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On the Behaviour of Potentials in Consolidation and Coupled Thermoelasticity

By

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Abstract

Behaviour of various potentials in Biot's linear theory of consolidation and linear coupled thermoelasticity is investigated for the case of general anisotropy. Initial behaviour of the solutions of these problems is discussed in detail.

1. Introduction

Biot's linear theory of consolidation[1], and the theory of quasistatic linear coupled thermoelasticity[2] are formulated into one and the same mathematical statement of the following form: To find a solution of the equations

$$\Delta^* \boldsymbol{u} - \nabla p = -\boldsymbol{f}\,,$$
$$\text{div}\, \dot{\boldsymbol{u}} + m\dot{p} - \boldsymbol{K}\cdot\nabla\nabla p = g\,, \quad \text{in } D\times(t>0)\,, \qquad (1a,\ b)$$

subject to an initial condition

$$(\text{div}\, \boldsymbol{u} + mp)\,|_{t=0} = \theta \quad \text{in } D \qquad (2)$$

and boundary conditions for $t>0$

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on } \partial D_u\,, \qquad (3)$$

$$\boldsymbol{s} = \boldsymbol{s}_0 \quad \text{on } \partial D_s\,, \ (\overline{\partial D_u \cup \partial D_s} = \partial D\,, \quad \partial D_u \cap \partial D_s = \phi) \qquad (4)$$

$$p = p_0 \quad \text{on } \partial D_p\,, \qquad (5)$$

$$r \equiv -\boldsymbol{n}\cdot\boldsymbol{K}\nabla p = r_0 \quad \text{on } \partial D_r\,, \ (\overline{\partial D_p \cup \partial D_r} = \partial D\,, \ \partial D_p \cap \partial D_r = \phi) \qquad (6)$$

where D indicates a domain in R^N (N=2, 3), ∂D its boundary with an outward unit normal vector $\boldsymbol{n}$, ∂D_u, ∂D_s, ∂D_p, ∂D_r portions of ∂D, ($\boldsymbol{u}$, p) the unknown

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(vector, scalar) functions, $\cdot$ the differentiation with respect to t (time), Δ^* an operator defined by

$$(\Delta^* \boldsymbol{u})_i = C_{ipjq}\, u_{j,pq}\,,$$

C_{ijkl} a constant tensor having the usual symmetry ($C_{ijkl}=C_{jikl}=C_{klij}$) and positivity, K_{ij} a positive symmetric tensor, m a positive constant, $\boldsymbol{s}$ the boundary traction defined by

$$s_i = C_{ipjq}\, u_{j,q}\, n_p - p n_i\,,$$

and $\boldsymbol{f}$, g, θ, $\boldsymbol{u}_0$, $\boldsymbol{s}_0$, p_0 and r_0 given functions, respectively. In these formulae we have used the standard tensor notation including summation convention.

The solution to this problem is written as[3),4)]

$$\begin{aligned}
\tilde{\boldsymbol{u}}(\boldsymbol{x}, s) = &\int_{\partial D} \boldsymbol{U}_0(\boldsymbol{x}-\boldsymbol{y})\, \boldsymbol{s}(\boldsymbol{y}, s)\, \mathrm{d}S - \int_{\partial D} \boldsymbol{S}_0(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{u}(\boldsymbol{y}, s) \mathrm{d}S \\
&+\int_D \boldsymbol{U}_0(\boldsymbol{x}-\boldsymbol{y})\, \boldsymbol{f}(\boldsymbol{y}, s) \mathrm{d}V + \int_{\partial D}\int_0^s \dot{\mathcal{U}}(\boldsymbol{x}-\boldsymbol{y}, s-t)\, \boldsymbol{s}(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S \\
&-\int_{\partial D}\int_0^s \dot{\mathcal{S}}(\boldsymbol{x}, \boldsymbol{y}, s-t)\, \boldsymbol{u}(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S - \int_{\partial D}\int_0^s \boldsymbol{P}(\boldsymbol{x}-\boldsymbol{y}, s-t)\, r(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S \\
&+\int_{\partial D}\int_0^s \boldsymbol{R}(\boldsymbol{x}, \boldsymbol{y}, s-t)\, p(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S + \int_D \int_0^s \dot{\mathcal{U}}(\boldsymbol{x}-\boldsymbol{y}, s-t)\, \boldsymbol{f}(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}V \\
&+\int_D \int_0^s \boldsymbol{P}(\boldsymbol{x}-\boldsymbol{y}, s-t)\, g(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}V + \int_D \boldsymbol{P}(\boldsymbol{x}-\boldsymbol{y}, s)\, \theta(\boldsymbol{y}) \mathrm{d}V\,, \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{p}(\boldsymbol{x}, s) = &\int_{\partial D} \boldsymbol{V}_0(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{s}(\boldsymbol{y}, s)\, \mathrm{d}S - \int_{\partial D} \boldsymbol{T}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{y}, s) \mathrm{d}S \\
&+\int_D \boldsymbol{V}_0(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y}, s) \mathrm{d}V + \int_{\partial D}\int_0^s \dot{\mathcal{V}}(\boldsymbol{x}-\boldsymbol{y}, s-t) \cdot \boldsymbol{s}(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S \\
&-\int_{\partial D}\int_0^s \dot{\mathcal{T}}(\boldsymbol{x}, \boldsymbol{y}, s-t) \cdot \boldsymbol{u}(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S - \int_{\partial D}\int_0^s Q(\boldsymbol{x}-\boldsymbol{y}, s-t)\, r(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S \\
&+\int_{\partial D}\int_0^s W(\boldsymbol{x}, \boldsymbol{y}, s-t)\, p(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S + \int_D \int_0^s \dot{\mathcal{V}}(\boldsymbol{x}-\boldsymbol{y}, s-t) \cdot \boldsymbol{f}(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}V \\
&+\int_D \int_0^s Q(\boldsymbol{x}-\boldsymbol{y}, s-t)\, g(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}V + \int_D Q(\boldsymbol{x}-\boldsymbol{x}, s)\, \theta(\boldsymbol{y}) \mathrm{d}V\,, \\
&\boldsymbol{x} \in \partial D \qquad (8)
\end{aligned}$$

where

$$(\tilde{\boldsymbol{u}}, \tilde{p}) = \begin{cases} \boldsymbol{u}, p & \boldsymbol{x} \in D \\ \boldsymbol{0}, 0 & \boldsymbol{x} \in D^e, \quad (D^e = R^N \backslash (D \cup \partial D)) \end{cases} \qquad (9)$$

$$\boldsymbol{U}_0 = \mathcal{F}_\xi^{-1}\left(\Delta^{*-1}(\boldsymbol{\xi}) - \frac{\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\right), \tag{10*}$$

$$\boldsymbol{V}_0 = \mathcal{F}_\xi^{-1}\left(\frac{-i\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\right), \tag{11}$$

$$\boldsymbol{P}_0 = \mathcal{F}_\xi^{-1}\left(\frac{-i\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\right), \tag{12}$$

$$Q_0 = \mathcal{F}_\xi^{-1}\left(\frac{1}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\right), \tag{13}$$

$$\dot{\mathcal{U}} = \mathcal{F}_\xi^{-1}\left(\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}\frac{\boldsymbol{\xi}\cdot\boldsymbol{K}\boldsymbol{\xi}}{(m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\,\boldsymbol{\xi})^2}\,\mathrm{ex}(\boldsymbol{\xi}, t)\right), \tag{14}$$

$$\dot{\mathcal{V}} = \mathcal{F}_\xi^{-1}\left(i\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}\frac{\boldsymbol{\xi}\cdot\boldsymbol{K}\boldsymbol{\xi}}{(m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\,\boldsymbol{\xi})^2}\,\mathrm{ex}(\boldsymbol{\xi}, t)\right), \tag{15}$$

$$\boldsymbol{P} = -i\mathcal{F}_\xi^{-1}\left(\frac{\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\,\mathrm{ex}(\boldsymbol{\xi}, t)\right), \tag{16}$$

$$Q = \mathcal{F}_\xi^{-1}\left(\frac{1}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\,\mathrm{ex}(\boldsymbol{\xi}, t)\right), \tag{17}$$

$$\begin{bmatrix} S_{0ij}(\boldsymbol{x}, \boldsymbol{y}, s) \\ T_{0j}(\boldsymbol{x}, \boldsymbol{y}, s) \end{bmatrix} = \frac{\partial}{\partial y_l}\begin{bmatrix} U_{0ik}(\boldsymbol{x}-\boldsymbol{y}, s) \\ V_{0k}(\boldsymbol{x}-\boldsymbol{y}, s) \end{bmatrix} C_{jmkl}\, n_m(\boldsymbol{y}) + \begin{bmatrix} P_{0i}(\boldsymbol{x}-\boldsymbol{y}, s) \\ Q_0(\boldsymbol{x}-\boldsymbol{y}, s) \end{bmatrix} n_j(\boldsymbol{y}), \tag{18}$$

$$\begin{bmatrix} \dot{\mathcal{S}}_{ij}(\boldsymbol{x}, \boldsymbol{y}, s) \\ \dot{\mathcal{T}}_j(\boldsymbol{x}, \boldsymbol{y}, s) \end{bmatrix} = \frac{\partial}{\partial y_l}\begin{bmatrix} \dot{\mathcal{U}}_{ik}(\boldsymbol{x}-\boldsymbol{y}, s) \\ \dot{\mathcal{V}}_k(\boldsymbol{x}-\boldsymbol{y}, s) \end{bmatrix} C_{jmkl}\, n_m(\boldsymbol{y}) + \begin{bmatrix} \dot{P}_i(\boldsymbol{x}-\boldsymbol{y}, s) \\ \dot{Q}(\boldsymbol{x}-\boldsymbol{y}, s) \end{bmatrix} n_j(\boldsymbol{y}), \tag{19}$$

and

$$\begin{bmatrix} R_i(\boldsymbol{x}, \boldsymbol{y}, s) \\ W(\boldsymbol{x}, \boldsymbol{y}, s) \end{bmatrix} = -\frac{\partial}{\partial y_j}\begin{bmatrix} P_i(\boldsymbol{x}-\boldsymbol{y}, s) \\ Q(\boldsymbol{x}-\boldsymbol{y}, s) \end{bmatrix} K_{jk}\, n_k(\boldsymbol{y}). \tag{20}$$

In these formulae, $\mathcal{F}_\xi^{-1}$ indicates the Fourier inverse transform ($\boldsymbol{\xi}\to\boldsymbol{x}$), $\Delta^{*-1}(\boldsymbol{\xi})$ the inverse of the matrix obtained by replacing ∇ in Δ^* by $\boldsymbol{\xi}$, and

$$\mathrm{ex}(\boldsymbol{\xi}, t) = e^{-G(\boldsymbol{\xi})t},\quad G(\boldsymbol{\xi}) = \frac{\boldsymbol{\xi}\cdot\boldsymbol{K}\boldsymbol{\xi}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}.$$

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* For $N=2$, one would have to interpret non-integrable integrals in the expression for $\boldsymbol{U}_0$ as the finite part.

Also, we have used s for time for convenience.

The physics of the problem tells that the data and the solution, except for $\partial p/\partial n$, are piecewise smooth in $(D \cup \partial D_R) \times (t>0)$, where

$$\partial D_R = (\partial D_u \cup \partial D_s) \cap (\partial D_p \cup \partial D_r) \, . \tag{21}$$

However, $\partial p/\partial n$ (or r) may have a singularity proportional to t^β on ∂D_R, where $\beta > -1$.

From a numerical analytical point of view, it is important to investigate the mathematical structure of this singularity. Indeed, we would not be able to establish an accurate boundary integral equation method (BIEM) based on (7) and (8) unless we could incorporate the effect of this singularity into the analysis. However, it would be reasonable to expect that (7) and (8) themselves would provide a clue to the understanding of this singularity because (7) and (8) are no less than the explicit forms of the solutions of (1). Motivated by this consideration, we investigate the behaviour of the integrals in (7) and (8) near ∂D for a small t. We use the method of the Fourier transform to this end, which enables us to consider the full anisotropic case. As a matter of fact, part of this investigation has been carried out by Nishimura & Kobayashi[5], where the potentials independent of time have been considered. Their potentials include those integrals in (7) and (8) whose kernel functions have suffix 0. Indeed, their results yield

$$\lim_{x \to x_0} \int_{\partial D} \boldsymbol{U}_0(\boldsymbol{x}-\boldsymbol{y})\, \boldsymbol{s}(\boldsymbol{y}, s)\, dS = \int_{\partial D} \boldsymbol{U}_0(\boldsymbol{x}_0-\boldsymbol{y})\, \boldsymbol{s}(\boldsymbol{y}, s) dS \, , \tag{22}$$

$$\lim_{x \to x_0} \int_{\partial D} \boldsymbol{S}_0(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{u}(\boldsymbol{y}, s) dS$$
$$= \pm \frac{1}{2} \boldsymbol{u}(\boldsymbol{x}_0, s) + \text{v.p.} \int_{\partial D} \boldsymbol{S}_0(\boldsymbol{x}_0-\boldsymbol{y})\, \boldsymbol{u}(\boldsymbol{y}, s) dS \, , \tag{23}$$

$$\lim_{x \to x_0} \int_{D} \boldsymbol{U}_0(\boldsymbol{x}-\boldsymbol{y})\, \boldsymbol{f}(\boldsymbol{y}, s) dV = \int_{D} \boldsymbol{U}_0(\boldsymbol{x}_0-\boldsymbol{y})\, \boldsymbol{f}(\boldsymbol{y}, s) dV \, , \tag{24}$$

$$\lim_{x \to x_0} \int_{D} \boldsymbol{P}_0(\boldsymbol{x}-\boldsymbol{y})\, \theta(\boldsymbol{y}) dV = \int_{D} \boldsymbol{P}_0(\boldsymbol{x}_0-\boldsymbol{y})\, \theta(\boldsymbol{y}) dV \, , \tag{25}$$

$$\lim_{x \to x_0} \int_{\partial D} \boldsymbol{V}_0(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{s}(\boldsymbol{y}, s) dS = \pm \frac{1}{2} \frac{\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}{m + \boldsymbol{n} \cdot \Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}} \cdot \boldsymbol{s}(\boldsymbol{x}_0, s)$$
$$+ \text{v.p.} \int_{\partial D} \boldsymbol{V}_0(\boldsymbol{x}_0-\boldsymbol{y}) \cdot \boldsymbol{s}(\boldsymbol{y}, s) dS \, , \tag{26}$$

$$\lim_{x \to x_0} \int_{\partial D} \boldsymbol{T}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{y}, s) dS$$

$$= \mp \frac{1}{2} \frac{\partial}{\partial \xi_\alpha} \left(\frac{\boldsymbol{C}[\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi}]\, \boldsymbol{n} - \boldsymbol{n}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}} \right)_j \Bigg|_{\boldsymbol{\xi}=\boldsymbol{n}} \frac{\partial u_j}{\partial x_\alpha}(\boldsymbol{x}_0, s)$$

$$+\text{p.f.} \int_{\partial D} \boldsymbol{T}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{y}, s) \mathrm{d}S$$

$$= \mp \frac{1}{2} \left(\frac{\operatorname{div} \boldsymbol{u}(\boldsymbol{x}_0, s) - \Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}\cdot \boldsymbol{C}[\nabla \boldsymbol{u}(\boldsymbol{x}_0, s)]\, \boldsymbol{n}}{m+\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}} \right)$$

$$+\text{p.f.} \int_{\partial D} \boldsymbol{T}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{y}, s) \mathrm{d}S\,, \tag{27*}$$

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} \int_D \boldsymbol{V}_0(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y}, s) \mathrm{d}V = \int_D \boldsymbol{V}_0(\boldsymbol{x}_0-\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y}, s) \mathrm{d}V\,, \tag{28}$$

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} \int_D Q_0(\boldsymbol{x}-\boldsymbol{y})\, \theta(\boldsymbol{y}) \mathrm{d}V = \frac{1}{2} \left(C_Q \mp \frac{1}{m+\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}} \right) \theta(\boldsymbol{x}_0)$$

$$+\text{v.p.}^- \int_D Q_0(\boldsymbol{x}_0-\boldsymbol{y})\, \theta(\boldsymbol{y}) \mathrm{d}V\,, \quad \boldsymbol{x}_0 \in \partial D \tag{29}$$

where the upper (lower) sign indicates the approach from the exterior (interior) of D, v.p.$^-$ the principal value integral defined by

$$\text{v.p.}^- \int_D \cdot\, \mathrm{d}V = \lim_{\varepsilon \downarrow 0} \int_{D \setminus B_\varepsilon(\boldsymbol{x}_0)} \cdot\, \mathrm{d}V\,, \tag{30}$$

$B_\varepsilon(\boldsymbol{x}_0)$ a ball having a radius of ε and centred at $\boldsymbol{x}_0$, and C_Q a number defined by

$$C_Q = \frac{1}{|S_N|} \int_{S_N} \mathcal{F}(Q_0) d\boldsymbol{\xi}\,, \quad \begin{pmatrix} S_N\text{: } N \text{ dimensional unit sphere} \\ \mathcal{F}\text{ : Fourier transform} \end{pmatrix} \tag{31}$$

respectively. In (27) we have used a cartesian frame whose origin is at $\boldsymbol{x}_0$ and whose Nth axis points in the direction of $\boldsymbol{n}(\boldsymbol{x}_0)$ (See Fig. 1.). Hence, in this paper we will

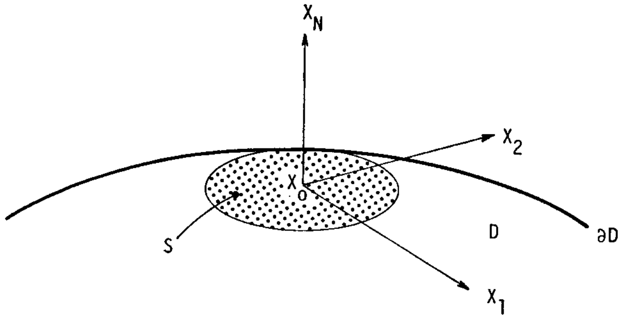

Fig. 1. Notation

* Greek index runs from 1 to $N-1$.

continue this subject by focusing on the integrals dependent on time. Specifically, we start by discussing an abstract problem, in which a class of potential functions including the time dependent integrals in (7) and (8) is considered. Use of the theory of the Fourier transform then determines the behaviour of these potential functions completely. We then proceed to the application of the obtained results to the specific potentials of interest. In particular, we establish a relation between the initial behaviour of p in (1) and the singularity of $\partial p/\partial n$ mentioned above. A few remarks concerning the implication of the present results in BIEM conclude this paper.

2. Statement of Problem

The motivation mentioned in the introduction suggests the computation of the following limits:

$$\lim_{s\downarrow 0}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_0^s\int_{\partial D}F(\boldsymbol{x}-\boldsymbol{y},s-t)\,\psi(\boldsymbol{y},t)dS_y\,dt\,, \tag{32}$$

$$\lim_{s\downarrow 0}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_0^s\int_{D}F(\boldsymbol{x}-\boldsymbol{y},s-t)\,\psi(\boldsymbol{y},t)dV_y\,dt\,, \tag{33}$$

$$\lim_{s\downarrow 0}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_{D}F(\boldsymbol{x}-\boldsymbol{y},s)\,\psi(\boldsymbol{y},0)dV_y\,, \tag{34}$$

where $\psi(\boldsymbol{y},t)$ is a density function having an asymptotic expansion

$$\psi(\boldsymbol{x},t)\sim\sum_i\varphi_i(\boldsymbol{x})\,t^{\beta_i}\quad\text{as }t\downarrow 0\,, \tag{35}$$

with exponents $-1<\beta_1<\beta_2<\cdots$, F is a kernel which has a (partial) Fourier transform

$$\int_{R^N}F(\boldsymbol{x},t)\,e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\,d\boldsymbol{x}=\hat{F}(\boldsymbol{\xi})\,e^{-G(\boldsymbol{\xi})t}\,, \tag{36}$$

and $\hat{F}$ and G are certain functions to be specified shortly. The reader should pay attention to the order of taking two limits in (32)–(34).

We next specify the forms of $\hat{F}$ and G by using the notion of (l, m) homogeneity: A function f is (l, m) homogeneous if

$$f(\lambda\boldsymbol{x})=\lambda^l|\lambda|^m f(\boldsymbol{x})$$

holds for $\lambda\in R$. With (10)–(20) in mind, we assume that G is (2, 0) homogeneous, and 'differentiable and positive' except at the origin. For $\hat{F}$, we require $(n, 0)$

homogeneity and smoothness, except at the origin. Finally, we set the parameters n and β_1 as in the following table:

	n	β_1	kernels (n)
surface-time integral (32)	$-1, 0$	$\beta_1 > -1$	$\boldsymbol{P}(-1)$, $\dot{\mathcal{U}}$, $\mathcal{Q}$, $\boldsymbol{R}(0)$
	$1, 2$	$\beta_1 = 0$	$\dot{\mathcal{V}}$, $\dot{\mathcal{S}}$, $W(1)$, $\dot{\mathfrak{T}}(2)$
volume-time integral (33)	$-1, 0, 1$	$\beta_1 = 0$	$\boldsymbol{P}(-1)$, $\dot{\mathcal{U}}$, $\mathcal{Q}(0)$, $\dot{\mathcal{V}}(1)$
volume integral (34)	$-1, 0$	$\beta_1 = 0$	$\boldsymbol{P}(-1)$, $\boldsymbol{Q}(0)$

It is readily shown that the above combinations cover all the possibilities one may encounter in the investigation of the integrals in (7) and (8). For example, one has to consider a negative β only in the surface-time integrals involving r in (7) and (8). Since these integrals have either $\boldsymbol{P}$ or $\mathcal{Q}$ as kernel functions, one may keep attention only to those cases with $n = -1, 0$.

3. Behaviour of Potentials

This section computes the limits in (32)–(34).

3.1 Surface-time integrals

We consider the limit shown in (32). As in Nishimura and Kobayashi[5], we assume that ∂D is locally plane and concentrate on the effect of the singularity of the kernel function F on (32). Since this singularity is localized at a boundary point $\boldsymbol{x}_0$, we may pay attention only to the contribution to (32) from the immediate vicinity of $\boldsymbol{x}_0$. This justifies our assumption that the domain of integration is R^{N-1}. Actually, we may approximate ψ by a function defined on a small planer disc S centred at $\boldsymbol{x}_0$ and tangent to ∂D, followed by putting $\psi = 0$ in $R^{N-1} \backslash S$. It is then convenient to use the cartesian frame shown in Fig. 1. With these tools thus introduced, we are now ready to investigate the limit in (32) by considering the following partial Fourier transform with respect to x_α*:

$$\lim_{s \downarrow 0} \lim_{x_N \to 0} \hat{E}_\beta(\xi_\alpha | x_N, s)$$

* Greek index runs from 1 to $N-1$.

$$= \lim_{s\downarrow 0} \lim_{x_N \to 0} \frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \int_0^s t^\beta \mathrm{d}t \int_{-\infty}^{\infty} e^{i\xi_N x_N - \varepsilon \xi_N^2} \hat{F}(\xi_\alpha, \xi_N)\, e^{-G(\xi_\alpha, \xi_N)(s-t)} \mathrm{d}\xi_N\,, \tag{37}$$

where β is one of the β_i's in (35) (β_1 in most cases). The limit in (32) is then obtained by multiplying (37) by the $N-1$ dimensional Fourier transform of ψ, followed by a Fourier inversion.

To begin with, we prepare a

Proposition. Let $\hat{F}$ be $(n, 0)$ homogeneous, with either $n=-1$, or '$n=0$ and $\hat{F}(\mathbf{0}, 1)=0$.' Also, let G be a positive $(2, 0)$ homogeneous function. Furthermore, $\hat{F}$ and G are assumed to be continuously differentiable except at the origin. We then have

$$\lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\infty} \hat{F}(\xi_\alpha, \xi_N)\, e^{-G(\xi_\alpha, \xi_N)\varepsilon} \mathrm{d}\xi_N$$
$$= \text{v.p.} \int_{-\infty}^{\infty} \left(\hat{F}(\xi_\alpha, \xi_N) - \frac{1}{\xi_N} H(\xi_\alpha)\right) \mathrm{d}\xi_N\,, \tag{38}$$

where H is a certain function independent of G. Hence, this limit is independent of the form of G. In particular,

$$\lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\infty} \hat{F}(\xi_\alpha, \xi_N)\, e^{-G(\xi_\alpha, \xi_N)\varepsilon} \mathrm{d}\xi_N = \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\infty} \hat{F}(\xi_\alpha, \xi_N)\, e^{-\varepsilon \xi_N^2} \mathrm{d}\xi_N\,.$$

Proof. We have

$$\int_{-\infty}^{\infty} \hat{F}(\xi_\alpha, \xi_N)\, e^{-G(\xi_\alpha, \xi_N)\varepsilon} \mathrm{d}\xi_N$$
$$= \text{v.p.} \int_{-\infty}^{\infty} \left(\hat{F}(\xi_\alpha, \xi_N) - \frac{1}{\xi_N} H(\xi_\alpha)\right) e^{-G(\xi_\alpha, \xi_N)\varepsilon}\, \mathrm{d}\xi_N$$
$$+ H(\xi_\alpha) \text{v.p.} \int_{-\infty}^{\infty} \frac{e^{-G(\xi_\alpha, \xi_N)\varepsilon}}{\xi_N} \mathrm{d}\xi_\alpha\,, \tag{39}$$

where v.p. stands for Cauchy's principal value, and

$$H(\xi_\alpha) = \begin{cases} \hat{F}(\mathbf{0}, 1) & (n=-1)\,, \\ \xi_\beta\, \partial/\partial\xi_\beta\, \hat{F}(\boldsymbol{\xi})\,|_{(\xi_\alpha, \xi_N)=(0,1)} & (n=0)\,. \end{cases}$$

The first integral on the RHS of (39) tends to the RHS of (38) as $\varepsilon \downarrow 0$, because the expression in the parentheses in (39) is $O(1/\xi_N^2)$ as $|\xi_N| \to \infty$. The integral in the second term on the RHS of (39) is rewritten as

$$\text{v.p.} \int_{-\infty}^{\infty} \frac{e^{-G(\xi_\alpha, \xi_N)\varepsilon}}{\eta} \mathrm{d}\eta = \int_0^{\infty} \frac{e^{-G(\sqrt{\varepsilon}\xi_\alpha, \eta)} - e^{-G(-\sqrt{\varepsilon}\xi_\alpha, \eta)}}{\eta} \mathrm{d}\eta\,,$$

which tends to zero as $\varepsilon \downarrow 0$ due to the assumption on G. This concludes the proof.

We now compute (32) for various combinations of n and β:

i) $n=-1, 0, \beta>-1$. We first rewrite the time integral in (37) into

$$\int_0^s e^{-G(\xi)(s-t)} t^\beta \mathrm{d}t = e^{-G(\xi)s} s^{1+\beta} \int_0^1 e^{G(\xi)su} u^\beta \mathrm{d}u\,. \quad (\beta>-1) \tag{40}$$

The asymptotic expansion for the incomplete gamma function then shows that the last integral in (40) is of the order of $s^\beta/G(\boldsymbol{\xi})$ for a large $|\boldsymbol{\xi}|$. We are thus justified (by Lebesgue's theorem) to write (37) as

$$\lim_{s\downarrow 0} \lim_{x_N\to 0} \hat{E}_\beta(\xi_\alpha | x_N, s) = \frac{1}{2\pi} \lim_{s\downarrow 0} \int_{-\infty}^{\infty} \mathrm{d}\xi_N \int_0^s \hat{F}(\xi_\alpha, \xi_N)\, e^{-G(\xi_\alpha, \xi_N)(s-t)} t^\beta \mathrm{d}t\,, \tag{41}$$

because the integrand (as a function of ξ_N) is of order $n-2$ ($\leqq -2$) at infinity. In passing, we present two other forms for the integral in (41), i.e.

$$s^{1+\beta} \int_0^1 u^\beta \mathrm{d}u \int_{-\infty}^{\infty} \hat{F}(\boldsymbol{\xi})\, e^{-G(\boldsymbol{\xi})s(1-u)} \mathrm{d}\xi_N \tag{42}$$

and

$$s^{1/2+\beta-n/2} \int_{-\infty}^{\infty} \mathrm{d}\eta \int_0^1 \hat{F}(\sqrt{s}\,\xi_\alpha, \eta)\, e^{-G(\sqrt{s}\xi_\alpha, \eta)(1-u)} u^\beta \mathrm{d}u\,, \tag{43}$$

for later use.

When $n=-1$ the proposition and (42) give

$$|\hat{E}_\beta(\xi_\alpha | 0, s)| \leqq s^{1+\beta} C(\xi_\alpha)\,,$$

where $C(\xi_\alpha)$ is a constant dependent on ξ_α but not on s. This means $\lim_{s\downarrow 0} \hat{E}_\beta(\xi_\alpha | 0, s)=0$ for $n=-1$ since $\beta>-1$. Hence, we conclude

$$\lim_{s\downarrow 0} \lim_{x\to x_0} \int_0^s \int_S F(\boldsymbol{x}-\boldsymbol{y}, s-t)\, \varphi(\boldsymbol{y})\, t^\beta \mathrm{d}S_y\, \mathrm{d}t = 0 \tag{44}$$

for $n=-1$ and $\beta>-1$ (see (36) and (37)), where φ indicates φ_1 in (35).

For $n=0$, (43) yields

$$\lim_{s\downarrow 0} s^{-(1/2+\beta)} \hat{E}_\beta(\xi_\alpha | 0, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\eta \int_0^1 \hat{F}(\mathbf{0}, \eta)\, e^{-G(\mathbf{0}, \eta)(1-u)} u^\beta \mathrm{d}u$$

$$= \frac{1}{2(\pi)^{1/2}} \frac{\hat{F}(\mathbf{0}, 1)}{(G(\mathbf{0}, 1))^{1/2}} B\left(\beta+1, \frac{1}{2}\right),$$

where $B(\cdot, \cdot)$ is the Beta function. This equation then shows that

$$\lim_{s\downarrow 0} \hat{E}_\beta(\xi_\alpha | 0, s)$$

is finite only if $\beta \geqq -1/2$ for a non-zero $\hat{F}(\mathbf{0}, 1)$. Moreover, this limit is non-zero only if $\beta = -1/2$, in which case we have

$$\lim_{s\downarrow 0} \hat{E}_\beta(\xi_\alpha | 0, s) = \frac{\sqrt{\pi}}{2} \frac{\hat{F}(\mathbf{0}, 1)}{(G(\mathbf{0}, 1))^{1/2}}. \quad \left(n = 0,\ \beta = -\frac{1}{2}\right)$$

Hence, we have the following result:

$$\lim_{s\downarrow 0} \lim_{x \to x_0} \int_0^s \int_S \hat{F}(\boldsymbol{x}-\boldsymbol{y}, s-t)\, \varphi(\boldsymbol{y})\, t^\beta dS_y\, dt$$

$$= \begin{cases} (\pi)^{1/2} \hat{F}(\boldsymbol{n})\, \varphi(\boldsymbol{x}_0)/2(G(\boldsymbol{n}))^{1/2}, & (\beta = -1/2) \\ 0. & (\beta > -1/2) \end{cases} \tag{45}$$

The limit in (45) cannot be finite for $\beta < -1/2$ in general.

ii) $n=1, 2$, $\beta=0$. We first perform the time integration in (37) to obtain

$$\hat{E}_\beta(\xi_\alpha | x_N, s) = \frac{1}{2(\pi)^{1/2}} \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\infty} e^{i\xi_N x_N - \varepsilon\xi_N^2} \left(\frac{\hat{F}(\boldsymbol{\xi})}{G(\boldsymbol{\xi})} - \frac{\hat{F}(\boldsymbol{\xi})\, e^{-G(\boldsymbol{\xi})s}}{G(\boldsymbol{\xi})} \right) d\xi_N. \tag{46}$$

An additional assumption, $\hat{F}(\mathbf{0}, 1) = 0$ for $n=2$, which our particular potentials of interest will be seen to satisfy, then transforms the first term on the RHS of (46) into

$$\lim_{x_N\to 0} \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\infty} e^{i\xi_N x_N - \varepsilon\xi_N^2} \frac{\hat{F}(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}\, d\xi_N$$

$$= \pm \hat{H}(\xi_\alpha) + \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\infty} e^{-\varepsilon\xi_N^2} \frac{\hat{F}(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}\, d\xi_N, \tag{47}$$

where

$$\hat{H}(\xi_\alpha) = \begin{cases} i(\hat{F}(\boldsymbol{\xi})/2G(\boldsymbol{\xi}))\,|_{(\xi_\alpha, \xi_N)=(0,1)}, & (n=1) \\ i\,\xi_\alpha(\partial/\partial\xi_\alpha)(\hat{F}(\boldsymbol{\xi})/2G(\boldsymbol{\xi}))\,|_{(\xi_\alpha, \xi_N)=(0,1)}, & (n=2) \end{cases} \tag{48}$$

and the $\pm$ is for the approach from $x_N > 0$ (upper) and $x_N > 0$ (lower)*. For a proof of (47) and (48) the reader is referred to Nishimura & Kobayashi[5].

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* This convention is used throughout this paper.

We next apply Lebesgue's theorem to the second term on the RHS of (46) to show

$$\lim_{x_N\to 0}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty} e^{i\xi_N x_N-\varepsilon\xi_N^2}\frac{\hat F(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}e^{-G(\boldsymbol{\xi})s}\mathrm{d}\xi_N=\int_{-\infty}^{\infty}\frac{\hat F(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}e^{-G(\boldsymbol{\xi})s}\mathrm{d}\xi_N\,.$$

This result and the proposition then prove

$$\begin{aligned}&\lim_{s\downarrow 0}\lim_{x_N\to 0}\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty} e^{i\xi_N x_N-\varepsilon\xi_N^2}\frac{\hat F(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}e^{-G(\boldsymbol{\xi})s}\,\mathrm{d}\xi_N\\&=\lim_{s\downarrow 0}\int_{-\infty}^{\infty}\frac{\hat F(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}e^{-G(\boldsymbol{\xi})s}\,\mathrm{d}\xi_N=\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\frac{\hat F(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}e^{-\varepsilon\xi_N^2}\,\mathrm{d}\xi_N\,.\end{aligned}\tag{49}$$

Hence, by combining (46), (47) and (49) we obtain

$$\lim_{s\downarrow 0}\lim_{x_N\to 0}\hat E_\beta(\xi_\alpha|x_N,s)=\pm\hat H(\xi_\alpha)\,.\tag{50}$$

Consequently, we have

$$\begin{aligned}&\lim_{s\downarrow 0}\lim_{x_N\to 0}\int_0^s\int_S F(\boldsymbol{x}-\boldsymbol{y},s-t)\,\varphi(\boldsymbol{y})\,\mathrm{d}S_y\mathrm{d}t\\&=\begin{cases}\pm(i/2)(\hat F(\boldsymbol{n})/G(\boldsymbol{n}))\,\varphi(\boldsymbol{x}_0)\,, & n=1\\ \pm(1/2)(\partial/\partial\xi_\alpha)(\hat F/G)\,|_{\boldsymbol{\xi}=\boldsymbol{n}}\,(\partial/\partial\xi_\alpha)\,\varphi(\boldsymbol{x}_0)\,, & n=2\end{cases}\end{aligned}\tag{51}$$

if the condition

$$\hat F(\boldsymbol{n})=0\tag{52}$$

is satisfied for $n=2$, where we have used (48) and (50). We remark that the condition in (52) is essential in the present application because of the e^{-Gt} term in (36). This is in contrast to the elastostatics case (see Nishimura & Kobayashi[5)]) where we did not have to assume (52).

One may wonder if terms with $\beta>0$ might give rise to any additional non-zero term to (32). That this is impossible is seen by using

$$\begin{aligned}&\hat E_\beta(\xi_\alpha|x_N,s)\\&=\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty} e^{i\xi_N x_N-\varepsilon\xi_N^2}\left(s^\beta\frac{\hat F(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}-\beta\frac{\hat F(\boldsymbol{\xi})}{G(\boldsymbol{\xi})}\int_0^s\frac{e^{-G(\boldsymbol{\xi})(s-1)}}{t^{1-\beta}}\,\mathrm{d}t\right),\end{aligned}\tag{53}$$

which one obtains from (37) and

$$\int_0^s e^{-G(\boldsymbol{\xi})(s-t)}t^\beta\,\mathrm{d}t=\frac{s^\beta}{G(\boldsymbol{\xi})}-\frac{\beta}{G(\boldsymbol{\xi})}\int_0^s\frac{e^{-G(\boldsymbol{\xi})(s-t)}}{t^{1-\beta}}\,\mathrm{d}t\,.$$

The first term in the integral in (53) is evaluated by using (47). Since $\beta>0$, however, this term vanishes as $s\downarrow 0$. For the second term, we use the proposition and (42) (with β replaced by $\beta-1$) to see that this also vanishes as $s\downarrow 0$ ($\hat{F}(\mathbf{0}, 1)=0$ for $n=2$ by assumption). This completes the proof.

3.2 Volume-time integrals

The method used in 3.1 yields

$$\lim_{s\downarrow 0}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_0^s\int_D F(\boldsymbol{x}-\boldsymbol{y},\ s-t)\varphi(\boldsymbol{y})t^{\beta}\mathrm{d}y\mathrm{d}t = 0\,.$$

$$\beta\geqq 0,\ -1\leqq n\leqq 1,\ \boldsymbol{x}_0\in\partial D \tag{54}$$

We shall, however, omit the proof in order to avoid repetition.

3.3 Volume integrals

In this section, we shall use $\boldsymbol{P}$ and Q (see (12), (13), (16) and (17)) for arbitrary (1, 0) and (0, 0) homogeneous kernels, respectively, in order to save symbols.

Again, the same reason as has been used in 3.1 gives

$$\lim_{s\downarrow 0}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_D \boldsymbol{P}(\boldsymbol{x}-\boldsymbol{y},\ s)\theta(\boldsymbol{y})\mathrm{d}V = \int_D \boldsymbol{P}_0(\boldsymbol{x}_0-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V, \tag{55}$$

$$\lim_{s\downarrow 0}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_D Q(\boldsymbol{x}-\boldsymbol{y},\ s)\theta(\boldsymbol{y})\mathrm{d}V = \frac{C_Q}{2}\theta(\boldsymbol{x}_0)+\text{v.p.}^-\int_D Q_0(\boldsymbol{x}_0-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V,$$

$$\boldsymbol{x}_0\in\partial D \tag{56}$$

where v.p.$^-$ and C_Q are defined in (40) and (41), respectively. Also, we can show

$$\lim_{s\downarrow 0}\int_D \boldsymbol{P}(\boldsymbol{x}-\boldsymbol{y},\ s)\theta(\boldsymbol{y})\mathrm{d}V = \int_D \boldsymbol{P}_0(\boldsymbol{x}-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V, \tag{57}$$

$$\lim_{s\downarrow 0}\int_D Q(\boldsymbol{x}-\boldsymbol{y},\ s)\theta(\boldsymbol{y})\mathrm{d}V = C_Q\theta(\boldsymbol{x})+\text{v.p.}\int_D Q_0(\boldsymbol{x}-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V, \tag{58}$$

for a point $\boldsymbol{x}\in D$.

4. Potentials in Consolidation and Coupled Thermoelasticity

Using the results given in (44), (45), (51) and (54) we obtain the following formulae in addition to (55)–(58):

$$\lim_{s\downarrow 0}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_{\partial D}\int_0^s \dot{\mathcal{U}}(\boldsymbol{x}-\boldsymbol{y},\ s-t)\ \boldsymbol{s}(\boldsymbol{y},\ t)\mathrm{d}t\mathrm{d}S = \mathbf{0}\,, \tag{59}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{\partial D}\int_0^s \dot{\mathcal{S}}(\boldsymbol{x},\boldsymbol{y},s-t)\boldsymbol{u}(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}S=\boldsymbol{0}\,, \tag{60}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{\partial D}\int_0^s \boldsymbol{P}(\boldsymbol{x}-\boldsymbol{y},s-t)r(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}S=\boldsymbol{0}\,, \tag{61}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{\partial D}\int_0^s \boldsymbol{R}(\boldsymbol{x},\boldsymbol{y},s-t)p(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}S=\boldsymbol{0}\,, \tag{62}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{D}\int_0^s \dot{\mathcal{U}}(\boldsymbol{x}-\boldsymbol{y},s-t)\boldsymbol{f}(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}V=\boldsymbol{0}\,, \tag{63}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{D}\int_0^s \boldsymbol{P}(\boldsymbol{x}-\boldsymbol{y},s-t)g(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}V=\boldsymbol{0}\,, \tag{64}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{\partial D}\int_0^s \dot{\mathcal{V}}(\boldsymbol{x}-\boldsymbol{y},s-t)\cdot\boldsymbol{s}(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}S$$
$$=\mp\frac{1}{2}\left(\frac{\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}{m+\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}\right)\cdot\boldsymbol{s}(\boldsymbol{x}_0,0)\,, \tag{65}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{\partial D}\int_0^s \dot{\mathcal{Q}}(\boldsymbol{x},\boldsymbol{y},s-t)\cdot\boldsymbol{u}(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}S$$
$$=\pm\frac{1}{2}\frac{\partial}{\partial\xi_\alpha}\left(\frac{\boldsymbol{C}(\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi})\boldsymbol{n}-\boldsymbol{n}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\right)_j\Bigg|_{\boldsymbol{\xi}=\boldsymbol{n}}\frac{\partial u_j}{\partial x_\alpha}(\boldsymbol{x}_0)\,, \tag{66}$$

(Note that (52) is satisfied.)

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{\partial D}\int_0^s Q(\boldsymbol{x}-\boldsymbol{y},s-t)r(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}S$$
$$=\begin{cases}\text{divergent}, & -1<\beta<-1/2\\ -(\pi\boldsymbol{n}\cdot\boldsymbol{K}\boldsymbol{n})^{1/2}r(\boldsymbol{x}_0)/2(m+\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n})^{1/2}, & \beta=1/2\\ 0 & -1/2<\beta\,,\end{cases} \tag{67}$$

where β and $r(x_0)$ are the exponent of the lowest power and the corresponding coefficient of the asymptotic expansion of $(\partial p/\partial n)\,(\boldsymbol{x}_0,t)$ near $t=0$, i.e.,

$$\frac{\partial p}{\partial n}(\boldsymbol{x}_0,t)=t^\beta r(\boldsymbol{x}_0)+o(t^\beta)\qquad\text{as}\quad t\downarrow 0\,,$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{\partial D}\int_0^s W(\boldsymbol{x},\boldsymbol{y},s-t)p(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}S=\mp\frac{1}{2}p(\boldsymbol{x}_0,0)\,, \tag{68}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{D}\int_0^s \dot{\mathcal{V}}(\boldsymbol{x}-\boldsymbol{y},s-t)\cdot\boldsymbol{f}(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}V=0\,, \tag{69}$$

$$\lim_{s\downarrow 0}\lim_{x\to x_0}\int_{D}\int_0^s Q(\boldsymbol{x}-\boldsymbol{y},s-t)g(\boldsymbol{y},t)\mathrm{d}t\mathrm{d}V=0\,. \tag{70}$$

For the definition of symbols, see (7), (8) and (10)–(20).

5. Behaviour of Solution on the Boundary

In this section, we discuss the initial behaviour of the solution of (1) on the boundary. We shall begin by comparing the limits

$$(\boldsymbol{u}(\boldsymbol{x}_0, 0), p(\boldsymbol{x}_0, 0)) := \lim_{s\downarrow 0} \lim_{\boldsymbol{x}(\in D)\to \boldsymbol{x}_0} (\boldsymbol{u}(\boldsymbol{x}, s), p(\boldsymbol{x}, s)) \tag{71}$$

and

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} (\boldsymbol{u}(\boldsymbol{x}, 0), p(\boldsymbol{x}, 0) := \lim_{\boldsymbol{x}(\in D)\to\boldsymbol{x}_0} \lim_{s\downarrow 0} (\boldsymbol{u}(\boldsymbol{x}, s), p(\boldsymbol{x}, s)) \tag{72}$$

for a point $\boldsymbol{x}_0 \in \partial D$.

The limit in (71) for $\boldsymbol{u}$ on ∂D_R (see (21)) is evaluated as

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x}_0, 0) &:= \lim_{s\downarrow 0} \boldsymbol{u}(\boldsymbol{x}_0, s) = 2\Big(\int_{\partial D} \boldsymbol{U}_0(\boldsymbol{x}_0-\boldsymbol{y})\boldsymbol{s}(\boldsymbol{y}, 0)\mathrm{d}S \\ &-\text{v.p.}\int_{\partial D} \boldsymbol{S}_0(\boldsymbol{x}_0, \boldsymbol{y})\boldsymbol{u}(\boldsymbol{y}, 0)\mathrm{d}S + \int_D \boldsymbol{U}_0(\boldsymbol{x}_0-\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y}, 0)\mathrm{d}V \\ &+\int_D \boldsymbol{P}_0(\boldsymbol{x}_0-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V\Big), \qquad \boldsymbol{x}_0\in\partial D_R\end{aligned} \tag{73}$$

where we have used (7), (22)–(24), (59)–(64) and (55). Also, (71) for p is

$$\begin{aligned}p(\boldsymbol{x}_0, 0) &:= \lim_{s\downarrow 0} p(\boldsymbol{x}_0, s) \\ &= 2\Big(\text{v.p.}\int_{\partial D} \boldsymbol{V}_0(\boldsymbol{x}_0-\boldsymbol{y})\cdot\boldsymbol{s}(\boldsymbol{y}, 0)\mathrm{d}S - \text{p.f.}\int_{\partial D} \boldsymbol{T}_0(\boldsymbol{x}_0, \boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y}, 0)\mathrm{d}S \\ &+\int_D \boldsymbol{V}_0(\boldsymbol{x}_0-\boldsymbol{y})\cdot\boldsymbol{f}(\boldsymbol{y}, 0)\mathrm{d}V + \frac{C_Q}{2}\theta(\boldsymbol{x}_0) + \text{v.p.}^-\int_D Q_0(\boldsymbol{x}_0-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V \\ &-\lim_{s\downarrow 0}\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_{\partial D}\int_0^s Q(\boldsymbol{x}-\boldsymbol{y}, s-t)r(\boldsymbol{y}, t)\mathrm{d}t\mathrm{d}S\Big), \qquad \boldsymbol{x}_0\in\partial D_R\end{aligned} \tag{74}$$

where p.f. indicates the finite part of a divergent integral. In deriving (74) we have used (8), (26)–(28), (65)–(70) and (56). For the computation of the limit in (72) we start with the formula

$$\begin{aligned}\tilde{\boldsymbol{u}}(\boldsymbol{x}, 0) &= \lim_{s\to 0} \tilde{\boldsymbol{u}}(\boldsymbol{x}, s) = \int_{\partial D} \boldsymbol{U}_0(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{s}(\boldsymbol{y}, 0)\mathrm{d}S \\ &-\int_{\partial D} \boldsymbol{S}_0(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{u}(\boldsymbol{y}, 0)\mathrm{d}S + \int_0 \boldsymbol{U}_0(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y}, 0)\mathrm{d}V \\ &+\int_D \boldsymbol{P}_0(\boldsymbol{x}-\boldsymbol{y})\theta(\boldsymbol{y}, 0)\mathrm{d}V,\end{aligned} \tag{75}$$

for $\boldsymbol{x}\in\partial D$, which one obtains from (7) and (57). We then let $\boldsymbol{x}$ tend to $\boldsymbol{x}_0$ in (75), using (22)–(25), to have

$$\lim_{x\to x_0}\tilde{\boldsymbol{u}}(\boldsymbol{x},0)=\mp\frac{1}{2}\boldsymbol{u}(\boldsymbol{x}_0,0)+\int_{\partial D}\boldsymbol{U}_0(\boldsymbol{x}_0-\boldsymbol{y})\boldsymbol{s}(\boldsymbol{y},0)\mathrm{d}S$$
$$-\text{v.p.}\int_{\partial D}\boldsymbol{S}_0(\boldsymbol{x}_0,\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y},0)\mathrm{d}S+\int_D\boldsymbol{U}_0(\boldsymbol{x}_0-\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y},0)\mathrm{d}V$$
$$+\int_D\boldsymbol{P}_0(\boldsymbol{x}_0-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V.\qquad \boldsymbol{x}_0\in D\partial_R \tag{76}$$

Similarly, (8), together with (26)–(29) and (58), gives

$$\lim_{x\to x_0}\tilde{p}(\boldsymbol{x},0)=\pm\frac{1}{2}\frac{\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}{m+\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}\cdot\boldsymbol{s}(\boldsymbol{x}_0,0)$$
$$\pm\frac{1}{2}\frac{\partial}{\partial\xi_\alpha}\left(\frac{\boldsymbol{C}(\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi})\boldsymbol{n}-\boldsymbol{n}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\right)_j\Bigg|_{\xi=n}\frac{\partial u_j}{\partial x_\alpha}(\boldsymbol{x}_0,0)$$
$$+\left(\frac{C_Q}{2}\mp\frac{1}{2}\frac{1}{m+\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}\right)\theta(\boldsymbol{x}_0)$$
$$+\text{v.p.}\int_{\partial D}\boldsymbol{V}_0(\boldsymbol{x}_0-\boldsymbol{y})\cdot\boldsymbol{s}(\boldsymbol{y},0)\mathrm{d}S-\text{p.f.}\int_{\partial D}\boldsymbol{T}_0(\boldsymbol{x}_0,\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y},0)\mathrm{d}S$$
$$+\int_D\boldsymbol{V}_0(\boldsymbol{x}_0-\boldsymbol{a})\boldsymbol{f}(\boldsymbol{y},0)\mathrm{d}V+\text{v.p.}^-\int_D Q_0(\boldsymbol{x}_0-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V.$$
$$\boldsymbol{x}_0\in\partial D_R \tag{77}$$

In (76) and (77), the approach $\boldsymbol{x}\to\boldsymbol{x}_0$ may be either from within D or from within D^e (See (9).) Since $\tilde{\boldsymbol{u}}(\boldsymbol{x},0)=\boldsymbol{0}$ for $\boldsymbol{x}\in D^e$ from (7) and (9), we obtain the limit in (72) for $\boldsymbol{u}$ as

$$\lim_{x(\in D)\to x_0}\boldsymbol{u}(\boldsymbol{x},0)=\lim_{x(\in D)\to x_0}\tilde{\boldsymbol{u}}(\boldsymbol{x},0)+\lim_{x(\in D^e)\to x_0}\tilde{\boldsymbol{u}}(\boldsymbol{x},0)$$
$$=2\left(\int_{\partial D}\boldsymbol{U}_0(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{s}(\boldsymbol{y},0)\,\mathrm{d}S-\text{v.p.}\int_{\partial D}\boldsymbol{S}_0(\boldsymbol{x}_0,\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y},0)\mathrm{d}S\right.$$
$$\left.+\int_D\boldsymbol{U}_0(\boldsymbol{x}_0-\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y},0)\,\mathrm{d}V+\int_D\boldsymbol{P}_0(\boldsymbol{x}_0-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V\right),$$
$$\boldsymbol{x}_0\in\partial D_R \tag{78}$$

where we have used (76). Analogously, we use (77) to obtain

$$\lim_{x(\in D)\to x_0}p(\boldsymbol{x},0)=2\left(\frac{C_Q}{2}\theta(\boldsymbol{x}_0)+\text{v.p.}\int_{\partial D}\boldsymbol{V}_0(\boldsymbol{x}_0-\boldsymbol{y})\cdot\boldsymbol{s}(\boldsymbol{y},0)\mathrm{d}S\right.$$
$$-\text{p.f.}\int_{\partial D}\boldsymbol{T}_0(\boldsymbol{x}_0,\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y},0)\,\mathrm{d}S+\int_D\boldsymbol{V}_0(\boldsymbol{x}_0-\boldsymbol{y})\cdot\boldsymbol{f}(\boldsymbol{y},0)\mathrm{d}V$$
$$\left.+\text{v.p.}^-\int_D Q_0(\boldsymbol{x}_0-\boldsymbol{y})\theta(\boldsymbol{y})\mathrm{d}V\right).\qquad \boldsymbol{x}_0\in\partial D_R \tag{79}$$

Hence, we have obtained expressions for the limits in (71) and (72).

We now proceed to the comparison between the two limits in (71) and (72). From (73) and (78), or directly from (76), we have

$$\lim_{x(\in D)\to x_0} \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}(\boldsymbol{x}_0, 0)\,. \qquad \boldsymbol{x}_0 \in \partial D_R \tag{80}$$

Also, an analogous comparison between (74) and (79) yields

$$\lim_{x(\in D)\to x_0} p(\boldsymbol{x}, 0) = p(\boldsymbol{x}_0, 0) + 2 \lim_{s\downarrow 0} \lim_{x\to x_0} \int_{\partial D} \int_0^s Q(\boldsymbol{x}-\boldsymbol{y}, s-t) r(\boldsymbol{y}, t) \mathrm{d}t \mathrm{d}S\,. \qquad \boldsymbol{x}_0 \in \partial D_R \tag{81}$$

Equation (80) is a natural consequence, meaning that the limit to the boundary of the initial displacement coincides with the limit to $t=0$ of the boundary displacement. On the other hand, (81), together with (67), rules out the possibility of $-1<\beta<-1/2$, because p has to be finite. Also, we see that (81) reduces to

$$\lim_{x(\in D)\to x_0} p(\boldsymbol{x}, 0) = p(\boldsymbol{x}_0, 0) - \left(\frac{\pi \boldsymbol{n}\cdot \boldsymbol{K}\boldsymbol{n}}{m+\boldsymbol{n}\cdot \Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}\right)^{1/2} \gamma(\boldsymbol{x}_0)\,, \tag{82}$$

with the help of (67), where

$$\gamma(\boldsymbol{x}_0) = \lim_{s\downarrow 0} \frac{\partial p}{\partial n}(\boldsymbol{x}_0, s)\sqrt{s}\,. \tag{83}$$

Hence, we conclude

$$\beta = -\frac{1}{2}$$

as far as the limits

$$\lim_{x(\in D)\to x_0} p(\boldsymbol{x}, 0) \quad \text{and} \quad \lim_{s\downarrow 0} p(\boldsymbol{x}_0, s) \tag{84}$$

are both finite and different. In this case, the coefficient γ of the $s^{-1/2}$ singularity in $\partial p/\partial n$ (see (83)) is obtained from (82). If $\beta>-1/2$ holds, or, in particular, if $\boldsymbol{x}_0 \in \partial D_r$ and $\gamma_0(\boldsymbol{x}_0, t)$ is bounded as a function of t (see (6)), we necessarily have

$$\lim_{x(\in D)\to x_0} p(\boldsymbol{x}, 0) = p(\boldsymbol{x}_0, 0)\,, \qquad \boldsymbol{x}_0 \in \partial D_R$$

because $\gamma(\boldsymbol{x}_0)=0$ in (82) and (83).

We finally remark that the relation

$$\lim_{x(\in D)\to x_0} p(\boldsymbol{x}, 0) \neq p(\boldsymbol{x}_0, 0) \tag{85}$$

is not as queer as it might seem. Actually, we can show that

$$\lim_{\boldsymbol{x}(\in D)\to \boldsymbol{x}_0} p(\boldsymbol{x}, 0)$$

is independent of p_0 or r_0. (See (5) and (6).) Indeed, (77) and (9) yield

$$\begin{aligned}\lim_{\boldsymbol{x}(\in D)\to \boldsymbol{x}_0} p(\boldsymbol{x}, 0) &= \lim_{\boldsymbol{x}(\in D)\to \boldsymbol{x}_0} \tilde{p}(\boldsymbol{x}, 0) - \lim_{\boldsymbol{x}(\in D^e)\to \boldsymbol{x}_0} \tilde{p}(\boldsymbol{x}, 0) \\ &= -\frac{\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}{m+\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}\cdot \boldsymbol{s}(\boldsymbol{x}_0, 0) \\ &\quad -\frac{\partial}{\partial \xi_\alpha}\left(\frac{\boldsymbol{C}(\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi})\boldsymbol{n}-\boldsymbol{n}}{m+\boldsymbol{\xi}\cdot\Delta^{*-1}(\boldsymbol{\xi})\boldsymbol{\xi}}\right)_j\Bigg|_{\boldsymbol{\xi}=\boldsymbol{n}} \frac{\partial u_j}{\partial x_\alpha}(\boldsymbol{x}_0, 0) \\ &\quad + \frac{\theta(\boldsymbol{x}_0)}{m+\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}, \end{aligned} \tag{86}$$

which tells that this limit is determined by $\boldsymbol{u}(\boldsymbol{x}, 0)$, $\boldsymbol{s}(\boldsymbol{x}, 0)$ and θ on ∂D. On the other hand, $\boldsymbol{u}(\boldsymbol{x}, 0)$ ($\boldsymbol{s}(\boldsymbol{x}, 0)$) on ∂D_s (∂D_u) is determined by an integral equation obtained from (3), (4), (9) and the exterior limit in (76). Hence the limit in (76) is determined only by $\boldsymbol{u}_0$, $\boldsymbol{s}_0$, $\boldsymbol{f}$ and θ. (See (1)–(4).) On the other hand, we are supposed to specify $p(\boldsymbol{x}_0, 0)=p_0$ on ∂D_p arbitrarily. Therefore, $p(\boldsymbol{x}_0, 0)$ on ∂D_p is independent of the data in (2)–(4) and, hence, independent of (86). Therefore, we generally have (85).

Example

In the case of isotropy we have

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \qquad K_{ij} = k\delta_{ij},$$

where λ, μ, and k are positive constants. Furthermore, we assume $m=0$. The one dimensional motion

$$u_1 = u_1(x_1, t), \quad u_{2,3} = 0, \quad p = p(x_1, t)$$

with the initial and boundary conditions (see (2)–(6))

$$\left.\frac{du_1}{dx_1}\right|_{t=0} = 0, \quad (0<x_1<h)$$

$$s_1 = p_0 \text{ (constant)}, \quad p = 0 \quad \text{on} \quad x_1 = 0$$

$$u_1 = 0, \quad \frac{\partial p}{\partial n} = 0 \qquad \text{on} \quad x_1 = h \ \ (h>0\text{: constant})$$

produces p given by[6)]

$$p(x_1) = \sum_{m=0}^{\infty} \frac{2p_0}{M} \sin \frac{Mx_1}{h} \exp(-M^2 T_v), \tag{87}$$

where $M=(2m+1)\pi/2$, $T_v=c_v t/h^2$ and $c_v=k(\lambda+2\mu)$. At $x_1=0$, we have

$$\frac{\partial p}{\partial n} = -\sum_{m=0}^{\infty} \frac{2p_0}{h} \exp(-M^2 T_v),$$

which approaches asymptotically to

$$-\frac{2p_0}{h} T_v^{-1/2} \int_0^{\infty} e^{-t^2} \mathrm{d}t = -\frac{2p_0}{\pi (c_v t)^{1/2}} \cdot \frac{\sqrt{\pi}}{2} = -p_0 (\pi c_v t)^{-1/2}$$

as $t \downarrow 0$. Therefore, (83) gives $\gamma(0) = -p_0/\sqrt{\pi c_v}$. Also, (87) yields $p(x_1, 0)=p_0$ $(0<x_1<h)$. On the other hand, the boundary condition at $x_1=0$ says $p(0, 0)=0$. Hence, by noting

$$\left(\frac{\boldsymbol{n}\cdot\Delta^{*-1}(\boldsymbol{n})\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{K}\boldsymbol{n}}\right)^{1/2} = \sqrt{k(\lambda+2\mu)}$$

we conclude that (82) is satisfied.

6. Concluding Remarks

1. The main results of this paper are the formulae in **4** and the discussion on (82) and (83) in **5**, in which the formulae in **4** are shown to be useful in considering the behaviour of the solution of (1).

2. A very accurate numerical calculation based on (7) and (8) (usually called the boundary integral equation method, or BIEM) is possible only when one takes the $t^{-1/2}$ singularity (see **5**) of $\partial p/\partial n$ into consideration. The multiple γ (see (83)) of this singular term on ∂D_p is obtained numerically in two steps. Namely, we first utilize the conventional BIEM (or the BIEM for incompressible elasticity[7] if $m=0$) to determine the limit in (72). We then use (82) and (5) to determine γ.

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