# Parameter Identification of Image Models by the Recursive Maximum Likelihood Method

by

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**Abstract**

This paper considers the problem of identifying the blur parameters of the observed image. It is assumed that the original image is a sample from the homogeneous random field described by a two-dimensional (2-D) semicausal model, and that the point spread function (PSF) characterizing the image blur is symmetric. It is also assumed that the observation noise is negligibly small. By applying the discrete sine transform, we derive a set of nearly uncorrelated ARMA models, which are of non-minimum phase, for the blurred image. Although all-pass components of the MA part of the models can not be estimated, we show that the parameters of the non-minimum phase MA part can be restored by exploiting the fact that the PSF is symmetric. We develop a new algorithm for identifying the blur parameters of the image model from the MA parameters estimated by the recursive maximum likelihood (RML) method. Simulation studies are also included to show the feasibility of the algorithm.

## 1. Introduction

For the past decades, considerable interest has been directed to the restoration of blurred or degraded images by applying the statistical estimation theory.[4,7-9,12,15] In the case where all the a priori information about the original images and the degradation process such as the AR coefficients of the original image model or the point spread function (PSF) that characterizes the degradation is completely known, it is easy to implement most algorithms and a considerable improvement ratio has been achieved. In certain cases, we can obtain information about the PSF from the mechanism of the imaging system. However, in many other cases, such information or a mathematical model for the PSF will not be available. So we have to identify the blur parameters directly from the observed image before applying the restoration algorithm.

To the best knowledge of the authors, the earlier contribution to the identification of the unknown image blur is due to Murphy and Silverman,[13] in which a maximum likelihood (ML) formulation is discussed based on the 1-D causal Markov realization of

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the image developed in 12). Tekalp, Kaufman and Woods[14] proposed a method of identifying the AR and blur parameters based on the ML method in the presence of the observation noise. More recently, Biemond, Putten and Woods[3] derived a parallel identification procedure of the blur where the vector state-space models are decomposed into a set of 1-D complex ARMA models by the discrete Fourier transform (DFT). In this paper, we consider the problem of identifying the blur parameters of the homogeneous image based on the semicausal model under the condition that the observation noise is negligibly small, and that the statistical properties of the original image are known.

This paper is organized as follows. The problem is formulated in Section 2. It is assumed that the PSF is symmetric ; this assumption plays a fundamental role in identifying the PSF. We also derive a vector state-space model with multiple delays. By the discrete sine transform (DST), the vector state-space model is decomposed into a set of nearly uncorrelated scalar subsystems, from which we obtain a set of ARMA models by eliminating the state variables.

In Section 3, we consider the identification and the invertibility condition for the ARMA models.[1,11] We show that the MA part of the model, which corresponds to the image blur, is not a minimum phase ; namely, the transfer function does not satisfy the invertibility condition. Hence the MA parameters can not be recovered from the observed data, since the estimated transfer function is the minimum phase version of the original transfer function. It is well known that the original parameter of a non-minimum phase ARMA model can be restored for the case of a non-Gaussian white noise using non-quadratic error criteria,[2] and for the Gaussian white noise case with a known non-stationary covariance function using the ML approach.[10] Since in the present situation, the noise is assumed to be stationary Gaussian white, the original parameters of the PSF can not be recovered without some other information about the structure of the MA part. It is shown that any symmentric PSF can be recovered by exploiting the fact that the polynomial associated with the MA part is self-reciprocal.

In Section 4, we develop a new algorithm for identifying the PSF from the MA parameters obtained in Section 3. Since the parameters of the ARMA model are expressed as polynomials in sine functions which form the basis of DST, we can calculate the blur parameters of the observed image from the coefficients of the polynomial.

Some simulation studies are included in Section 5 to show the efficiency of the present algorithm. It is shown that the algorithm works quite well in the case where the MA parameters are identified accurately.

In Section 6, an extension is discussed to the case where the AR parameters of the original image are also unknown. In this case, the algorithm becomes a little more complicated, and accurate estimates of the ARMA parameters are needed.

## 2. Image Model

We consider a discrete monochromatic image with $N \times N$ array of pixels. The gray level of the original and observed images at point $(n, m)$ is denoted by $x(n, m)$ and $y(n, m)$, where $n$ and $m$ are the vertical and horizontal position variables, respectively, where $n, m = 1, \cdots, N$. We assume that the original image $x(n, m)$ is a zero mean, homogeneous random field described by a semicausal model[7)]

$$x(n, m) = \sum_{i=-p}^{p} a_{i0} x(n-i, m) + \sum_{i=-p}^{p} \sum_{j=1}^{q} a_{ij} x(n-i, m-j) + w(n, m) \tag{2.1}$$

where $a_{00}=0$, $a_{-i,j}=a_{ij}$, $i=1, \cdots, p$, $j=0, \cdots, q$, and the non-negative integers $p$ and $q$ denote the window of the semicausal model. The $w(n, m)$ is a zero mean homogeneous random field with the autocovariance function

$$R_{ww}(i, j) = E\{w(n, m) w(n+i, m+j)\}$$
$$= \begin{cases} \sigma_w^2 & ; i=j=0 \\ -a_{i0}\sigma_w^2 & ; j=0,\ i=\pm 1, \cdots, \pm p \\ 0 & ; \text{otherwise} \end{cases} \tag{2.2}$$

where $E$ denotes the mathematical expection.

We assume that the image blur is modeled by a space-invariant, finite-area moving average on the original image as in 4), 6), 8), 9), 12), 15). Thus the observed image $y(n, m)$ is expressed as

$$y(n, m) = \sum_{i=-t}^{t} \sum_{j=-s}^{s} c_{ij} x(n-i, m-j) + v(n, m) \tag{2.3}$$

where $v(n, m)$ is the Gaussian white noise with mean zero and variance $\sigma_v^2$, the coefficients $c_{ij}$ define the PSF of the blur and the non-negative integers $t$ and $s$ represent the extent of the blur. For the sake of simplicity, it is assumed that $c_{ij}$ are symmetric in both $i$ and $j$, satisfying

$$\sum_{i=-t}^{t} \sum_{j=-s}^{s} c_{ij} = 1 \tag{2.4}$$

Since the observation $y(n, m)$ of (2.3) involves a noncausal combination of the $x(n, m)$'s, we wish to reduce (2.3) to a semicausal representation similar to (2.1) by introducing the delay in the observation as $\bar{y}(n, m) = y(n, m-s)$. Also, assuming that the effect of the observation noise is so small that it is negligible, (2.3) becomes

$$\bar{y}(n, m) = \sum_{i=-t}^{t} \sum_{l=0}^{L} c_{i,l-s} x(n-i, m-l) \tag{2.5}$$

where $L=2s$.

### 2.1 State Space Model

In order to derive a state-space model for the blurred image, we define $N\times 1$ vectors of image columns as

$$\begin{aligned} x(m) &= [x(1, m), x(2, m), \cdots, x(N, m)]^T \\ w(m) &= [w(1, m), w(2, m), \cdots, w(N, m)]^T \\ \bar{y}(m) &= [\bar{y}(1, m), \bar{y}(2, m), \cdots, \bar{y}(N, m)]^T \end{aligned} \tag{2.6}$$

where the superscript $(\cdot)^T$ denotes the transpose. Let the $N\times N$ matrices $A_j$, $j=0, 1, \cdots, q$ and $C_l$, $l=0, 1, \cdots, L$ be given by

$$A_0=\begin{bmatrix} 1 & -a_{10} & \cdots & -a_{p0} & & & 0 \\ -a_{10} & \ddots & \ddots & & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ -a_{p0} & & \ddots & \ddots & \ddots & & -a_{p0} \\ & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & \ddots & -a_{10} \\ 0 & & & -a_{p0} & \cdots & -a_{10} & 1 \end{bmatrix} \tag{2.7a}$$

$$A_j=\begin{bmatrix} a_{0j} & a_{1j} & \cdots & a_{pj} & & & 0 \\ a_{1j} & \ddots & \ddots & & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ a_{pj} & & \ddots & \ddots & \ddots & & a_{pj} \\ & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & \ddots & a_{1j} \\ 0 & & & a_{1j} & \cdots & a_{pj} & a_{0j} \end{bmatrix}, \quad j=1,\cdots,q \tag{2.7b}$$

and

$$C_l=\begin{bmatrix} c_{0,l-s} & c_{1,l-s} & \cdots & c_{t,l-s} & & & 0 \\ c_{1,l-s} & \ddots & \ddots & & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ c_{t,l-s} & & \ddots & \ddots & \ddots & & c_{t,l-s} \\ & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & \ddots & c_{1,l-s} \\ 0 & & & c_{t,l-s} & \cdots & c_{1,l-s} & c_{0,l-s} \end{bmatrix}, \quad l=0, \cdots, L. \tag{2.7c}$$

It should be noted that $A_j$ and $C_l$ are the banded Toeplitz matrices, since the parameters $a_{ij}$ and $c_{ij}$ are symmetric in $i$.

Then, by using (2.6), (2.7a) - (2.7c), we see that (2.1) and (2.5) become

$$x(m)=\sum_{j=1}^{q} A_0^{-1}A_jx(m-j)+A_0^{-1}w(m) \tag{2.8}$$

$$\bar{y}(m)=\sum_{l=0}^{L} C_lx(m-l) \tag{2.9}$$

where it is assumed that $x(n, m)=0$ for $n\leq 0$ and $n\geq N+1$, and that $\det A_0\neq 0$. It may be noted that although $w(n, m)$ defined by (2.2) is not a white noise, $w(m)$ in (2.8) is a white noise vector with a zero mean and a covariance matrix

$$E\{w(m)w^T(k)\}=\sigma_w^2 A_0\delta_{mk} \tag{2.10}$$

where $\delta_{mk}$ is the Kronecker delta. Therefore, (2.8) and (2.9) form a horizontally causal state-space model with multiple delays for the blurred image, where $x(m)$ is the state vector and $\bar{y}(m)$ is the output vector.

## 2. 2 Decomposition of State-Space Model

According to the derivation of the parallel restoration algorithm for blurred images,[8,9] we decompose the vector state-space models of (2.8) and (2.9) into a set of nearly uncorrelated scalar subsystems. Then by eliminating the state variables, we obtain a set of ARMA models, to which we can apply the recursive maximum likelihood (RML) method to identify the ARMA parameters.

Let $D$ be a $N\times N$ banded Toeplitz matrix of the form

$$D=\begin{bmatrix} d_0 & d_1 & \cdots & d_K & & & 0 \\ d_1 & \ddots & \ddots & & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ d_K & & \ddots & \ddots & \ddots & & d_K \\ & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & \ddots & d_1 \\ 0 & & & d_K & \cdots & d_1 & d_0 \end{bmatrix} \tag{2.11}$$

and let $Q$ be the $N\times N$ tridiagonal matrix of the form

$$Q=\begin{bmatrix} 0 & 1 & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{bmatrix} \tag{2.12}$$

**Lemma 1** (Jain[6]) : A symmetric banded Toeplitz matrix $D$ is factored as

$$D=f(Q)+D_b \tag{2.13}$$

where $f(Q)$ is the polynominal in $Q$ given by

$$f(Q)=\sum_{i=0}^{K}\bar{d}_i Q^i \tag{2.14}$$

and where

$$\bar{d}_i=\sum_{k=0}^{i(K)} d_{2k+i}\gamma_{k,i},\ i(K)=\left[\frac{K-i}{2}\right] \tag{2.15}$$

where $[x]$ denotes the integer part of $x$, and

$$\gamma_{k,i}=\frac{(-1)^k(2k+1)}{k}\binom{k+1-1}{k-1},\ k=1,\ \cdots,\ i(K), \tag{2.16}$$

and $\gamma_{0,i}=1$. The matrix $D_b$ in (2.13) is a sparse Hankel matrix of rank at most $2(K-1)$.

**Lemma 2** : The state-space models of (2.8) and (2.9) can be decomposed into a set of scalar subsystems as

$$\theta_i(m)\cong\sum_{j=1}^{q}f_0^{-1}(\lambda_i)f_j(\lambda_i)\,\theta_i(m-j)+\xi_i(m) \tag{2.17}$$

$$\eta_i(m)\cong\sum_{l=0}^{L}g_l(\lambda_i)\,\theta_i(m-l) \tag{2.18}$$

where $\theta_i(m)$, $\xi_i(m)$ and $\eta_i(m)$ are the $i$-th component of $N\times 1$ vectors $\theta(m)$, $\xi(m)$ and $\eta(m)$, which are the DST's of the image columns $x(m)$, $w(m)$ and $\bar{y}(m)$, respectively. Also, $f_j(\lambda_i)$, $j=0,\ \cdots,\ q$ and $g_l(\lambda_i)$, $l=0,\ \cdots,\ L$ are the polynomials defined similarly to (2.14) in Lemma 1 with deg $f_j(\lambda_i)=p$ and deg $g_l(\lambda_i)=t$ where $\lambda_i=2\cos(i\pi/(N+1))$ are the eigenvalues of the matrix $Q$. The covariance of $\xi_i(m)$ is given by

$$E\{\xi_i(m)\,\xi_j(k)\}\cong\sigma_w^2 f_0^{-1}(\lambda_i)\,\delta_{ij}\delta_{mk}\equiv\sigma_{\xi_i}^2\delta_{ij}\delta_{mk}. \tag{2.19}$$

**Proof** : A derivation is given in 6). □

From (2.19), we see that $\xi_i(m)$ and $\xi_j(m)$ are nearly uncorrelated, so are $\theta_i(m)$ and $\theta_j(m)$, where $i\neq j$. Thus the systems of (2.17) and (2.18) form a set of nearly uncorrelated subsystems, so that we can process the image data $\eta_i(m)$, $m=1,\ \cdots,\ N$, separately for each $i$, $i=1,\ \cdots,\ N$.

Eliminating $\theta_i(m)$'s from (2.17) and (2.18) yields

$$\eta_i(m)=\sum_{j=1}^{q}f_0^{-1}(\lambda_i)f_j(\lambda_i)\,\eta_i(m-j)+\sum_{l=0}^{L}g_l(\lambda_i)\,\xi_i(m-l) \tag{2.20}$$

Here, let us define

$$e_i(m)\equiv g_0(\lambda_i)\,\xi_i(m) \tag{2.21a}$$

$$a_j(i)\equiv f_j(\lambda_i)/f_0(\lambda_i),\ j=1,\ \cdots,\ q \tag{2.21b}$$

$$d_l(i)\equiv g_l(\lambda_i)/g_0(\lambda_i),\ l=1,\ \cdots,\ L. \tag{2.21c}$$

Then (2.20) can be reduced to the following ARMA $(q,\ L)$ model

$$\eta_i(m)=\sum_{j=1}^{q}a_j(i)\,\eta_i(m-j)+e_i(m)+\sum_{l=1}^{L}d_l(i)\,e_i(m-l) \tag{2.22}$$

where $e_i(m)$ is a white noise with the variance

$$\sigma_e^2(i)=(g_0(\lambda_i))^2\sigma_\xi^2(i). \tag{2.23}$$

We observe that (2.22) is a set of $N$ nearly uncorrelated ARMA models for the blurred image, where $a_j(i)$ and $d_l(i)$ are the unknown parameters to be estimated. Since it is assumed that the coefficients $c_{ij}$ are symmetric in $j$, we see from (2.7c) that $g_l(\lambda_i)=$

$g_{L-l}(\lambda_i)$, $l=0, 1, \cdots, [(L-1)/2]$. Thus it follows from (2.21c) that the MA coefficients $d_l(i)$ are symmetric in $l$, namely, $d_l(i)=d_{L-l}(i)$, $l=0, 1, \cdots, [(L-1)/2]$, and $d_0(i)=1$.

The objective of present paper is to develop a method of identifying the unknown paremeters $a_{ij}$, $\sigma_w^2$ and $c_{ij}$ from the observations $y(n, m)$. To this end, we have to identify the ARMA parameters of (2.21a) - (2.21c) from the DST image.

## 3. Identification of ARMA Model

In this section, we present a method of identifying the ARMA parameters by applying the recursive identification algorithm.

### 3.1 Recursive Algorithm[5]

Consider the parameter identification problem for the ARMA model

$$y(t)=\sum_{i=1}^{na}a_i y(t-i)+e(t)+\sum_{j=1}^{nc}c_j e(t-j) \tag{3.1}$$

where $t$ denotes the time variable, and $e(t)$ is a white noise sequence with zero mean and the variance $\sigma_e^2$. The RELS and RML methods are commonly employed for identifying the parameters of the ARMA model of (3.1). Recently, Friedlander[5] has developed an algorithm that unifies the RELS and the RML methods by expoloiting the advantage of both methods.

Let

$$\theta=[a_1, \cdots, a_{na}, c_1, \cdots, c_{nc}]^T \tag{3.2}$$

$$\phi^T(t)=[y(t-1), \cdots, y(t-n_a), e(t-1), \cdots, e(t-n_c)] \tag{3.3}$$

Then (3.1) becomes

$$y(t)=\phi^T(t)\theta+e(t) \tag{3.4}$$

The recursive algorithm due to Friedlander is summarized as follows:

*RML Algorithm with a Modified Prefilter:*

Let

$$\hat{\theta}(t)=[\hat{a}_1(t), \cdots, \hat{a}_{na}(t), \hat{c}_1(t), \cdots, \hat{c}_{nc}(t)]^T \tag{3.5}$$

be the estimate of $\theta$ based on the data $y(1), \cdots, y(t)$. Then the parameter estimation algorithm is given by

$$\varepsilon(t+1)=y(t+1)-\phi^T(t+1)\hat{\theta}(t) \tag{3.6}$$

$$\hat{\theta}(t+1)=\hat{\theta}(t)+P(t+1)\psi(t+1)\varepsilon(t+1) \tag{3.7}$$

$$P(t+1)=P(t)-\frac{P(t)\psi(t+1)\psi^T(t+1)P(t)}{1+\psi^T(t+1)P(t)\psi(t+1)} \tag{3.8}$$

$$e(t+1)=y(t+1)-\phi^T(t+1)\hat{\theta}(t+1) \tag{3.9}$$

where $\varepsilon(t)$ and $e(t)$ are the a priori and a posterior prediction errors, respectively , and where $\psi(t)$ is defined by

$$\psi^T(t) = [\tilde{y}(t-1), \cdots, \tilde{y}(t-n_a), \tilde{e}(t-1), \cdots, \tilde{e}(t-n_c)] \tag{3.10}$$

In (3.10), $\tilde{y}(t)$ and $\tilde{e}(t)$ are generated by

$$\tilde{y}(t) = \frac{1}{D_t(z^{-1})} y(t), \quad \tilde{e}(t) = \frac{1}{D_t(z^{-1})} e(t) \tag{3.11}$$

where

$$D_t(z^{-1}) \equiv \widehat{C}_t(K(t) z^{-1}) = 1 + \sum_{l=1}^{nc} K(t)^l \hat{c}_l(t) z^{-l} \tag{3.12}$$

$$K(t+1) = \lambda K(t) + (1-\lambda), \; K(0) = 0 \tag{3.13}$$

and where $z^{-1}$ is the backward shift operator and $\lambda$ is a constant with $0<\lambda<1$. □

If we set $K(t) \equiv 0$ in (3.12), then $D_t(z^{-1}) = 1$, so that the above algorithm reduces to the RELS method. But if we set $K(t) \equiv 1$, namely if $D_t(z^{-1}) = \widehat{C}_t(z^{-1})$, then we have the RML algorithm. Since $K(t)$ of (3.13) move from 0 to 1, the above algorithm may have the transient property of the RELS and the asymptotic efficiency of the RML.

### 3. 2 Invertibility Condition of ARMA Model

Now define

$$A(z^{-1}) = 1 - \sum_{k=1}^{na} a_k z^{-k} \tag{3.14}$$

$$C(z^{-1}) = 1 + \sum_{l=1}^{nc} c_l z^{-l} \tag{3.15}$$

Then the ARMA model of (3.1) is expressed as

$$A(z^{-1}) y(t) = C(z^{-1}) e(t) \tag{3.16}$$

where $z^{-1}$ is the backward shift operator and we assume that $A(\cdot)$ and $C(\cdot)$ are coprime. The following conditions must be imposed for the identification of the ARMA model of (3.16).[1,11]

(C1) The $e(t)$ is a Gaussian white noise with a mean zero and the finite variance $\sigma_e^2$.

(C2) All the zeros of $A(z^{-1})$ lie inside the unit circle.

(C3) All the zeros of $C(z^{-1})$ lie inside the unit circle.

Condition (C2) guarantees the stationarity of the output $y(t)$, and condition (C3) implies that $C(z^{-1})$ is invertible, so that the transfer function $G(z) \equiv C(z^{-1})/A(z^{-1})$ is of a minimum phase.

We consider the conditions (C1) - (C3) for the ARMA model of (2.22). Let $n_a \equiv q$, $n_c \equiv L$, $a_j \equiv a_j(i)$ and $c_l \equiv d_l(i)$. Then the ARMA model of (2.22) is reduced to (3.1) [or (3.16)]. Condition (C1) is clearly satisfied. Also, condition (C2) is satisfied, since the original image field is a sample from a homogeneous random field. But as shown below, the invertibility condition (C3) does not hold. As mentioned at the end of Section 2.2, we have $c_l = c_{n_c-l}$, $l=0, 1, \cdots, [(n_c-1)/2]$, and $c_{n_c}=1$. Thus, the coefficients of the polynomial $C(z^{-1})$ of (3.15) are symmetric, so that the reciprocal polynomical $C^*(z^{-1}) \equiv z^{-n_c}C(z)$ is identical to the original $C(z^{-1})$. This implies that if $z_0$ is a zero of the polynomial $C(z^{-1})$ then the reciprocal $z_0^{-1}$ is also a zero of $C(z^{-1})$. Therefore, the polynomial $C(z^{-1})$ has zeros both inside and outside the unit circle, except for the case where all the zeros of $C(z^{-1})$ are on the unit circle. Hence, in general, the invertibility conditions does not hold for the ARMA model for the blurred image, so that the estimated parameters $c_l$ do not converge to the true values. But we can recover the desired estimate of $C(z^{-1})$ by exploiting the assumption that its cofficients are symmetric.

Let $G(z) \equiv C(z^{-1})/A(z^{-1})$, and let $S_{yy}(z)$ be the spectral density of the stationary process $y(t)$. Then it follows from (3.16) that

$$S_{yy}(z) = G(z)\,G(z^{-1})\,\sigma_e^2,\ z=\exp(j\omega),\ -\pi\leq\omega\leq\pi. \tag{3.18}$$

By the canonical spectral factorization, we have

$$S_{yy}(z) = S_y^+(z)\,S_y^-(z)\,\sigma_e^2 \tag{3.19}$$

where the canonical factor $S_y^+(z)$ is a minimum phase function and $S_y^-(z) = S_y^+(z^{-1})$. Therefore, we see that the estimated transfer function from the output observations $y(t)$ will be the minimum phase version of the true transfer function $G(z)$ so that the zeros of the estimated polynomial $\hat{C}(z) \equiv 1 + \sum_{l=1}^{n_c}\hat{c}_l z^{-l}$ coincide with the stable zeros of $C(z^{-1})\,C(z)$. In other words, we can not estimate all-pass components of the transfer functions by the algorithm based on the second-order statistics of the given data.

But since the zeros of $C(z^{-1})$ are reciprocal as well as complex conjugate, we see that all the zeros of the estimated polynomial $\hat{C}(z^{-1})$ will be multiple except for possible pairs of zeros on the unit circle as shown in Fig. 1 (a). Therefore by taking a mirror image of each one of the multiple zeros of $\hat{C}(z^{-1})$ with respect to the unit circle as shown in Fig. 1 (b), we can recover the desired polynomial $C(z^{-1}) = 1+\sum_{l=1}^{n_c}c_l z^{-l}$. Thus, we can uniquely estimate parameters $a_j$ and $c_l$ by the above modification of MA parameters.

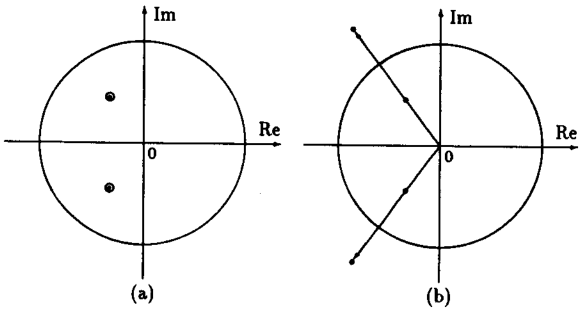


Fig. 1 Location of Zeros.

## 4. Algorithm for Identifying Blur Parameters

In this section, we derive a new algorithm for identifying the blur parameters by using the ARMA parameters estimated in the previous section. In this section, it is assumed that the orders $p$, $q$, $t$, $s$ of the image and blurred models are known. Although not difficult theoretically, the joint identification of AR and MA parameters is not easy from our numerical experience.

**Identification Algorithm A**

*Step 1* : From the DST image of (2.22), compute the estimates of the MA parameters $\widehat{d}_l(i)$, $l=1, \cdots, L$ and $\widehat{\sigma}^2{}_e(i)$ for each $i$, $i=1, \cdots, N_d$, by the recursive algorithm in Section 3, where $N_d<N$ and is usually taken to be $N/2$ since the $S/N$ ratio of the transformed image $\eta_i(m)$ is very small for large $i$. □

*Step 2*: Modify $\widehat{d}_l(i)$ to $\widetilde{d}_l(i)$ by reflecting out each one of the multiple zeros of $\widehat{D}_i(z^{-1})$ so that $\widetilde{D}_i(z^{-1})$ becomes self-reciprocal for each $i=1, \cdots, N_d$. □

It should be noted that before computing $\widetilde{d}_l(i)$ we must adjust the parameters $\widehat{d}_l(i)$ so that $\widehat{D}_i(z^{-1})$ has exact multiple zeros. Since we rarely have an exact zero numerically, it is assumed that a multiple zero is located at the midpoint of a pair of the nearest zeros whose imaginary parts have the same sign.

As mentioned in Section 2.2, $f_j(\lambda_i)$ and $g_l(\lambda_i)$ are of the order $p$ and $t$, respectively, so that we have

$$f_j(\lambda_i)=\sum_{k=0}^{p}\alpha_k^{(j)}\lambda_i^k,\ j=0, \cdots, q \tag{4.1}$$

$$g_l(\lambda_i)=\sum_{k=0}^{t}\beta_k^{(l)}\lambda_i^k,\ l=0, \cdots, L \tag{4.2}$$

where the coefficients $\alpha_k^{(j)}$ and $\beta_k^{(l)}$ are connected with the components of the banded Toeplitz matrices $A_j$ and $C_l$ through (2.15) and (2.16) in Lemma 1.

From (2.23) and (4.2), it follows that

$$\sum_{k=0}^{t}\beta_k^{(0)}\lambda_i^k=\sqrt{\frac{\sigma_e^2(i)}{\sigma_\xi^2(i)}},\ i=1, \cdots, N_d \tag{4.3}$$

Define the matrix $\Lambda_{N_d}$ and the vectors $\beta$ and $r$ as

$$\Lambda_{N_d}=\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^t \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^t \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_{N_d} & \lambda_{N_d}^2 & \cdots & \lambda_{N_d}^t \end{bmatrix}\quad [N_d\times(t+1)] \tag{4.4a}$$

$$\beta=[\beta^{(0)}{}_0,\ \beta^{(0)}{}_1, \cdots, \beta^{(0)}{}_t]^T \qquad [(t+1)\times 1] \tag{4.4b}$$

$$r=\left[\sqrt{\frac{\sigma_e^2(1)}{\sigma_\xi^2(1)}},\ \sqrt{\frac{\sigma_e^2(2)}{\sigma_\xi^2(2)}},\ \cdots,\ \sqrt{\frac{\sigma_e^2(N_d)}{\sigma_\xi^2(N_d)}}\right]^T \quad (N_d\times 1) \tag{4.4c}$$

where it may be noted that $\Lambda_{N_d}$ is a Vandermonde-like matrix. Then, (4.3) can be rewritten as $\Lambda_{N_d}\beta=r$. Since it is assumed that the statistical properties of the original image is known, $\sigma_\xi^2(i)$ in (4.4c) can be obtained from (2.19) up to a constant $\sigma_w^2$. It may be also noted that the actual value of $\sigma_w^2$ is irrelevant, since the PSF coefficients are normalized in Step 6.

*Step 3*: Compute the least-squares estimate of the parameter vector $\beta$ as

$$\hat{\beta}=(\Lambda_{N_d}^T\Lambda_{N_d})^{-1}\Lambda_{N_d}^T r. \ \square \tag{4.5}$$

*Step 4*: By using $\hat{\beta}^{(0)}_k$ , $k=0, \cdots, t$, we can compute $\hat{g}_0(\lambda_i)$ from (4.2). □

*Step 5*: From(2.21c) and (4.2), we have

$$g_l(\lambda_i)=\sum_{k=0}^{t}\beta_k^{(l)}\lambda_i^k=g_0(\lambda_i)\,d_l(i),\ i=1,\ \cdots,\ N_d \tag{4.6}$$

Therefore, by using $\tilde{d}_l(i)$, $l=1, \cdots, L$ in Step 2 and $\hat{g}_0(\lambda_i)$ in Step 4, we can obtain $\{\hat{\beta}_k^{(l)}, k=0, \cdots, t, l=0, \cdots, L\}$ by applying the least-squares method to (4.6). □

*Step 6*: Calculate the PSF $\hat{c}_{ij}$ from the relation of (2.15) in Lemma 1, and normalize $\hat{c}_{ij}$ as

$$\sum_{i=-t}^{t}\sum_{j=-s}^{s}\hat{c}_{ij}=1. \ \square \tag{4.7}$$

This completes the estimation of the PSF confficients $C_l$. After estimating the unknown parameters by the above algorithum, we can restore the blurred image by using the parallel restoration algorithms.[8,9)]

## 5. Simulation Studies

We present some simulation results using a synthetic pesudo-random image and the Moon image with the size 256×256 to show the feasibility of the algorithm of Steps 1—6. We assume that the image obeys the semicausal model of (2.2), and that the image has a separable exponential autocovariance function

$$R_{xx}(i,j)=E\{x(n,m)x(n+i,m+j)\}$$
$$=\sigma_x^2 a_1^{|i|}a_2^{|j|} \tag{5.1}$$

where $\sigma_x^2$ is the variance of the original image, and $a_1$ and $a_2$ are the vertical and horizontal correlations, respectively. It is shown[7)] that the autocovariance function (5.1) can be realized by the semicausal model of (2.1) with $p=q=1$, and

$$a_{10}=\frac{a_1}{1+a_1^2},\ a_{01}=a^2,\ a_{11}=\frac{a_1a_2}{1+a_1^2},$$

$$\sigma_w^2=\frac{(1-a_1^2)(1-a_2^2)}{1+a_1^2}\sigma_x^2 \tag{5.2}$$

Thus it follows from (2.7a) and (2.7b) that

$$A_0=I-a_{10}Q,\ A_1=a_2A_0 \tag{5.3}$$

where $I$ denotes the identity matrix and $Q$ is the tridiagonal matrix defined by (2.12). Hence

$$f_0(\lambda_i)=1-a_{10}\lambda_i,\ f_1(\lambda_i)=a_2f_0(\lambda_i) \tag{5.4}$$

where $\lambda_i$'s are the eigenvalues of $Q$. Also,

$$\sigma_\xi^2(i)=\frac{\sigma_w^2}{1-a_{10}\lambda_i} \tag{5.5}$$

We assume that the PSF is given by

$$(c_{ij})=\frac{1}{74}\begin{bmatrix}1&2&3&2&1\\2&3&5&3&2\\3&5&10&5&3\\2&3&5&3&2\\1&2&3&2&1\end{bmatrix}$$

$$=\begin{bmatrix}0.01351&0.02703&0.04054&0.02703&0.01351\\0.02703&0.04054&0.06757&0.04054&0.02703\\0.04054&0.06757&0.13514&0.06757&0.04054\\0.02703&0.04054&0.06757&0.04054&0.02703\\0.01351&0.02703&0.04054&0.02703&0.01351\end{bmatrix} \tag{5.6}$$

where $t=s=2$, $L=4$. Then the polynomials $g_l(\lambda_i)$, $l=0, 1, \cdots, 4$ are given by

$$\begin{aligned}g_0(\lambda_i)&=g_4(\lambda_i)=\frac{1}{74}(1+2\lambda_i+\lambda_i^2)\\g_1(\lambda_i)&=g_3(\lambda_i)=\frac{1}{74}(1+3\lambda_i+2\lambda_i^2)\\g_2(\lambda_i)&=\frac{1}{74}(4+5\lambda_i+3\lambda_i^2)\end{aligned} \tag{5.7}$$

Before performing simulations, we have numerically verified that if we are given the true values of parameters $d_l$, $l=0, 1, \cdots, 4$ of (2.22), then the PSF parameters can be restored correctly. This implies that the identification of MA parameters is the crucial from the practical point of view. But it is well known that the convergence of the estimates of MA parameters is very slow.[11]

We generate a pseudo-random image by the causal model as

$$x(n, m)=a_1x(n-1, m)+a_2x(n, m-1)-a_1a_2x(n-1, m-1)+\tilde{w}(n, m) \tag{5.8}$$

where $a_1=a_2=0.9$, and $\sigma_{\tilde{w}}^2=1.0$, so that the parameters of the semicausal model of (2.1) are given by $a_{10}=a_{-1,0}=0.4972$, $a_{01}=0.9$, $a_{11}=a_{-1,1}=-0.4475$ and $\sigma_w^2=0.5525$.

Figs. 2 and 3 show the original pseudo-random image and the blurred image, respectively. The PSF identified by the algorithm of Section 4 is shown in Table 1. Then we restore the blurred image of Fig. 3 by the parallel restoration algorithm.[9] Table 2 shows the comparison of the restoration results for the pseudo-random image with the

Table 1 Result of Identification

$$\begin{bmatrix} 0.01016 & 0.02600 & 0.03366 & 0.02600 & 0.01016 \\ 0.02287 & 0.04707 & 0.06244 & 0.04707 & 0.02287 \\ 0.04831 & 0.05083 & 0.18513 & 0.05083 & 0.04831 \\ 0.02287 & 0.04707 & 0.06244 & 0.04707 & 0.02287 \\ 0.01016 & 0.02600 & 0.03366 & 0.02600 & 0.01016 \end{bmatrix}$$

Table 2 Result of Restoration for Pseudo-Random Image

| | True PSF | Estimated PSF |
|---|---|---|
| $e_B$ | 2.7463 | |
| $e_A$ | 1.4866 | 1.5125 |
| $\eta_{BA}$(dB) | 2.69 | 2.62 |

known blurred parameters and with the estimated parameters, where $e_B$ and $e_A$ denote the mean square error before and after restoration, respectively, and where $\eta_{BA}$, the improvement in SNR, is defined by

Table 3 Result of Restoration for Moon Image

| | True PSF | Estimated PSF |
|---|---|---|
| $e_B$ | 57.239 | |
| $e_A$ | 37.124 | 37.650 |
| $\eta_{BA}$(dB) | 1.88 | 1.82 |

$$\eta_{BA}=10\log_{10}\frac{e_B}{e_A}\,(dB) \tag{5.9}$$

We also show the restoration result for the Moon image in Table 3. The restoration is performed with the PSF of Table 1 estimated from the pesudo-random image. We consider the pseudo-random image as a test image from which we can obtain the character-

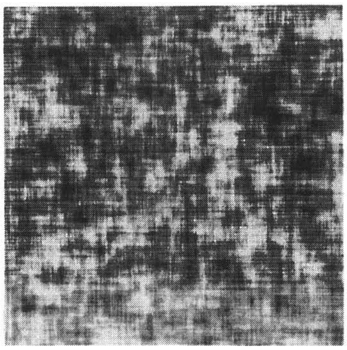

Fig. 2 Original Pseudo-Random Image.

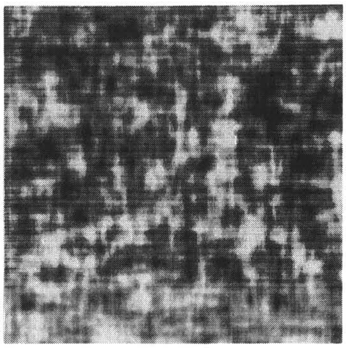

Fig. 3 Blurred Pseudo-Random Image.

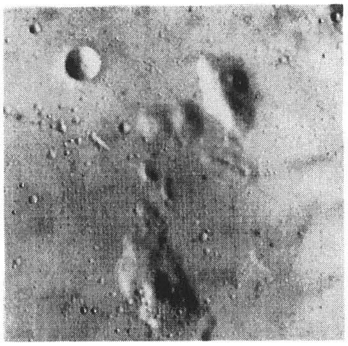

Fig. 4 Original Moon Image.

Fig. 5 Blurred Moon Image.

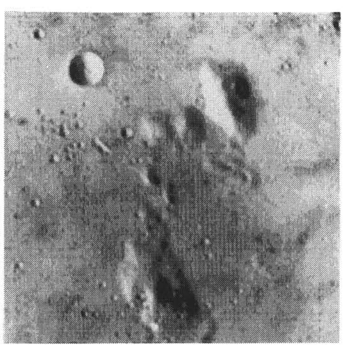

Fig. 6 Restored Moon Image.

istic of our imaging system. We observe from Tables 2 and 3 that there is little difference in the improvement in SNR between the two restorations. Also, Figs. 4 - 6 show the original, blurred and restored Moon images, respectively. There are little noticeable distinctions in the visibility between the original image of Fig. 4 and the restored image of Fig. 6.

## 6. An Extension

In Section 4, we developed a method of identifying MA parameters assuming that the AR parameters are known a priori. Here, we present an extension of the identification algorithm to the case where the AR parameters are not known.

By squaring and expanding (4.3), we have

$$\sum_{k=0}^{2t}\gamma_k\lambda_i^k=\frac{\sigma_e^2(i)}{\sigma_\xi^2(i)} \tag{6.1}$$

where the coefficients $\gamma_k$ are given by

$$\gamma_k=\begin{cases}\sum_{j=0}^{k}\beta_j^{(0)}\beta_{k-j}^{(0)}, & 0\leq k\leq t\\ \sum_{j=0}^{k-t}\beta_j^{(0)}\beta_{k-j}^{(0)}, & t+1\leq k\leq 2t\end{cases} \tag{6.2}$$

Also, from (2.19) and (4.1)

$$\frac{1}{\sigma_\xi^2(i)}=\frac{1}{\sigma_w^2}\sum_{r=0}^{p}\alpha_r^{(0)}\lambda_i^r \tag{6.3}$$

Therefore, (6.1) can be reduced to

$$\sum_{k=0}^{2t}\gamma_k\lambda_i^k-\sum_{r=1}^{p}\frac{\alpha_r^{(0)}}{\sigma_w^2}\sigma_e^2(i)\lambda_i^r=\frac{\alpha_0^{(0)}}{\sigma_w^2}\sigma_e^2(i),\ i=1,\cdots,N_d. \tag{6.4}$$

Hence, if we define $\Lambda_{Nd}$, $\beta$ and $\xi$ as

$$\Lambda_{Nd}=\begin{bmatrix}1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{2t} & \sigma_e^2(1)\lambda_1 & \cdots & \sigma_e^2(1)\lambda_1^p\\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{2t} & \sigma_e^2(2)\lambda_2 & \cdots & \sigma_e^2(2)\lambda_2^p\\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots\\ 1 & \lambda_{N_d} & \lambda_{N_d}^2 & \cdots & \lambda_{N_d}^{2t} & \sigma_e^2(N_d)\lambda_{N_d} & \cdots & \sigma_e^2(N_d)\lambda_{N_d}^p\end{bmatrix} \tag{6.5a}$$

$[N_d\times(2t+p+1)]$

$$\beta=\left[\gamma_0,\ \gamma_1,\ \cdots,\ \gamma_{2t},\ -\frac{\alpha_1^{(0)}}{\sigma_w^2},\ -\frac{\alpha_2^{(0)}}{\sigma_w^2},\ \cdots,\ -\frac{\alpha_p^{(0)}}{\sigma_w^2}\right]^T\quad[(2t+p+1)\times 1] \tag{6.5b}$$

$$\xi=\frac{\alpha_0^{(0)}}{\sigma_w^2}[\sigma_e^2(1),\ \sigma_e^2(2),\ \cdots,\ \sigma_e^2(N_d)]^T\quad[N_d\times 1] \tag{6.5c}$$

Thus, (6.4) is written as $\Lambda_{Nd}\beta=\xi$.

Steps in Section 4 should be modified as follows.

**Identification Algorithm B**

*Step 1*: From the DST image of (2.22), compute the estimates of the ARMA parameters $\hat{a}(i)$, $i=1, \cdots, q$, $\hat{d}_l(i)$, $l=1, \cdots, L$ and $\hat{\sigma}^2{}_e(i)$ for each $i$, $i=1, \cdots, N_d$, by the recursive algorithm in Section 3. □

*Step 2*: Same as Step 2 of Algorithm A. □

*Step 3*: Compute the least-squares estimate

$$\hat{\beta}=(\Lambda_{N_d}^T\Lambda_{N_d})^{-1}\Lambda_{N_d}^T\xi. \quad \square \tag{6.6}$$

Although the value $\alpha_0^{(0)}/\sigma_0^2$ in (6.5c) is not known, it may tentatively be assumed, say, to be unity, so that the $\hat{\beta}$ obtained in Step 3 is the least-squares estimate of $\beta$ up to some constant multiplication.

*Step 4*: By using $\hat{\gamma}_k$, $k=0, \cdots, 2t$, which are the first $2t+1$ components of the vector $\hat{\beta}$ obtained by (6.6), and the relation of (6.2), we can compute $\{\hat{\beta}^{(0)}{}_k\ ,\ k=0, \cdots, t\}$, from which we obtain $\hat{g}(\lambda_i)$ up to a constant multiplication. □

*Step 5*: Same as Step 5 of Algorithm A. □

*Step 6*: Same as Step 6 of Algorithm A. □

*Step 7*: By recalculating $\beta_k^{(0)}$ from the normalized $\hat{c}_{ij}$ we obtain the estimate of $\hat{g}(\lambda_i)$. Then, we obtain $\hat{\sigma}^2{}_\xi(i)$ by using (6.1) and the estimate $\hat{\sigma}^2{}_e(i)$. □

Steps 1 - 7 are the modification of Steps 1 - 6 of the algorithm in Section 4. Hereafter, we derive the algorithm for identifying the AR parameters.

*Step 8*: From (6.3), we have

$$\sum_{r=0}^{p}\alpha_r^{(0)}\lambda_i^r=\frac{\sigma_w^2}{\sigma_{\xi_i}^2} \tag{6.7}$$

to which we can apply the least-squares method to obtain the estimates of the parameters $\{\alpha_k^{(0)},\ k=0, 1, \cdots, p\}$, where the estimates of $\hat{\sigma}^2{}_{\xi_i}$ are obtained in Step 7 and where $\sigma_w^2$ is tentatively assumed to be unity. □

*Step 9*: From the relation of (2.15), we calculate $\{\hat{a}_{00}, \hat{a}_{10}, \cdots, \hat{a}_{p0}\}$ which are the estimates of the parameters in the first row of the coefficient matrix $A_0$. Since the diagonal components of $A_0$ are unity, the estimate of the first row of $A_0$ is given by

$$\left\{1, \frac{\hat{a}_{10}}{\hat{a}_{00}}, \cdots, \frac{\hat{a}_{p0}}{\hat{a}_{00}}, 0, \cdots, 0\right\}. \quad \square \tag{6.8}$$

*Step 10*: Since the polynomial $\hat{f}_0(\lambda_i)$ is determined by

$$\frac{1}{\hat{a}_{00}}\{\hat{\alpha}_0^{(0)}, \cdots, \hat{\alpha}_p^{(0)}\} \tag{6.9}$$

it follows from (6.7) that the estimate of $\sigma_w^2$ is given by

$$\hat{\sigma}^2{}_w=\frac{1}{N_d}\sum_{i=1}^{Nd}\hat{f}_0(\lambda_i)\,\sigma_\xi^2(i). \;\square \tag{6.10}$$

*Step 11* : From (2.21b) and (6.7), we have

$$\sum_{r=0}^{p}\alpha_r^{(j)}\lambda_i^r=\frac{\sigma_w^2}{\sigma_\xi^2(i)}a_j(i),\; j=1,\;\cdots,\;q \tag{6.11}$$

Then, we can compute $\{\hat{\alpha}^{(j)}_k\,,\;k=0,\cdots,\;p,\;j=1,\;\cdots,\;q\}$ similarly to Step 8. □

*Step 12*: Compute the estimates of $A_j$, $j=1,\;\cdots,\;q$ similarly to Step 9. □

**Remark** : We have verified numerically that the above algorithm yields good estimates of the PSF parameters as well as the system parameters, provided that accurate estimates of parameters of a set of ARMA models of (2.22) are given. At present, however, we have not yet succeeded in obtaining good estimates from both the synthetic and the real images of Figs. 2 and 4.

## 7. Conclusions

In this paper, we have developed a new algorithm for identifying the blur parameters of the observed image. It is assumed that the original image can be expressed by a 2-D semicausal model, and that the PSF of the blur is symmetric. We derive a set of nearly uncorrelated ARMA models, which are of a non-minimum phase, by applying the DST to the blurred image. Although the MA parameters are not invertible, it is shown that the parameters of the non-minimum phase ARMA models can be recovered by exploiting the fact that the coefficients of MA part are symmetric. We have presented a new algorithm for identifying the blur parameters from the estimated MA parameters by using the least-squares method. Some numerical results are included to show the efficiency of the present algorithm. An extension to the case where both the AR and MA parameters are unknown is briefly discussed.


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