qquad \text{for} \quad \begin{bmatrix}x_N>0\\ x_N=0\\ x_N<0\end{bmatrix}.$$

Therefore, by letting $\delta\downarrow 0$ we have

$$\hat{F}(\xi_\alpha|x_N) = \frac{i}{2}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}\hat{F}(0, 1) + \frac{1}{2\pi}\,\mathrm{v.p.}\int_{-\infty}^{\infty}\left(\hat{F}(\boldsymbol{\xi}) - \frac{\hat{F}(\mathbf{0}, 1)}{\xi_N}\right)e^{i\xi_N x_N}d\xi_N$$

according as $x_N>0$ (upper), $x_N=0$ (middle), or $x_N<0$ (lower), where v.p. indicates the integral in the sense of Cauchy's principal value. Note that the function (of ξ_α) defined by

$$\hat{F}(\xi_\alpha|0) = \frac{1}{2\pi}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\hat{F}(\xi_\alpha, \xi_N)e^{-\varepsilon\xi_N^2}d\xi_N$$

is $(n, 1)$ homogeneous (for $n=-1, 0$). In particular, for the present case of $(-1, 1)$ homogeneity, we have

$$\int_{S_{N-1}}\hat{F}(\xi_\alpha|0)dS = 0$$

from the symmetry. This shows that $\mathcal{F}^{-1}_{\xi_\alpha}(\hat{F}(\xi_\alpha|0))(x_\alpha)^*$, as an $N-1$ dimensional distribution, is expressed as a principal value of a $(1, -N)$ homogeneous function.[2] In addition, $\mathcal{F}^{-1}_{\xi_\alpha}(\hat{F}(\xi_\alpha|0))(x_\alpha)$ coincides with $F(x_\alpha, 0)$ — a $(1, -N)$ homogeneous function — away from the origin. Hence we conclude

$$\lim_{x\to x_0}\int_S F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}S = \pm\frac{i}{2}\hat{F}(\boldsymbol{n})\varphi(\boldsymbol{x}_0)+\text{v.p.}\int_S F(\boldsymbol{x}_0-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}S. \quad (13)$$

The upper (lower) sign is for the approach from the positive (negative) side, with positive side indicating $x_N>0$. This convention will be used throughout this paper.

iii) $n=0$. Without loss of generality we may assume

$$\hat{F}(\boldsymbol{0}, 1) = 0 .$$

Actually, if this is not to be the case we may modify the definition of $\hat{F}(\boldsymbol{\xi})$ by subtracting $\hat{F}(\boldsymbol{0}, 1)$ from $\hat{F}(\boldsymbol{\xi})$. This process changes F in (9) by $-\hat{F}(\boldsymbol{0}, 1)\delta(\boldsymbol{x}-\boldsymbol{y})$, but this term vanishes for $\boldsymbol{x}\notin\partial D$ and $\boldsymbol{y}\in\partial D$, thus keeping (9) unchanged. With this assumption, we use (11) and an expansion for $\hat{F}$ as in ii) to obtain

$$2\pi\hat{F}(\xi_\alpha|x_N) = \xi_\alpha\frac{\partial}{\partial\xi_\alpha}\hat{F}(\boldsymbol{0}, 1)\begin{bmatrix}\pi i\\ 0\\ -\pi i\end{bmatrix}$$
$$+\text{v.p.}\int_{-\infty}^{\infty}\left(\hat{F}(\boldsymbol{\xi})-\frac{\xi_\alpha}{\partial_N}\frac{\partial}{\partial\xi_\alpha}\hat{F}(\boldsymbol{0}, 1)\right)e^{i\xi_N x_N}\mathrm{d}\xi_N , \quad (14)$$

and hence

$$\lim_{x_N\to 0}\hat{F}(\xi_\alpha|x_N) = \pm\frac{i}{2}\xi_\alpha\frac{\partial}{\partial\xi_\alpha}\hat{F}(\boldsymbol{0}, 1)+F(\xi_\alpha|0) , \quad (15)$$

where we have used the same notation as has been used in ii).

We now proceed to the interpretation of $\mathcal{F}^{-1}_{\xi_\alpha}(\hat{F}(\xi_\alpha|0))$, or the Fourier inverse transform of the second term in (15). To this end we note that

$$\hat{F}(\xi_\alpha|0) = \frac{1}{2\pi}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\hat{F}(\xi_\alpha, \xi_N)e^{-\varepsilon\xi_N^2}\mathrm{d}\xi_N$$

is a well-defined $(0, 1)$-homogeneous function of ξ_α. Its Fourier inverse transform on ξ_α is equal to the restriction to $x_N=0$ of $F(x_\alpha, x_N)$ for non-zero x_α; a $(0, N)$-homogeneous function of x_α. From these observations, we can show that $\mathcal{F}^{-1}_{\xi_\alpha}(\hat{F}(\xi_\alpha|0))$ (as an $N-1$ dimensional distribution) is to be understood as a finite

* $\mathcal{F}^{-1}_{\xi_\alpha}$ denotes the Fourier inverse transform. $(\xi_\alpha\to x_\alpha)$

part (p.f.) defined by

$$\text{p.f.}\int_{R^{N-1}} F(x_\alpha, 0)\varphi(x_\alpha)\mathrm{d}x_\alpha = \lim_{\varepsilon\downarrow 0}\Big[\int_{R^{N-1}\setminus B_\varepsilon(0)} F(x_\alpha, 0)\varphi(x_\alpha)\mathrm{d}x_\alpha - \frac{\varphi(0)}{\varepsilon}\int_{S_{N-1}} F(x_\alpha, 0)\mathrm{d}S\Big],$$

where S_{N-1} is an $N-1$ dimensional unit sphere. To see this, we start from an observation that p.f. $F(x_\alpha, 0)$ and $\mathcal{F}^{-1}_{\xi_\alpha}(\hat{F}(\xi_\alpha|0))$ coincide on R^{N-1} except at the origin. We therefore have

$$\mathcal{F}^{-1}_{\xi_\alpha}(\hat{F}(\xi_\alpha|0)) = \sum_{|\alpha|\leq M} C^\alpha D^\alpha \delta(\boldsymbol{x}) + \text{p.f.}\, F(x_\alpha) \tag{16}$$

where C^α is a certain constant,

$$D^\alpha := \partial^{\alpha_1}_{x_1}\partial^{\alpha_1}_{x_2}\cdots\partial^{\alpha_{N-1}}_{x_{N-1}},$$
$$\alpha_i = (\alpha_1, \alpha_2, \cdots, \alpha_{N-1})$$

and

$$|\alpha| := \sum_i \alpha_i\,.$$

Indeed, a distribution having a point support is known to have the form given in the first term of (16)[2]. In addition, we see that

$$\begin{aligned}\mathcal{F}_{x_\alpha}\,\text{p.f. }(F(x_\alpha, 0))(\lambda\xi_\alpha) &= \lim_{\varepsilon\downarrow 0}\Big[\int_{R^{N-1}\setminus B_\varepsilon(0)} F(x_\alpha, 0)e^{-i\lambda\xi_\alpha x_\alpha}\mathrm{d}x_\alpha - \frac{1}{\varepsilon}\int_{S_{N-1}} F(x_\alpha, 0)\mathrm{d}S\Big] \qquad (\lambda x_\alpha = y_\alpha)\\ &= \lim_{|\lambda|\varepsilon\downarrow 0}\Big[\int_{R^{N-1}\setminus B_{|\lambda|\varepsilon}} F(y_\alpha, 0)e^{-i\xi_\alpha y_\alpha}\mathrm{d}y_\alpha - \frac{1}{|\lambda|\varepsilon}\int_{S_{N-1}} F(x_0, 0)\mathrm{d}S\Big]|\lambda|\\ &= |\lambda|\mathcal{F}_{x_\alpha}(\text{p.f. }F(x_\alpha, 0))(\xi_\alpha)\,, \qquad (\lambda x_\alpha = y_\alpha)\end{aligned}$$

which show that

$$\mathcal{F}_{x_\alpha}(\text{p.f. }F(x_\alpha, 0))$$

is (0, 1)-homogeneous. On the other hand, $\mathcal{F}_{x_\alpha}(D^\alpha\delta(\boldsymbol{x}))$ ($|\alpha|=1$) is (1, 0)-homogeneous. Hence, we conclude that $C^{\forall\alpha}=0$, which proves our statement. We thus have

$$\lim_{x\to x_0}\int_S F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}S = \pm\frac{1}{2}\frac{\partial}{\partial\xi_\alpha}\hat{F}(\boldsymbol{0}, 1)\frac{\partial}{\partial x_\alpha}\varphi(\boldsymbol{x}_0) + \text{p.f.}\int_S F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}S\,. \tag{17}$$

2.2 Volume integrals

We next investigate the limits of the form

$$\lim_{x\to x_0}\int_B F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,,$$

where $F(\boldsymbol{x})$ is the same function as in **2.1**, and $\varphi(\boldsymbol{y})$ is a smooth density function. As in **2.1**, we replace the domain of integration by R^N with the help of an extension $\varphi(\boldsymbol{y})=0$ for $\boldsymbol{y}\in R^N\backslash B$. We then have the following results for different n:

i) $n=-2,\ -1$. We easily see that

$$\lim_{x\to x_0}\int_B F(\boldsymbol{x}-\boldsymbol{y})\,\varphi(\boldsymbol{y})\mathrm{d}\boldsymbol{y}=\int_B F(\boldsymbol{x}_0-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\,. \tag{18}$$

ii) $n=0$. We decompose $\hat{F}(\boldsymbol{\xi})$ into

$$\hat{F}(\boldsymbol{\xi})=\mathrm{C}_F+\hat{\grave{F}}(\boldsymbol{\xi})\,,$$

where

$$\mathrm{C}_F=\frac{1}{|S_N|}\int_{S_N}\hat{F}(\boldsymbol{\xi})\mathrm{d}S\,,\quad \hat{\grave{F}}(\boldsymbol{\xi})=\hat{F}(\boldsymbol{\xi})-\mathrm{C}_F\,. \tag{19a, b}$$

The inverse transform of C_F is $\mathrm{C}_F\delta(\boldsymbol{x})$. Also, we know that $\mathscr{F}_\xi^{-1}\hat{\grave{F}}(\boldsymbol{\xi})$ is equal to v.p. $\grave{F}(\boldsymbol{x})$ where $\grave{F}(\boldsymbol{x})$ is a $(0,\,-N)$-homogeneous function. Hence, we see that the Fourier 'integral'

$$\mathscr{F}_\xi^{-1}\{\hat{F}(\boldsymbol{\xi})\phi(\boldsymbol{\xi})\}$$

has the following expression

$$\mathrm{C}_F\varphi(\boldsymbol{x})+\text{v.p.}\int_D F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V, \tag{20a}$$

for $\boldsymbol{x}\in D$, and

$$\int_D F(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V, \tag{20b}$$

for $\boldsymbol{x}\in D^c\backslash\partial D$, where C_F is given in (19a). In these formulae we have dropped ` because $F=\grave{F}$ in the classical sense.

In order to further investigate $\grave{F}(\boldsymbol{x})$, we use the partial Fourier transform in the following manner:

$$\int_B \grave{F}(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}\boldsymbol{y}=\mathscr{F}_{\xi_\alpha}^{-1}\Big(\int_{-\rho}^{0}\hat{\grave{F}}(\xi_\alpha|x_N-y_N)\phi(\xi_\alpha|y_N)\mathrm{d}y_N\Big)(x_\alpha)\,,$$

$$(x_N\neq 0) \tag{21}$$

where

$$\phi(\xi_\alpha | y_N) = \int_{(y_\alpha, y_N)\in B} e^{-i\xi_\alpha y_\alpha} \varphi(y_\alpha, y_N) \mathrm{d}y_\alpha .$$

Into $\hat{\grave{F}}(\cdot|\cdot)$ in (21) we substitute an analogue of (14) given by

$$2\pi\hat{\grave{F}}(\xi_\alpha | x_N) = \sqrt{\pi} \lim_{\varepsilon \downarrow 0} \frac{e^{-x_N^2/4\varepsilon}}{\varepsilon^{1/2}} \hat{\grave{F}}((\mathbf{0}, 1) + \xi_\alpha \frac{\partial}{\partial \xi_\alpha} \hat{\grave{F}}(\mathbf{0}, 1) \begin{bmatrix} \pi i \\ 0 \\ -\pi i \end{bmatrix}$$

$$+\text{v.p.} \int_{-\infty}^{\infty} \left(\hat{\grave{F}}(\boldsymbol{\xi}) - \hat{\grave{F}}(\mathbf{0}, 1) - \frac{\xi_\alpha}{\xi_N} \frac{\partial}{\partial \xi_\alpha} \hat{\grave{F}}(\mathbf{0}, 1) \right) e^{i\xi_N x_N} \mathrm{d}\xi_N ,$$

which differs from (14) because $\hat{\grave{F}}(\mathbf{0}, 1) \neq 0$ in general. Use of this formula,

$$\frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \sqrt{\pi} \int_{-\rho}^{0} \frac{e^{-(x_N - y_N)^2/4\varepsilon}}{\varepsilon^{1/2}} \phi(\xi_\alpha | y_N) \mathrm{d}y_N = \begin{bmatrix} 0 \\ \phi(\xi_\alpha | 0)/2 \\ \phi(\xi_\alpha | x_N) \end{bmatrix} \text{ for } \begin{bmatrix} x_N > 0 \\ x_N = 0 \\ x_N > 0 \end{bmatrix},$$

and

$$\mathscr{F}_{\xi_\alpha}^{-1}\left(\int_{-\rho}^{0} \hat{\grave{F}}(\xi_\alpha | x_N - y_N) \phi(\xi_\alpha | y_N) \mathrm{d}y_N \right)(x_\alpha) = \mathscr{F}_{\xi}^{-1}(\hat{\grave{F}}(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}))(\boldsymbol{x})$$

then transforms (21) into

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \int_B \grave{F}(\boldsymbol{x} - \boldsymbol{y}) \varphi(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = \mp \frac{\hat{\grave{F}}(\mathbf{0}, 1)}{2} \varphi(\boldsymbol{x}_0) + \mathscr{F}_{\xi}^{-1}(\hat{\grave{F}}(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}))(\boldsymbol{x}_0) ,$$

$$(\boldsymbol{x}_0 = (0, 0, \cdots 0)) , \tag{22}$$

where $\phi(\boldsymbol{\xi})$ is the Fourier transform of $\varphi(\boldsymbol{x})$ on B. Hence, we are left with the interpretation of the last term in (22). To this end, we note the following relations which follow from the symmetry and (19):

$$\int_{S_N^+} \grave{F}(\boldsymbol{x}) \mathrm{d}S = \int_{S_N^-} \grave{F}(\boldsymbol{x}) \mathrm{d}S = 0 , \tag{23}$$

where $S_N^{+,-} = S_N \cap \{x_N >, < 0\}$ (signs and $>$, $<$ are to be taken in the same order). Since

$$\mathscr{F}_{\xi}^{-1}(\hat{\grave{F}}(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}))(\boldsymbol{x}_0) = \lim_{\varepsilon \downarrow 0} \int_{R^N \setminus B_\varepsilon(x_0)} \grave{F}(\boldsymbol{x}_0 - \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d}V_y$$

$$= \text{v.p.}^- \int_B \grave{F}(\boldsymbol{x}_0 - \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d}V_y , \tag{24}$$

we obtain

$$\lim_{x\to x_0}\mathscr{F}^{-1}(\hat{F}(\xi)\hat{\phi}(\xi))(x) = \lim_{x\to x_0}\left\{\begin{bmatrix} 0 \\ C_F \end{bmatrix}\varphi(x)+\text{v.p.}\int_B F(x-y)\varphi(y)\mathrm{d}V_y\right\}$$

$$= \left(\frac{C_F}{2}\mp\frac{\hat{F}(n)}{2}\right)\varphi(x_0)+\text{v.p.}^-\int_B F(x_0-y)\varphi(y)\mathrm{d}V_y\,, \tag{25}$$

where

$$\text{v.p.}^-\int_B \cdot\,\mathrm{d}V = \lim_{\varepsilon\downarrow 0}\int_{B\setminus B_\varepsilon(x_0)} \cdot\,\mathrm{d}V.$$

Note that the special principal-value integral, denoted by v.p.$^-$ and defined above, is convergent due to (23). Actually, this is why (24) holds. We also have

$$\lim_{x-x_0}\text{v.p.}\int_B F(x-y)\,\varphi(y)\,\mathrm{d}V = \pm\left(\frac{C_F}{2}-\hat{F}(n)\right)\varphi(x_0)$$

$$+\text{v.p.}^-\int_B F(x_0-y)\,\varphi(y)\mathrm{d}V_y\,. \tag{26}$$

3. Elastic Potentials

We now apply the foregoing analysis to elastic potentials. Since $-\Delta^{*-1}(i\xi)$ is a $(-2, 0)$-homogeneous function, we have the following results.

3.1 Simple layer potential

We have $F=\Gamma$ and $\hat{F}(\xi)=\Delta^{*-1}(\xi)$ by definition. The comments below (7) (or (10))readily give

$$\lim_{x\to x_0}\int_{\partial D}\Gamma(x-y)\,\varphi(y)\mathrm{d}S = \int_{\partial D}\Gamma(x_0-y)\varphi(y)\mathrm{d}S\,. \tag{27}$$

As to the derivatives of this potential, we shall start with an identity

$$\lim_{x\to x_0}\nabla\int_{\partial D}\Gamma(x-y)\varphi(y)\mathrm{d}S = \lim_{x\to x_0}\Big[\int_{\partial D\cap B_\varepsilon(x_0)}\nabla\Gamma(x-y)\varphi(y)\mathrm{d}S$$

$$+\int_{\partial D\setminus(\partial D\cap B_\varepsilon(x_0))}\nabla\Gamma(x-y)\varphi(y)\mathrm{d}S\Big]\,. \tag{28}$$

After approximating $\partial D\cap B_\varepsilon(x_0)$ by a small plane segment, we apply (13) to the first integral on the right hand side of (28), where $\varepsilon>0$ is a sufficiently small number. This, together with the observation that the second integral has no singularity, yields

$$\lim_{x\to x_0}\nabla\int_{\partial D}\Gamma(x-y)\varphi(y)\mathrm{d}S$$

$$= \lim_{x \to x_0} \int_{\partial D} \mathcal{F}_{\xi}^{-1}(i\boldsymbol{\xi} \otimes \Delta^{*-1}(\boldsymbol{\xi}))(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{dS}$$

$$= \mp \frac{\boldsymbol{n}}{2} \otimes \Delta^{*-1}(\boldsymbol{n})\varphi(\boldsymbol{x}_0) + \text{v.p.} \int_{\partial D} \nabla_x \boldsymbol{\Gamma}(\boldsymbol{x}_0-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{dS}. \qquad (29)$$

In particular, we have the well-known formula[1)]

$$\lim_{x \to x_0} \mathrm{T}_x \int_{\partial D} \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{dS} = \mp \frac{1}{2}\varphi(\boldsymbol{x}_0) + \text{v.p.} \int_{\partial D} \mathrm{T}_x \boldsymbol{\Gamma}(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{dS},$$

since $C_{ipjq} n_p n_q = \Delta^*_{ij}(\boldsymbol{n})$.

3.2 Double layer potential

We now apply the same reasoning as we have used in **3.1** to obtain various limits relevant to double layer potentials. To begin with, we note from (4) and (5) that the Fourier transform of the double layer potential can locally be written as

$$-i\Delta^{*-1}_{ip}(\boldsymbol{\xi})\xi_q C_{pqrk} n_r,$$

since $\boldsymbol{n}$ is locally constant. This observation, together with (13), yields[1)]

$$\lim_{x \to x_0} \int_{\partial D} \boldsymbol{\Gamma}_{\mathrm{I}}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{dS} = \pm \frac{\varphi(\boldsymbol{x}_0)}{2} + \text{v.p.} \int_{\partial D} \boldsymbol{\Gamma}_{\mathrm{I}}\varphi(\boldsymbol{y})\mathrm{dS}. \qquad (30)$$

Also, we have

$$\lim_{x \to x_0} \partial_i \int_{\partial D} \Gamma_{\mathrm{I}jk}\varphi_k(\boldsymbol{y})\mathrm{dS}$$

$$= \pm \frac{1}{2} \frac{\partial}{\partial \xi_\alpha}(\xi_i \Delta^{*-1}_{jp}(\boldsymbol{\xi})\xi_q C_{pqrk})|_{\xi=n} n_r \frac{\partial}{\partial x_\alpha}\varphi_k(\boldsymbol{x}_0)$$

$$+ \text{p.f.} \int_{\partial D} \partial_i \Gamma_{\mathrm{I}\,jk}\varphi_k(\boldsymbol{y})\mathrm{dS}$$

$$= \pm \frac{1}{2}\left\{\delta_{i\alpha}\frac{\partial}{\partial x_\alpha}\varphi_j(\boldsymbol{x}_0) - n_i \Delta^{*-1}_{jr}(\boldsymbol{n})C_{rp\alpha q} n_p \frac{\partial}{\partial x_\alpha}\varphi_q(\boldsymbol{x}_0)\right\}$$

$$+ \text{p.f.} \int_{\partial D} \partial_i \Gamma_{\mathrm{I}\,jk}\varphi_k(\boldsymbol{y})\mathrm{dS}, \qquad (31)$$

where we have used (17). In particular, we obtain

$$\lim_{x \to x_0} \mathrm{T}_x \int_{\partial D} \boldsymbol{\Gamma}_{\mathrm{I}}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{dS} = \text{p.f.} \int_{\partial D} \mathrm{T}_x \boldsymbol{\Gamma}_{\mathrm{I}}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{dS} \qquad (32)$$

since

$$n_k C_{ijkl}\left(\delta_{i\alpha}\frac{\partial}{\partial x_\alpha}\varphi_j - n_j \Delta^{*-1}_{jr}(\boldsymbol{n})C_{rp\alpha q} n_p \frac{\partial}{\partial x_\alpha}\varphi_q\right) = 0.$$

Equation (32) is often called the generalised Lyapunov-Tauber theorem.[1]

We finally remark that the present derivation of (31) is not exact from a purely mathematical point of view because it does not take into consideration the possible effect of the curvature of ∂D. However, it is not difficult to see that the result is correct. Indeed, one starts from the well-known formula[3]

$$\partial_k \int_{\partial D} \Gamma_{Iij}\varphi_j \mathrm{d}S = e_{pqk}\int_{\partial D} \Gamma_{il,m} C_{lmqj} e_{pab} n_a \varphi_{j,b} \mathrm{d}S\,,$$

and then uses (13) to obtain (31), where e_{ijk} is the permutation symbol.

3.3 Volume potential

From (18) one readily sees that

$$\nabla \int_D \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V = \int_D \nabla\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V\,, \tag{33}$$

where $\boldsymbol{x}$ is a point in R^N. We also have (See (20).)

$$\nabla\nabla \int_D \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V = \begin{cases} \boldsymbol{C}_\Gamma\varphi(\boldsymbol{x}) + \text{v.p.} \int \nabla\nabla\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V & \boldsymbol{x}\in D \\ \int \nabla\nabla\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V & \boldsymbol{x}\in R^N\backslash\bar{D}\,, \end{cases} \tag{34}$$

where

$$C_{\Gamma ijkl} = -\frac{1}{|S_N|}\int_{S_N} \xi_i\xi_j \Delta^{*-1}_{kl}(\boldsymbol{\xi})\mathrm{d}S \qquad \text{(See (19}a\text{))}\,.$$

In particular, we obtain

$$\begin{aligned}(\Delta^* \int_D \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V)_i &= C_{iprq}\partial_p\partial_q(\int_D \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V)_r \\ &= \begin{cases} -\varphi_i(\boldsymbol{x}) & \boldsymbol{x}\in D \\ 0 & \boldsymbol{x}\in R^N\backslash\bar{D}\,. \end{cases}\end{aligned}$$

This well-known result is usually called the Poisson formula.[1]

The second derivative

$$\nabla\nabla \int_D \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V$$

jumps as the point $\boldsymbol{x}$ crosses ∂D. Actually, we see from (25) that

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} \nabla\nabla \int_D \boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}V = \frac{1}{2}(\boldsymbol{C}_\Gamma \pm \boldsymbol{n}\otimes\boldsymbol{n}\otimes\Delta^{*-1}(\boldsymbol{n}))\varphi(\boldsymbol{x}_0)$$

$$+\text{v.p.}^{-}\int_{D}\nabla\nabla\boldsymbol{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\mathrm{d}V \tag{35}$$

holds. Also, from (26), we have

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\text{v.p.}\int_{D}\nabla\nabla\boldsymbol{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\mathrm{d}V=\frac{1}{2}(\boldsymbol{C}_{\Gamma}+\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{\Delta}^{*-1}(\boldsymbol{n}))\boldsymbol{\varphi}(\boldsymbol{x}_0)$$

$$+\text{v.p.}^{-}\int_{D}\nabla\nabla\boldsymbol{\Gamma}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\mathrm{d}V. \tag{36}$$

Note that these results are consistent with the Poisson formula, since

$$-C_{ijkl}C_{\Gamma jklm}=\frac{1}{|S_N|}\int_{S_N}C_{ijkl}n_j n_k\Delta^{*-1}_{lm}(\boldsymbol{\xi})\mathrm{d}S=\delta_{im}.$$

We finally remark that the potentials of the form

$$\int\Gamma_{ij,k_y}(\boldsymbol{x}-\boldsymbol{y})\varphi_{jk}(\boldsymbol{y})\mathrm{d}V$$

can be viewed as

$$-\partial_{k_x}\int\Gamma_{ij}(\boldsymbol{x}-\boldsymbol{y})\varphi_{jk}(\boldsymbol{y})\mathrm{d}V,$$

which is exactly the derivative of a volume potential. Therefore, we can apply the foregoing analysis for volume potentials to the potential of this type, which plays an essential role in BIEM for elastoplasticity.[4]

4. Concluding Remarks

1. It is easy to see that (3), (13), (17) and (18) yield the following identity on ∂D[5]:

$$\tau_{pq}:=C_{pqij}\partial_i u_j=2(\text{v.p.}\int_{\partial D}C_{pqij}\partial_i\Gamma_{jk}t_k dS$$

$$-\text{p.f.}\int_{\partial D}C_{pqij}\partial_i\Gamma_{\text{I}jk}u_k dS+\int_{\partial D}C_{pqij}\partial_i\Gamma_{jk}F_k dV),$$

where τ_{pq} stands for the stress on the boundary. Cruse & Van Buren obtained this formula for the isotropic case by using a direct calculation.[6] Our formula generalizes their result to the anisotropic cases.

2. Since the behaviour of the Fourier transform of a function f at infinity reflects the singularity of f, we see that our formulae (10), (13), (17), (20) and (26) are valid also in time harmonic elastodynamics[1]. This is because the Fourier transforms of the fundamental solutions of elastostatics and elastodynamics behave similarly at

infinity.

3. Some of the results in this paper have been published without detailed proof in reports by Nishimura & Kobayashi[4] and Kobayashi & Nishimura[5].

References

1) Kupradze, V.D., Gegelia, T.G., Basheleishvili, M.O. and Burchuladze, T.V.: Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, North-Holland, Amsterdam-New York-Oxford, 1979.
2) Mizohata, S.: Theory of Partial Differential Equations, Iwanami, Tokyo, 1965. (in Japanese)
3) Sladek, V. and Sladek, J.: Appl. Math. Model., Vol. 6, p. 374, 1982.
4) Nishimura, N. and Kobayashi, S.: Elastoplastic analysis by indirect methods, In; Developments in Boundary Element Methods-III, (Eds. Banerjee, P.K. & Mukherjee, S.), Appl. Sci. Publ., London, 1984.
5) Kobayashi, S. and Nishimura, N.: Preprint for J.S.C.E. Kansai Chap. Convention, p. 1, 1983. (in Japanese)
6) Cruse, T.A., and Van Buren, W.: Int. J. Fracture Mech., Vol. 7, p. 1, 1971.