Solutions to Wiener Filtering and Stationary LQG Problems via H_2 Control Theory–Part I: Continuous–Time System

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Abstract

This paper derives solutions to the multivariable Wiener filtering and the stationary LQG problems using the H_2 optimal control theory and the state–space technique. Inner–outer factorization and spectral factorization results arising in H_2/H_∞ optimal controls are also derived by the state–space technique.

1. Introduction

Motivated by the work of Wilson [11], this paper develops solutions to the multivariable Wiener filtering and the stationary LQG problems by applying the state–space technique for the model matching problem developed by Doyle [3-4], and Francis [6]. It has been shown that the general $H_a(a=2$ or $\infty)$ control problem is reduced to the model matching problem of finding a stable transfer function $Q(s)$ such that

$$J=\|T_1(s)-T_2(s)Q(s)T_3(s)\|_a=\text{minimum} \tag{1.1}$$

where $T_1(s)$, $T_2(s)$, $T_3(s)$ are also stable. For $a=2$, the norm of a matrix function $G(s)$ with no poles on the imaginary axis is defined by

$$\|G(s)\|_2^2=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\mathrm{tr}[G^*(s)G(s)]ds \tag{1.2}$$

where $G^*(s):=G^T(-s)$, and where $(\bullet)^T$ denotes the transpose.

Let $T_2(s)$ and $T_3(s)$ be factored as (see Section 2.2)

$$T_2(s)=T_{2i}(s)T_{2o}(s),\quad T_{2i}:\ \text{inner},\ T_{2o}:\ \text{outer} \tag{1.3}$$

and

$$T_3(s)=T_{3co}(s)T_{3ci}(s),\quad T_{3co}:\ \text{co–outer},\ T_{3ci}:\ \text{co–inner} \tag{1.4}$$

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Then the optimal solution $Q(s)$ is given by [5], [9]

$$Q(s)=T_{2o}^{-1}(s)[T_{2i}^{*}(s)T_{1}(s)T_{3ci}^{*}(s)]_{+}T_{3co}^{-1}(s) \tag{1.5}$$

where $[\bullet]_{+}$ denotes the stable part of a matrix function by partial fraction expansion. We employ the formula (1.5) to derive solutions to the Wiener filtering and the stationary LQG problems.

As preliminary, we derive an inner–outer factorization by the state–space technique. We also develop an algorithm of spectral factorization arising from the H_∞ optimization [6].

In Section 2, we begin with a summary of the useful results of operations on transfer function matrices and present an inner–outer factorization of a stable transfer function. In Section 3, we then proceed to a derivation of the solution of the Wiener filtering problem. We present a classical solution based on the spectral factorization and additive decomposition. For the case where the spectral densities are rational, the problem is embedded in the model matching problem to derive a state–space solution. In Section 4, the same technique is applied to the stationary LQG problem for which the optimal controller is derived by using inner–outer and co–inner–outer factorizations of transfer functions appearing in the model matching problem. Section 5 provides a new proof for the spectral factorization algorithm that arises from the H_∞ optimization.

2. Mathematical Preliminaries

In this section, we summarize some useful results for continuous–time transfer function matrices and inner–outer factorization.

2.1 Transfer Functions

We consider proper, real–rational transfer function matrices described by the state–space representation

$$G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]:=D+C(sI-A)^{-1}B \tag{2.1}$$

where A, B, C, D are constant matrices of dimensions $n\times n$, $n\times m$, $p\times n$, $p\times m$, respectively. A transfer function matrix $G(s)$ is stable, if it is analytic in $\mathrm{Re}[s]\geq 0$; namely, $G(s)$ is stable if and only if the eigenvalues of A lie in the open left half–plane $\mathrm{Re}[s]<0$.

Let RL^∞ be the matrix functions of s which are bounded on the imaginary axis. Let RH_+^∞ be the class of stable, proper functions. Thus, if $G(s)$ in RL^∞ is analytic in $\mathrm{Re}[s] \geq 0$, it belongs to RH_+^∞. The complementary space RH_-^∞ is the class of functions in RL^∞ which are analytic in $\mathrm{Re}[s] < 0$.

The following formulae collect useful operations on transfer function matrices [3], [4], [6].

(a) For a nonsingular T,

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] \tag{2.2}$$

(b) Suppose that $G(s)$ is square and D is nonsingular. Then, we get

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^{-1} = \left[\begin{array}{c|c} A-BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right] \tag{2.3}$$

(c) A product of transfer function matrices is expressed as

$$\begin{aligned} \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] &= \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right] \end{aligned} \tag{2.4}$$

(d) For $G^*(s) := G^T(-s)$, we get

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^* = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] \tag{2.5}$$

2.2 Inner–Outer Factorization

A matrix function $G(s)$ in RH_+^∞ is called inner if $G^*(s)G(s) = I_m$. Thus an inner function $G(s)$ must be tall, namely, $p \geq m$. A matrix function $G(s)$ in RH_+^∞ is outer if $G(s)$ has a right-inverse which is analytic in $\mathrm{Re}[s] > 0$. For a square $G(s)$, if both $G(s)$, $G^{-1}(s)$ are in RH_+^∞, then $G(s)$ is outer. An inner–outer factorization of $G(s)$ in RH_+^∞ is given by

$$G(s) = G_i(s)G_o(s), \quad G_i: \text{ inner, } G_o: \text{ outer} \tag{2.6}$$

A matrix $G(s)$ is co–inner or co–outer if $G^T(s)$ is inner or outer, respectively. Thus a co–inner–outer factorization is given by

$$G(s) = G_{co}(s)G_{ci}(s), \quad G_{co}: \text{ co–outer, } G_{ci}: \text{ co–inner} \tag{2.7}$$

It is easy to see that a co–inner–outer factorization of $G(s)$ is derived from an inner–outer factorization of $G^T(s)$.

Now we consider an inner-outer factorization of a stable transfer matrix. Suppose that $G(s)$ is stable and $G(j\omega)$ is of maximum column rank for all $0 \leq \omega \leq \infty$. Let a minimal realization of $G(s)$ be given by (2.1). Define $\Delta := D^T D$ and a Hamiltonian matrix

$$\mathscr{H} = \begin{bmatrix} A - B\Delta^{-1}D^T C & -B\Delta^{-1}B^T \\ -C^T C + C^T D\Delta^{-1}D^T C & -(A - B\Delta^{-1}D^T C)^T \end{bmatrix} \tag{2.8}$$

It should be noted that $\mathscr{H}$ equals the A-matrix of $[G^*(s)G(s)]^{-1}$. Moreover, let the algebraic Riccati equation (ARE) associated with the Hamiltonian $\mathscr{H}$ be

$$\begin{aligned}&(A - B\Delta^{-1}D^T C)^T X + X(A - B\Delta^{-1}D^T C) - XB\Delta^{-1}B^T X \\ &\qquad + C^T C - C^T D\Delta^{-1}D^T C = 0\end{aligned} \tag{2.9}$$

Theorem 2.1

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization with A stable and $\Delta = D^T D > 0$.

Then an inner-outer factorization of $G(s)$ is given by (2.6) with

$$G_o(s) = \left[\begin{array}{c|c} A & B \\ \hline -\Delta^{1/2}K & \Delta^{1/2} \end{array}\right] \tag{2.10}$$

$$G_i(s) = \left[\begin{array}{c|c} A + BK & B\Delta^{-1/2} \\ \hline C + DK & D\Delta^{-1/2} \end{array}\right] \tag{2.11}$$

where $K = -\Delta^{-1}(B^T X + D^T C)$ and $\Delta = \Delta^{T/2}\Delta^{1/2}$.

Proof: Although a proof is found in [6], we provide a different proof. Using (2.4),

$$\begin{aligned} G^*(s)\,G(s) &= \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ -C^T C & -A^T & -C^T D \\ \hline D^T C & B^T & \Delta \end{array}\right] \end{aligned} \tag{2.12}$$

Thus from (2.3), we get

$$[G^*(s)\,G(s)]^{-1} = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] \tag{2.13}$$

where $\tilde{A}=\mathscr{H}$ and

$$\tilde{B}=\begin{bmatrix} B\Delta^{-1} \\ -C^TD\Delta^{-1} \end{bmatrix},\ \tilde{C}=-[\Delta^{-1}D^TC \ \ \Delta^{-1}B^T],\ \tilde{D}=\Delta^{-1} \tag{2.14}$$

From hypotheses, $\mathscr{H}$ of (2.8) has no eigenvalues on the imaginary axis, and $(A-B\Delta^{-1}D^TC,\ B\Delta^{-1}B^T)$ is controllable. Hence, the ARE of (2.9) has a unique positive definite solution X and $A+BK$ is stable [6]. Introducing the basis change $T=\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$, we get

$$\begin{aligned} T^{-1}\tilde{A}T &= \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A-B\Delta^{-1}D^TC & -B\Delta^{-1}B^T \\ -C^TC+C^TD\Delta^{-1}D^TC & -(A-B\Delta^{-1}D^TC)^T \end{bmatrix}\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \\ &= \begin{bmatrix} A+BK & -B\Delta^{-1}B^T \\ 0 & -(A+BK)^T \end{bmatrix} \end{aligned} \tag{2.15}$$

$$T^{-1}\tilde{B}=\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} B\Delta^{-1} \\ -C^TD\Delta^{-1} \end{bmatrix}=\begin{bmatrix} B\Delta^{-1} \\ K^T \end{bmatrix} \tag{2.16}$$

$$\begin{aligned} \tilde{C}T &= -[\Delta^{-1}D^TC \ \ \Delta^{-1}B^T]\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \\ &= [K \ \ -\Delta^{-1}B^T] \end{aligned} \tag{2.17}$$

Hence, from (2.13) and (2.15)–(2.17), we have

$$\begin{aligned} G^*(s)\,G(s) &= \left[\begin{array}{c|c} T^{-1}\tilde{A}T & T^{-1}\tilde{B} \\ \hline \tilde{C}T & \tilde{D} \end{array}\right]^{-1} \\ &= \left[\begin{array}{cc|c} A+BK & -B\Delta^{-1}B^T & B\Delta^{-1} \\ 0 & -(A+BK)^T & K^T \\ \hline K & -\Delta^{-1}B^T & \Delta^{-1} \end{array}\right]^{-1} \\ &= \left[\begin{array}{cc|c} A & 0 & B \\ -K^T\Delta K & -A^T & K^T\Delta \\ \hline -\Delta K & B^T & \Delta \end{array}\right] \end{aligned} \tag{2.18}$$

We observe from (2.18) and (2.12) that (2.18) has a factorization of the from

$$\begin{aligned} G^*(s)\,G(s) &= \left[\begin{array}{c|c} A & B \\ \hline Z & \Delta^{1/2} \end{array}\right]^* \left[\begin{array}{c|c} A & B \\ \hline Z & \Delta^{1/2} \end{array}\right] \\ &= \left[\begin{array}{c|c} -A^T & -Z^T \\ \hline B^T & \Delta^{T/2} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline Z & \Delta^{1/2} \end{array}\right] \end{aligned}$$

$$=\left[\begin{array}{cc|c} A & 0 & B \\ -Z^TZ & -A^T & -Z^T\Delta^{1/2} \\ \hline \Delta^{T/2}Z & B^T & \Delta \end{array}\right] \tag{2.19}$$

Comparing (2.12) and (2.19) gives

$$Z=\Delta^{-T/2}(B^TX+D^TC)=-\Delta^{1/2}K \tag{2.20}$$

We can also show that $G_o(s)$ of (2.10) is an outer function. An inner function is therefore obtained by $G_i(s)=G(s)G_o^{-1}(s)$, which is given by (2.11). This completes the proof of Theorem 2.1. □

In the above proof, we utilized the fact that $T^{-1}\tilde{A}T$ has a block upper triangular form with $A+BK$ stable and $-(A+BK)^T$ antistable and that $T=\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ does not change the diagonal block elements A and $-A^T$ of the A-matrix of $G^*(s)G(s)$. The present method of proof can also be applied to the derivation of a spectral factor $G_o(s)$ such that $G_o^*(s)G_o(s)=G^*(s)G(s)$, even if A is not stable.

In fact, we assume that A is not stable, but has no eigenvalues on the imaginary axis. We define the ARE associated with the transpose of the A-matrix, a Hamiltoninan matrix, of (2.19) as

$$AP+PA^T-PZ^TZP=0 \tag{2.21}$$

Let the stabilizing solution of (2.21) be P with $A_P:=A-PZ^TZ$ stable. Then we can show that the application of $T=\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix}$ to (2.19) gives

$$G^*(s)\,G(s)=\left[\begin{array}{cc|c} A_P & 0 & B_P \\ -Z^TZ & -A_P^T & -Z^T\Delta^{1/2} \\ \hline \Delta^{T/2}Z & B_P^T & \Delta \end{array}\right] \tag{2.22}$$

where $B_P:=B-PZ^T\Delta^{1/2}$. Since (2.22) has the same form as (2.18) or (2.19), it can be factored as in (2.19). Actually, the outer function is given by

$$G_O(s)=\left[\begin{array}{c|c} A_P & B_P \\ \hline -\Delta^{1/2}K & \Delta^{1/2} \end{array}\right] \tag{2.23}$$

where

$$\begin{aligned} A_P&=A-PK^T\Delta K \\ B_P&=B+PK^T\Delta \end{aligned} \tag{2.24}$$

3. Multivariable Wiener Filtering Problem

In this section, we consider the multivariable Wiener filtering problem. First we describe the optimal Wiener filtering problem and present its solution based on the spectral factorization and additive decomposition [1], [7]. Then for the case where the spectral density matrices are rational, we embed the Wiener filtering problem in the standard model matching problem. The optimal solution is derived by applying the formula (1.5) and the state-space technique [3], [6].

3.1 Problem Statement and Classical Solution

Suppose that we observe the signal $y(t)$ which is the sum of the desired signal $\theta(t)$ and the noise $\nu(t)$; namely,

$$y(t)=\theta(t)+\nu(t) \tag{3.1}$$

where $y(t)$, $\theta(t)$, $\nu(t)$, $-\infty<t<\infty$ are p-dimensional zero-mean second order jointly stationary processes. It is assumed that the signal $\theta(t)$ and the noise $\nu(t)$ are uncorrelated. Let the spectral density matrices of $y(t)$, $\theta(t)$, and $\nu(t)$ be given by $S_{yy}(s)$, $S_{\theta\theta}(s)$, and $S_{\nu\nu}(s)$, respectively. Then we have

$$S_{yy}(s)=S_{\theta\theta}(s)+S_{\nu\nu}(s) \tag{3.2}$$

The Wiener filtering problem is to find the least-squares estimate (LSE) of the desired signal $\theta(t)$ based on the past observations $Y^t=\{y(\tau),\ -\infty<\tau<t\}$. As shown in Fig. 1, if we denote the LSE by $\hat{\theta}(t)$, the problem is to find the

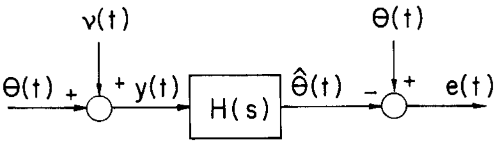

Fig. 1 Wiener filtering problem

causal filter $H(s)=\mathscr{L}\ \{h(t)\}$ minimizing the mean square error

$$J=E\{\|\theta(t)-\hat{\theta}(t)\|^2\} \tag{3.3}$$

where $E\ \{\cdot\}$ denotes the mathematical expectation, and the LSE $\hat{\theta}(t)$ is given by

$$\hat{\theta}(t)=\int_0^\infty h(\tau)y(t-\tau)d\tau$$

We now assume that the spectral density matrix $S_{yy}(s)$ has the canonical spectral factorization

$$S_{yy}(s)=\Phi_y(s)\Phi_y^T(-s) \tag{3.4}$$

The canonical factor $\Phi_y(s)$ is a $p\times p$ outer function and is unique up to the right multiplication by an orthogonal matrix [12].

Theorem 3.1

The transfer function matrix $H(s)$ of the optimal Wiener filter is given by

$$H(s)=[S_{\theta\theta}(s)\Phi_y^{-T}(-s)]_+\Phi_y^{-1}(s) \tag{3.5}$$

where $\Phi_y^{-T}(s):=(\Phi_y^{-1}(s))^T$.

Proof: A proof is given in [1], [7], [8]. □

We assume further that $\nu(t)$ is a white noise with $S_{\nu\nu}(s)=R$, positive definite, and that $S_{\theta\theta}(s)\to 0$ as $s\to\infty$. Then it follows that [1]

$$S_{yy}(s)=S_{\theta\theta}(s)+R \;\to\; R \text{ as } s\to\infty \tag{3.6}$$

and

$$\Phi_y(s)\to L \text{ as } s\to\infty \tag{3.7}$$

where L is a nonsingular matrix such that $LL^T=R$. From (3.4) and (3.6),

$$S_{\theta\theta}(s)=\Phi_y(s)\Phi_y^T(-s)-R \tag{3.8}$$

and hence

$$S_{\theta\theta}(s)\Phi_y^{-T}(-s)=\Phi_y(s)-R\Phi_y^{-T}(-s) \tag{3.9}$$

It follows from (3.7) that the r. h. s. of (3.9) is decomposed into the sum of the stable function $\Phi_y(s)$ and the antistable function $-R\Phi_y^{-T}(-s)$, where both functions tend to the non-zero constant matrix L as $s\to\infty$. Adding and subtracting this constant matrix to the r. h. s. of (3.9) yield

$$S_{\theta\theta}(s)\Phi_y^{-T}(-s)=[\Phi_y(s)-L]+L[I-L^T\Phi_y^{-T}(-s)] \tag{3.10}$$

This is an additive decomposition for which each term in the r. h. s. vanishes at $s=\infty$, so that the stable part is given by [7]

$$[S_{\theta\theta}(s)\Phi_y^{-T}(-s)]_+=\Phi_y(s)-L \tag{3.11}$$

Theorem 3.2

The transfer function of the optimal Wiener filter is given by

$$H(s)=I_p-L\Phi_y^{-1}(s) \tag{3.12}$$

Proof: A proof is immediate from (3.5) and (3.11). □

The above derivation of the optimal Wiener filter is due to Barrett [1] and Shaked [8]. In the following, we wish to derive the same result for the case where the spectral density functions are rational by using the optimal H_2 control theory, after converting the problem into a standard model matching problem [3], [4], [6].

3.2 Standard Model Matching Problem

In this section, we assume that the desired signal $\theta(t)$ has a rational spectral density matrix, so that it is generated by a state–space model

$$\dot{x}(t)=Ax(t)+G\xi(t),\ A: \text{stable} \tag{3.13}$$

$$\theta(t)=Cx(t) \tag{3.14}$$

where $x(t)$ is the $n\times 1$ state vector, $\xi(t)$ is the $q\times 1$ white noise with a mean zero and covariance matrix I_q, and A, G, C are $n\times n$, $n\times q$, $p\times n$ constant matrices, respectively. We assume that (A, G, C) is minimal. Define $\Phi(s):=(sI-A)^{-1}$. The spectral density matrix $S_{yy}(s)$ is then given by

$$S_{yy}(s)=R+C\Phi(s)GG^T\Phi^T(-s)C^T \tag{3.15}$$

We see from Fig. 1 that, in s–domain,

$$\begin{aligned} e(s)&=\theta(s)-\hat{\theta}(s)=C\Phi(s)G\xi(s)-\hat{\theta}(s)\\ y(s)&=\theta(s)+\nu(s)=C\Phi(s)G\xi(s)+L\eta(s)\\ \hat{\theta}(s)&=H(s)y(s) \end{aligned} \tag{3.16}$$

where $\nu(s)=L\eta(s)$, and where η is a white noise with $N(0, I_p)$.

According to the general framework of the model matching problem [3], [4], [6], (3.16) is rewritten as (see Fig. 2)

$$\begin{aligned} e(s)&=P_{11}v(s)+P_{12}\hat{\theta}(s)\\ y(s)&=P_{21}v(s)+P_{22}\hat{\theta}(s) \end{aligned} \tag{3.17}$$

and

$$\hat{\theta}(s)=H(s)y(s) \tag{3.18}$$

where

$$v(s)=\begin{bmatrix}\xi(s)\\ \eta(s)\end{bmatrix}, \tag{3.19}$$

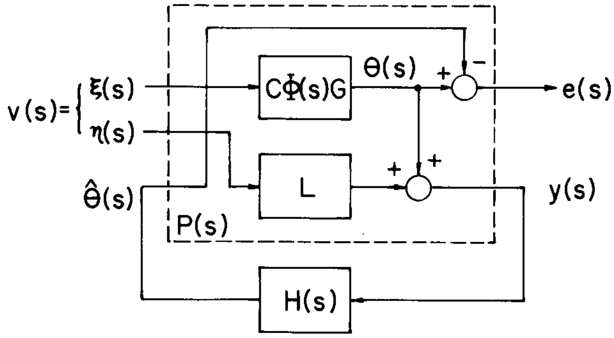

Fig. 2 Standard block diagram for Wiener filtering

$$P_{11}=[C\Phi(s)G\ \ 0],\qquad P_{12}=-I_p$$
$$P_{21}=[C\Phi(s)G\ \ L],\qquad P_{22}=0 \tag{3.20}$$

Moreover, the error $e(s)$ is expressed as

$$e(s)=[P_{11}(s)-H(s)P_{21}(s)]v(s) \tag{3.21}$$

Thus the optimal Wiener filtering problem reduces to a standard model matching problem that minimizes the H_2 norm of the transfer function from $v(s)$ to $e(s)$, namely,

$$J=\|P_{11}(s)-H(s)P_{21}(s)\|_2=\text{minimum}. \tag{3.22}$$

It may be noted in (3.20) that a doubly coprime factorization of P_{22} is not necessary, since $P_{22}=0$ in the present problem.

For simplicity, we define $T_1(s):=P_{11}(s)$ and $T_2(s):=P_{21}(s)$. Let $T_2(s)=T_{2\infty}(s)T_{2ci}(s)$ be a co-inner-outer factorization of $T_2(s)$, where $T_{2\infty}$ is co-outer and T_{2ci} is co-inner. It follows from (1.5) that the optimal filter $H(s)$ is expressed as

$$H(s)=[T_1(s)T_{2ci}^*(s)]_+T_{2co}^{-1}(s) \tag{3.23}$$

In the following, we derive the optimal filter transfer function by computing the r. h. s. of (3.23) via the state-space technique.

3.3 Solution to Wiener Filtering Problem

We see that realizations of $T_1(s)$ and $T_2(s)$ are respectively given by

$$T_1(s)=\left[\begin{array}{c|cc} A & G & 0 \\ \hline C & 0 & 0 \end{array}\right] \tag{3.24}$$

and

$$T_2(s)=\left[\begin{array}{c|cc} A & G & 0 \\ \hline C & 0 & L \end{array}\right] \tag{3.25}$$

Since $T_2(j\omega)$ is of maximum row rank for $0 \leq \omega \leq \infty$, a co-inner-outer factorization of $T_2(s)$ is easily obtained by Theorem 2.1. It follows from (2.8) that the Hamiltonian matrix associated with the inner-outer factorization of $T_2(s)^T$ is given by

$$\mathscr{H}=\begin{bmatrix} A^T & -C^TR^{-1}C \\ -GG^T & -A \end{bmatrix} \tag{3.26}$$

Also, from (2.9), the ARE associated with $\mathscr{H}$ is given by

$$AY+YA^T-YC^TR^{-1}CY+GG^T=0 \tag{3.27}$$

It is well known that since (A, G, C) is minimal, the ARE of (3.27) has a unique positive definite solution Y, and $A-YC^TR^{-1}C$ is stable. Thus it follows from (2.10) that an co-outer function of $T_2(s)$ is given by

$$T_{2co}(s)=\left[\begin{array}{c|c} A & YC^TL^{-T} \\ \hline C & L \end{array}\right] \tag{3.28}$$

where $T_2(s)T_2(-s)^T=T_{2co}(s)T_{2co}(-s)^T$, so that a co-inner function is obtained as

$$\begin{aligned} T_{2ci}(s) &= [T_{2co}(s)]^{-1}T_2(s) \\ &= \left[\begin{array}{c|c} A & YC^TL^{-T} \\ \hline C & L \end{array}\right]^{-1} \left[\begin{array}{c|cc} A & G & 0 \\ \hline C & 0 & L \end{array}\right] \\ &= \left[\begin{array}{c|c} A-YC^TR^{-1}C & YC^TR^{-1} \\ \hline -L^{-1}C & L^{-1} \end{array}\right] \left[\begin{array}{c|cc} A & G & 0 \\ \hline C & 0 & L \end{array}\right] \\ &= \left[\begin{array}{cc|cc} A-YC^TR^{-1}C & YC^TR^{-1}C & 0 & YC^TL^{-T} \\ 0 & A & G & 0 \\ \hline -L^{-1}C & L^{-1}C & 0 & I_p \end{array}\right] \end{aligned} \tag{3.29}$$

By the basis change $T=\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$, we get

$$T_{2ci}(s)=\left[\begin{array}{c|cc} A-YC^TR^{-1}C & G & -YC^TL^{-T} \\ \hline L^{-1}C & 0 & I_p \end{array}\right] \tag{3.30}$$

Hence, from (3.24) and (3.30),

$$T_1(s)\,T^*_{2ci}(s)=\left[\begin{array}{c|cc} A & G & 0 \\ \hline C & 0 & 0 \end{array}\right]\left[\begin{array}{c|c} -A^T+C^TR^{-1}CY & -C^TL^{-T} \\ \hline G^T & 0 \\ -L^{-1}CY & I_p \end{array}\right]$$

$$=\left[\begin{array}{cc|c} A & GG^T & 0 \\ 0 & -A^T+C^TR^{-1}CY & -C^TL^{-T} \\ \hline C & 0 & 0 \end{array}\right] \tag{3.31}$$

Introducing the basis change $T=\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$ yields

$$T_1(s)\,T^*_{2ci}(s)=\left[\begin{array}{cc|c} A & 0 & YC^TL^{-T} \\ 0 & -A^T+C^TR^{-1}CY & -C^TL^{-T} \\ \hline C & CY & 0 \end{array}\right]$$

$$\begin{aligned} &=C(sI-A)^{-1}YC^TL^{-T} \\ &\quad -CY(sI+A^T-C^TR^{-1}CY)^{-1}C^TL^{-T} \end{aligned} \tag{3.32}$$

Since the first term in the r. h. s. of (3.32) is stable, but the second term is antistable, the stable part of $T_1(s)T^*_{2ci}(s)$ is given by

$$[T_1(s)\,T^*_{2ci}(s)]_+=\left[\begin{array}{c|c} A & YC^TL^{-T} \\ \hline C & 0 \end{array}\right] \tag{3.33}$$

It follows from (3.23), (3.28) and (3.33) that the optimal transfer function is given by

$$\begin{aligned} H(s)&=\left[\begin{array}{c|c} A & YC^TL^{-T} \\ \hline C & 0 \end{array}\right]\left[\begin{array}{c|c} A & YC^TL^{-T} \\ \hline C & L \end{array}\right]^{-1} \\ &=\left\{\left[\begin{array}{c|c} A & YC^TL^{-T} \\ \hline C & L \end{array}\right]-L\right\}\left[\begin{array}{c|c} A & YC^TL^{-T} \\ \hline C & L \end{array}\right]^{-1} \\ &=I_p-L\left[\begin{array}{c|c} A & YC^TL^{-T} \\ \hline C & L \end{array}\right]^{-1} \\ &=\left[\begin{array}{c|c} A-YC^TR^{-1}C & YC^TR^{-1} \\ \hline C & 0 \end{array}\right] \end{aligned} \tag{3.34}$$

$$=C(sI-A+YC^TR^{-1}C)^{-1}YC^TR^{-1} \tag{3.35}$$

This is the transfer function of the well-known steady-state Kalman filter, and is exactly the transfer function of the Wiener filter [7]. It may be also noted that the expression (3.34) is directly obtained from (3.12) and (3.28).

4. Stationary LQG Problem

In this section, we consider the stationary LQG problem via the H_2 optimal control theory. The problem is transformed into a model matching problem by using a doubly coprime factorization, and the optimal controller is derived by applying an inner–outer and co–inner outer factorizations.

4.1 Problem Statement

Consider a linear stochastic system described by

$$\dot{x}(t)=Ax(t)+Bu(t)+G\xi(t)$$
$$y(t)=Cx(t)+L\eta(t) \tag{4.1}$$

where $x(t)$ is the $n\times 1$ state vector, $u(t)$ is the $m\times 1$ control vector, $y(t)$ is the $p\times 1$ observation vector, $\xi(t)$ is the $q\times 1$ process noise, and $\eta(t)$ is the $p\times 1$ observation noise. A, B, C, G, L are constant matrices of dimensions $n\times n$, $n\times m$, $p\times n$, $n\times q$, $p\times p$ respectively. The noise processes $\xi(t)$, $\eta(t)$ are white Gaussian with means zero and

$$E\{\xi(t)\xi^T(\tau)\}=I_q\delta(t-\tau) \tag{4.2a}$$
$$E\{\eta(t)\eta^T(\tau)\}=I_p\delta(t-\tau) \tag{4.2b}$$

Moreover, x_0, $\xi(t)$, and $\eta(t)$ are assumed to be independent.

We consider the stationary LQG control problem that minimizes the steady–state average cost

$$J=E\{x^T(t)Qx(t)+u^T(t)Ru(t)\},\qquad t\to\infty \tag{4.3}$$

where the closed–loop system is required to be internally asymptotically stable, and where $Q\geq 0$, $R>0$. The admissible control $u(t)$ can only depend on the past observations $\{y(\tau),\ \tau<t\}$. Also, we assume that $(A, B, Q^{1/2})$, (A, G, C) are minimal, where $Q^{1/2}$ is a matrix such that $Q=Q^{T/2}Q^{1/2}$.

4.2 Model Matching Problem

In order to rewrite the LQG control problem as a standard H_2 control problem, we define

$$e(t) := \begin{bmatrix} Q^{1/2}x(t) \\ R^{1/2}u(t) \end{bmatrix}, \quad v(t) := \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} \tag{4.4}$$

It follows from (4.1) and (4.4) that, in s-domain,

$$\begin{bmatrix} e(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} v(s) \\ u(s) \end{bmatrix}$$

$$= \begin{bmatrix} Q^{1/2}\Phi G & 0 & Q^{1/2}\Phi B \\ 0 & 0 & R^{1/2} \\ C\Phi G & L & C\Phi B \end{bmatrix} \begin{bmatrix} \xi(s) \\ \eta(s) \\ u(s) \end{bmatrix} \tag{4.5}$$

where $\Phi(s) = (sI-A)^{-1}$. The admissible control is expressed as

$$u(s) = K(s)y(s) \tag{4.6}$$

where $K(s)$ is an $m \times p$ rational transfer function in RH_+^∞. Hence, a standard block diagram of the stationary LQG problem becomes as shown in Fig. 3.

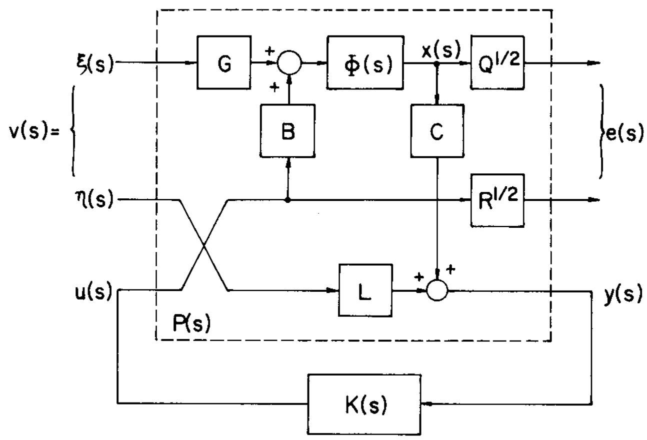

Fig. 3 Standard block diagram for stationary LQG problem

From (4.5) and (4.6), we get

$$e(s) = [P_{11} + P_{12}K(I-P_{22}K)^{-1}P_{21}]v(s) \tag{4.7}$$

We see from (4.4) that minimizing J of (4.3) is equivalent to minimizing

$$\|e(s)\|_2^2 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} e^T(-s)e(s)ds \tag{4.8}$$

with respect to $K(s)$. The LQG control problem therefore reduces to minimizing the H_2 norm

$$J=\|P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}\|_2 \tag{4.9}$$

A doubly coprime factorization of $P_{22}(s)=C\Phi(s)B$ is given by [6], [9]

$$P_{22}(s)=N_2M_2^{-1}=\tilde{M}_2^{-1}\tilde{N}_2 \tag{4.10}$$

where

$$\begin{bmatrix} \tilde{X}_2 & -\tilde{Y}_2 \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix}\begin{bmatrix} M_2 & Y_2 \\ N_2 & X_2 \end{bmatrix}=\begin{bmatrix} M_2 & Y_2 \\ N_2 & X_2 \end{bmatrix}\begin{bmatrix} \tilde{X}_2 & -\tilde{Y}_2 \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix}=I_{p+m} \tag{4.11}$$

It has been shown that the set of all proper rational stabilizing controllers is parametrized as [3], [6], [13]

$$\begin{aligned} K(s) &= (Y_2(s)-M_2(s)Q(s))(X_2(s)-N_2(s)Q(s))^{-1} \\ &= (\tilde{X}_2(s)-Q(s)\tilde{N}_2(s))^{-1}(\tilde{Y}_2(s)-Q(s)\tilde{M}_2(s)) \end{aligned} \tag{4.12}$$

where $Q(s)$ is a stable rational function.

Now we define

$$\begin{aligned} T_1(s) &= P_{11}+P_{12}M_2\tilde{Y}_2P_{21} \\ T_2(s) &= P_{12}M_2 \\ T_3(s) &= \tilde{M}_2P_{21} \end{aligned} \tag{4.13}$$

Then $T_1(s)$, $T_2(s)$, $T_3(s)$ are stable, and the performance index (4.9) is expressed as (1.1) with $\alpha=2$ [6]. Hence the solution $Q(s)$ is given by (1.5). Moreover, if we choose F, H such that $A_F:=A+BF$, $A_H:=A+HC$ are stable, then the state-space realizations of M_2, N_2, $\tilde{M}_2$, $\tilde{N}_2$, X_2, Y_2, $\tilde{X}_2$, $\tilde{Y}_2$ are given by [6]

$$\begin{aligned} M_2 &= \left[\begin{array}{c|c} A_F & B \\ \hline F & I_m \end{array}\right], & N_2 &= \left[\begin{array}{c|c} A_F & B \\ \hline C & 0 \end{array}\right] \\ \tilde{M}_2 &= \left[\begin{array}{c|c} A_H & H \\ \hline C & I_p \end{array}\right], & \tilde{N}_2 &= \left[\begin{array}{c|c} A_H & B \\ \hline C & 0 \end{array}\right] \\ X_2 &= \left[\begin{array}{c|c} A_F & -H \\ \hline C & I_p \end{array}\right], & Y_2 &= \left[\begin{array}{c|c} A_F & -H \\ \hline F & 0 \end{array}\right] \\ \tilde{X}_2 &= \left[\begin{array}{c|c} A_H & -B \\ \hline F & I_m \end{array}\right], & \tilde{Y}_2 &= \left[\begin{array}{c|c} A_H & -H \\ \hline F & 0 \end{array}\right] \end{aligned} \tag{4.14}$$

Thus it follows from (4.13) that (see Appendix)

$$T_1(s)=\left[\begin{array}{cc|cc} A_F & -BF & G & 0 \\ 0 & A_H & G & HL \\ \hline Q^{1/2} & 0 & 0 & 0 \\ R^{1/2}F & -R^{1/2}F & 0 & 0 \end{array}\right] \tag{4.15a}$$

$$T_2(s)=\left[\begin{array}{c|c} A_F & B \\ \hline Q^{1/2} & 0 \\ R^{1/2}F & R^{1/2} \end{array}\right] \tag{4.15b}$$

$$T_3(s)=\left[\begin{array}{c|cc} A_H & G & HL \\ \hline C & 0 & L \end{array}\right] \tag{4.15c}$$

4.3 Solution to Stationary LQG Problem

To apply the formula (1.5), we need the inner–outer and co–inner–outer factorizations of $T_2(s)$ and $T_3(s)$, respectively. Since $T_2(j\omega)$ is of full column rank for $0\leq\omega\leq\infty$, an outer function such that $T_{2o}^*(s)T_{2o}(s)=T_2^*(s)T_2(s)$ is obtained from (2.10) :

$$T_{2o}(s)=\left[\begin{array}{c|c} A+BF & B \\ \hline R^{-T/2}(RF+B^TX) & R^{1/2} \end{array}\right] \tag{4.16}$$

where X is a unique stabilizing solution of the ARE

$$A^TX+XA-XBR^{-1}B^TX+Q=0 \tag{4.17}$$

If we take $F=-R^{-1}B^TX$, then $A_F:=A-BR^{-1}B^TX$ is stable, and the simplest form of the outer function is obtained as $T_{2o}(s)=R^{1/2}$. We therefore get an inner function from (4.15 b) as

$$T_{2i}(s)=T_2(s)R^{-1/2}=\left[\begin{array}{c|c} A_F & BR^{-1/2} \\ \hline Q^{1/2} & 0 \\ R^{1/2}F & I_p \end{array}\right] \tag{4.18}$$

Since $T_3(j\omega)$ is of row full rank for $0\leq\omega\leq\infty$, a co–inner–outer factorization of $T_3(s)$ is also obtained from Theorem 2.1. Applying an inner–outer factorization to $T_3(s)^T$, we have

$$T_{3co}(s)=\left[\begin{array}{c|c} A_H & HL+YC^TL^{-T} \\ \hline C & L \end{array}\right] \tag{4.19}$$

where Y is a unique stabilizing solution of the ARE

$$AY+YA^T-YC^T(LL^T)^{-1}CY+GG^T=0 \tag{4.20}$$

If we set $H=-YC^T(LL^T)^{-1}$, then (4.19) reduces to $T_{3co}(s)=L$. Hence, for $H=-YC^T(LL^T)^{-1}$, it follows from (4.15 c) that

$$T_{3ci}(s)=T_{3co}^{-1}(s)\,T_3(s)=\left[\begin{array}{c|cc} A_H & G & HL \\ \hline L^{-1}C & 0 & I_p \end{array}\right] \tag{4.21}$$

where $A_H:=A-YC^T(LL^T)^{-1}C$ is stable.

Now we compute $T_{2i}^{*}(s)T_1(s)T_{3ci}(s)$ by using the above results. In the following, F and H are fixed as

$$F=-R^{-1}B^TX,\ H=-YC^T(LL^T)^{-1} \tag{4.22}$$

It follows from (4.15 a) and (4.18) that

$$T_{2i}^{*}T_1=\left[\begin{array}{c|cc} -A_F^T & -Q^{T/2} & -F^TR^{T/2} \\ \hline R^{-T/2}B^T & 0 & I_p \end{array}\right]\left[\begin{array}{cc|cc} A_F & -BF & G & 0 \\ 0 & A_H & G & HL \\ \hline Q^{1/2} & 0 & 0 & 0 \\ R^{1/2}F & -R^{1/2}F & 0 & 0 \end{array}\right]$$

$$=\left[\begin{array}{ccc|cc} A_F & -BF & 0 & G & 0 \\ 0 & A_H & 0 & G & HL \\ -Q-F^TRF & F^TRF & -A_F^T & 0 & 0 \\ \hline R^{1/2}F & -R^{1/2}F & R^{-T/2}B^T & 0 & 0 \end{array}\right]$$

$$=\left[\begin{array}{ccc|cc} A_F & 0 & -HC & 0 & -HL \\ 0 & -A_F^T & 0 & -XG & 0 \\ 0 & 0 & A_H & G & HL \\ \hline 0 & R^{-T/2}B^T & -R^{1/2}F & 0 & 0 \end{array}\right] \tag{4.23a}$$

$$=\left[\begin{array}{cc|cc} -A_F^T & 0 & -XG & 0 \\ 0 & A_H & G & HL \\ \hline R^{-T/2}B^T & -R^{1/2}F & 0 & 0 \end{array}\right] \tag{4.23}$$

where (4.23 a) is derived by the basis change $T=\begin{bmatrix} I & 0 & I \\ 0 & 0 & I \\ X & I & X \end{bmatrix}$. Thus, from (4.18) and (4.23), we get

$$T_{2i}^{*}T_1T_{3ci}^{*}=\left[\begin{array}{cc|cc} -A_F^T & 0 & -XG & 0 \\ 0 & A_H & G & HL \\ \hline R^{-T/2}B^T & -R^{1/2}F & 0 & 0 \end{array}\right]\left[\begin{array}{c|c} -A_H^T & -C^TL^{-T} \\ \hline G^T & 0 \\ L^TH^T & I_p \end{array}\right]$$

$$=\left[\begin{array}{ccc|c} -A_H^T & 0 & 0 & -C^TL^{-T} \\ -XGG^T & -A_F^T & 0 & 0 \\ GG^T+HLL^TH^T & 0 & A_H & HL \\ \hline 0 & R^{-T/2}B^T & -R^{1/2}F & 0 \end{array}\right]$$

$$=\left[\begin{array}{ccc|c} -A_H^T & 0 & 0 & -C^TL^{-T} \\ -XGG^T & -A_F^T & 0 & 0 \\ 0 & 0 & A_H & 0 \\ \hline -R^{1/2}FY & R^{-T/2}B^T & -R^{1/2}F & 0 \end{array}\right] \tag{4.24a}$$

$$=\left[\begin{array}{cc|c} -A_H^T & 0 & -C^TL^{-T} \\ -XGG^T & -A_F^T & .0 \\ \hline -R^{1/2}FY & R^{-T/2}B^T & 0 \end{array}\right] \tag{4.24}$$

where (4.24 a) is obtained by the basis change $T=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ Y & 0 & I \end{bmatrix}$.

Since A_F, A_H are stable, the r. h. s. of (4.24) is antistable; hence we have

$$[T_{2i}^{*}(s)T_1(s)T_{3ci}^{*}(s)]_+=0 \tag{4.25}$$

It therefore follows from (1.5) that $Q_{opt}(s)=0$, so that from (4.12), the optimal controller is given by $K_{opt}(s)=Y_2(s)X_2^{-1}(s)=\tilde{X}_2^{-1}(s)\tilde{Y}_2(s)$. Thus, from (4.14),

$$K_{opt}(s)=\left[\begin{array}{c|c} A_F & -H \\ \hline F & 0 \end{array}\right]\left[\begin{array}{c|c} A_F & -H \\ \hline C & I_p \end{array}\right]^{-1}$$

$$=\left[\begin{array}{c|c} A_F & -H \\ \hline F & 0 \end{array}\right]\left[\begin{array}{c|c} A_F+HC & H \\ \hline C & I_p \end{array}\right]$$

$$=\left[\begin{array}{cc|c} A_F+HC & 0 & H \\ -HC & A_F & -H \\ \hline 0 & F & 0 \end{array}\right]$$

Introducing the basis change $T=\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$ yields

$$K_{opt}(s)=\left[\begin{array}{cc|c} A_F+HC & 0 & H \\ 0 & A_F & 0 \\ \hline -F & F & 0 \end{array}\right] \tag{4.26}$$
$$=-F(sI-A-BF-HC)^{-1}H$$

This is the transfer function of the well-known optimal LQG controller, which consists of the stationary Kalman filter and noise-free regulator [4], [8].

5. Spectral Factorization

In this section, digressing from H_2 problems, we consider a spectral factorization appearing in the H_∞ optimization [6]. Let a stable transfer function be given by a minimal realization

$$G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad A: \text{stable} \tag{5.1}$$

The H_∞ norm of $G(s)$ is defined by [4], [6]

$$\|G(s)\|_\infty:=\sup_{\omega} \bar{\sigma}[G(j\omega)], \quad 0\leq\omega\leq\infty \tag{5.2}$$

where $\bar{\sigma}[\cdot]$ denotes the maximum singular value.

Suppose that γ is a scalar constant such that $\|G(s)\|_\infty<\gamma$. An outer function $\Theta(s)$ is called a spectral factor of $\gamma^2I-G^*(s)G(s)$ if

$$\Theta^*(s)\Theta(s)=\gamma^2I-G^*(s)G(s) \tag{5.3}$$

In the following, we present a factorization algorithm for $P(s):=\gamma^2I-G^*(s)G(s)$. The derivation of the algorithm is similar to that of Theorem 2.1.

Referring to (2.12),

$$P(s)=\left[\begin{array}{cc|c} A & 0 & -B \\ -C^TC & -A^T & C^TD \\ \hline D^TC & B^T & \gamma^2I-D^TD \end{array}\right] = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \Delta \end{array}\right] \tag{5.4}$$

where we assume that $\Delta:=\gamma^2I-D^TD$ is positive definite. Define

$$E:=A+B\Delta^{-1}D^TC, \quad \Sigma:=B\Delta^{-1}B^T, \quad \Pi:=C^TC+C^TD\Delta^{-1}D^TC \tag{5.5}$$

Then, from (5.4),

$$P^{-1}(s)=\left[\begin{array}{cc|c} E & \Sigma & B\Delta^{-1} \\ -\Pi & -E^T & -C^T D\Delta^{-1} \\ \hline \Delta^{-1}D^T C & \Delta^{-1}B^T & \Delta^{-1} \end{array}\right] \tag{5.6}$$

$$=\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$$

We see that $\tilde{A}$ is a Hamiltonian matrix, so that it is denoted by

$$\mathscr{H}=\begin{bmatrix} E & \Sigma \\ -\Pi & -E^T \end{bmatrix} \tag{5.7}$$

We assume that $\mathscr{H}$ has no eigenvalues on the imaginary axis. Then, $\mathscr{H}$ has n eigenvalues in $\mathrm{Re}[s]<0$ and n in $\mathrm{Re}[s]>0$. Hence there exists an orthogonal matrix $U=\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ such that $\mathscr{H}U=US$, where $S=\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$ is a block upper triangular form with S_{11} stable and S_{22} antistable. Thus we get

$$EU_{11}+\Sigma U_{21} = U_{11}S_{11} \tag{5.8a}$$

$$-\Pi U_{11}-E^T U_{21} = U_{21}S_{11} \tag{5.8b}$$

Theorem 5.1 [3], [6]

Suppose that $\mathscr{H}$ of (5.7) has no eigenvalues on the imaginary axis, and (E, Σ) is stabilizable. Then U_{11} is invertible, and $X:=U_{21}U_{11}^{-1}$ is symmetric and satisfies the ARE

$$E^T X+XE+X\Sigma X+\Pi=0 \tag{5.9}$$

Moreover, $E+\Sigma X$ is stable.

Proof: For the proof we need some claims.

Claim 1 : First we show that $U_{21}^T U_{11}$ is symmetric. Taking the transpose of (5.8 b), and post-multiplying this by U_{11} yield

$$-U_{11}^T\Pi U_{11}-U_{21}^T E U_{11}=S_{11}^T U_{21}^T U_{11} \tag{5.10}$$

Substituting EU_{11} of (5.8 a) into (5.10) gives a Lyapunov equation for $U_{21}^T U_{11}$:

$$S_{11}^T(U_{21}^T U_{11})+(U_{21}^T U_{11})S_{11}=-U_{11}^T\Pi U_{11}+U_{21}^T\Sigma U_{21} \tag{5.11}$$

We observe that the r. h. s. of (5.11) is symmetric. Thus $U_{21}^T U_{11}$ is symmetric, since S_{11} is stable.

Claim 2: We claim that $N:=\ker U_{11}$ is S_{11}-invariant. Take a nonzero $z\in N$. Then, from (5.8 a), $\Sigma U_{21}z=U_{11}S_{11}z$. Pre-multiplying this by $z^T U_{21}^T$ yields

$$z^T U_{21}^T\Sigma U_{21}z=z^T U_{21}^T U_{11}S_{11}z \tag{5.12}$$

But from Claim 1, the r. h. s. of this equation is zero, since it equals $z^T U_{11}^T U_{21} S_{11} z = 0$. Thus, from (5.12), we have $\Sigma U_{21} z = 0$, so that $U_{11} S_{11} z = 0$. This completes the proof of the claim.

Claim 3: We show that U_{11} is invertible. Suppose that U_{11} is singular, so that $N := \ker U_{11}$ is non-empty. Since N is invariant under S_{11}, we get $S_{11}N = N\Lambda$ with Λ stable. By using a Jordan form J, we get $\Lambda = T^{-1}JT$ for some T. Hence, we have $S_{11}(NT^{-1}) = (NT^{-1})J$, so that the first column of this relation gives

$$S_{11} z_1 = \lambda_1 z_1,\ z_1 \in N,\ \mathrm{Re}[\lambda_1] < 0 \tag{5.13}$$

It follows from (5.8 b) and (5.13) that $E^T U_{21} z_1 = -U_{21} S_{11} z_1 = -\lambda_1 U_{21} z_1$. Also, from (5.8 a), $\Sigma U_{21} z_1 = U_{11} S_{11} z_1 = 0$. Consequently, we have

$$E^T U_{21} z_1 = (-\lambda_1) U_{21} z_1,\ \Sigma U_{21} z_1 = 0,\ \mathrm{Re}[-\lambda_1] > 0 \tag{5.14}$$

Since (E, Σ) is stabilizable, $U_{21} z_1 = 0$. But, since $U_{11} z_1 = 0$, and since $\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$ is full rank, we have $z_1 = 0$. This is a contradiction, so that U_{11} is invertible.

We now prove Theorm 5.1. From Claim 1, $U_{21}^T U_{11} = U_{11}^T U_{21}$. This implies that $X = U_{21} U_{11}^{-1} = U_{11}^{-T} U_{21}^T = X^T$, so that X is symmetric. Moreover, we see from (5.8) that

$$E + \Sigma U_{21} U_{11}^{-1} = U_{11} S_{11} U_{11}^{-1} \tag{5.15a}$$
$$-\Pi - E^T U_{21} U_{11}^{-1} = U_{21} S_{11} U_{11}^{-1} \tag{5.15b}$$

Noting that $(U_{21} S_{11} U_{11}^{-1})(U_{11} S_{11} U_{11}^{-1})^{-1} = X$, we get from (5.15),

$$-(\Pi + E^T X)(E + \Sigma X)^{-1} = X \tag{5.16}$$

This implies (5.9). Finally, from (5.15 a), we see that $E + \Sigma X = U_{11} S_{11} U_{11}^{-1}$ is stable, since S_{11} is stable. □

It should be noted that ARE of (5.9) has many solutions, but the solution X with $E + \Sigma X$ stable is unique.

We now turn to the derivation of the factorization algorithm. Introducing the basis change $T = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$, we can easily show that

$$T^{-1} \tilde{A} T = \begin{bmatrix} E + \Sigma X & \Sigma \\ 0 & -(E + \Sigma X)^T \end{bmatrix} \tag{5.17}$$

$$T^{-1} \tilde{B} = \begin{bmatrix} B\Delta^{-1} \\ -K^T \end{bmatrix} \tag{5.18}$$

$$\tilde{C} T = [K \ \ \Delta^{-1} B^T] \tag{5.19}$$

where

$$K=\Delta^{-1}(B^TX+D^TC) \tag{5.20}$$

and where K defined above satisfies $E+\Sigma X=A+BK$. Taking the inverse of (5.6) using (5.17) - (5.19) yields

$$P(s)=\left[\begin{array}{c|c} T^{-1}\tilde{A}T & T^{-1}\tilde{B} \\ \hline \tilde{C}T & \tilde{D} \end{array}\right]^{-1}$$

$$=\left[\begin{array}{cc|c} A+BK & \Sigma & B\Delta^{-1} \\ 0 & -(A+BK)^T & -K^T \\ \hline K & \Delta^{-1}B^T & \Delta^{-1} \end{array}\right]^{-1}$$

$$=\left[\begin{array}{cc|c} A & 0 & B \\ K^T\Delta K & -A^T & -K^T\Delta \\ \hline -\Delta K & -B^T & \Delta \end{array}\right] \tag{5.21}$$

We observe that (5.21) is factored as

$$P(s)=\left[\begin{array}{c|c} -A^T & Z^T \\ \hline -B^T & \Delta^{T/2} \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline Z & \Delta^{1/2} \end{array}\right]$$

$$=\left[\begin{array}{cc|c} A & 0 & B \\ Z^TZ & -A^T & Z^T\Delta^{1/2} \\ \hline \Delta^{T/2}Z & -B^T & \Delta \end{array}\right] \tag{5.22}$$

The main theorem of this section is the following.

Theorem 5.2

Let $G(s)$ be given by (5.1). Suppose that

$$P(j\omega)=\gamma^2I-G^*(j\omega)G(j\omega)>0,\ 0\leq\omega\leq\infty \tag{5.23}$$

Then, a spectral factor $\Theta(s)$ of (5.3) can be computed by

$$\Theta(s)=\left[\begin{array}{c|c} A & B \\ \hline -\Delta^{1/2}K & \Delta^{1/2} \end{array}\right] \tag{5.24}$$

where $\Delta:=\gamma^2I-D^TD$, and K is given by

$$K=\Delta^{-1}(B^TX+D^TC) \tag{5.25}$$

and where X is the unique stabilizing solution of the ARE:

$$\begin{aligned}(A+B\Delta^{-1}D^TC)^TX+X(A+B\Delta^{-1}D^TC)+XB\Delta^{-1}B^TX \\ +C^TC+C^TD\Delta^{-1}D^TC=0\end{aligned} \tag{5.26}$$

Proof: The form of $\Theta(s)$ of (5.24) is immediate from (5.21) and (5.22). Also,

$(E, \Sigma)=(A+B\Delta^{-1}D^TC,\ B\Delta^{-1}B^T)$ is stabilizable, since $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is minimal and $\Delta>0$ from(5.23). Moreover, (5.23) implies that the Hamiltonian matrix $\mathscr{H}$ of (5.7) has no eigenvalues on the imaginary axis. It fact, suppose that $\mathscr{H}$ has an eigenvalue on the imaginary axis. Then, from (5.4),

$$(\bar{A}-\bar{B}\Delta^{-1}\bar{C})w=j\lambda_0 w,\ w\neq 0 \tag{5.27}$$

or

$$(j\lambda_0 I-\bar{A})w=-\bar{B}\Delta^{-1}\bar{C}w \tag{5.28}$$

We can show that $\bar{C}w\neq 0$. For, if $\bar{C}w=0$, then $\bar{A}$ has an eigenvalue on the imaginary axis; but this contradicts the assumption that A is stable. Define $v:=-\Delta^{-1}\bar{C}w\neq 0$. From (5.4) and (5.27), we get $P(j\lambda_0)v=[\Delta+\bar{C}(j\lambda_0 I-\bar{A})^{-1}\bar{B}]v=0$, a contradiction. Hence the assumptions of Theorem 5.1 are fulfilled, so that the ARE of (5.26) has the unique stabilizing solution. This completes the proof of the theorem. □

6. Conclusions

In this paper, for continuous-time systems, we have considered the multivariable Wiener filtering and the stationary LQG problems and related inner-outer and spectral factorizations. By embedding these problems in the model matching problem, solutions to the Wiener filtering and LQG problems are derived by applying the H_2 control theory and the state-space method. We have also provided a new proof for the spectral factorization that arises in the H_∞ optimization problem.

In Part II, we will develop parallel results for discrete-time systems.

Appendix

First we derive (4.15 c). It follows from (4.5) that

$$P_{21}(s)=[C\Phi G\ \ L]=\left[\begin{array}{c|cc} A & G & 0 \\ \hline C & 0 & L \end{array}\right] \tag{A1}$$

Thus, from (4.13), (4.14) and (A 1),

$$T_3(s)=\left[\begin{array}{c|c} A_H & H \\ \hline C & I_p \end{array}\right]\left[\begin{array}{c|cc} A & G & 0 \\ \hline C & 0 & L \end{array}\right] = \left[\begin{array}{cc|cc} A & 0 & G & 0 \\ HC & A_H & 0 & HL \\ \hline C & C & 0 & L \end{array}\right] \tag{A2}$$

By the basis change $T=\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$,

$$T_3(s)=\left[\begin{array}{cc|cc} A & 0 & G & 0 \\ 0 & A_H & G & HL \\ \hline 0 & C & 0 & L \end{array}\right]=\left[\begin{array}{c|cc} A_H & G & HL \\ \hline C & 0 & L \end{array}\right] \tag{A3}$$

Next we prove (4.15 b). We see from (4.5) that

$$P_{12}(s)=\left[\begin{array}{c|c} A & B \\ \hline Q^{1/2} & 0 \\ 0 & R^{1/2} \end{array}\right] \tag{A4}$$

so that from (4.13), (4.14) and (A 4),

$$T_2(s)=\left[\begin{array}{c|c} A & B \\ \hline Q^{1/2} & 0 \\ 0 & R^{1/2} \end{array}\right]\left[\begin{array}{c|c} A_F & B \\ \hline F & I_m \end{array}\right]$$

$$=\left[\begin{array}{cc|c} A_F & 0 & B \\ BF & A & B \\ \hline 0 & Q^{1/2} & 0 \\ R^{1/2}F & 0 & R^{1/2} \end{array}\right]$$

$$=\left[\begin{array}{cc|c} A_F & 0 & B \\ 0 & A & 0 \\ \hline Q^{1/2} & Q^{1/2} & 0 \\ R^{1/2}F & 0 & R^{1/2} \end{array}\right] \tag{A5}$$

by the basis change $T=\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$. But this proves (4.15 b).

Finally, it follows from (4.15 b) and (4.14) that

$$\tilde{T}_1:=P_{12}M_2\tilde{Y}_2P_{21}=T_2\tilde{Y}_2P_{21}$$

$$= \left[\begin{array}{c|c} A_F & B \\ \hline Q^{1/2} & 0 \\ R^{1/2}F & R^{1/2} \end{array}\right] \left[\begin{array}{c|c} A_H & H \\ \hline -F & 0 \end{array}\right] \left[\begin{array}{c|cc} A & G & 0 \\ \hline C & 0 & L \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_F & B \\ \hline Q^{1/2} & 0 \\ R^{1/2}F & R^{1/2} \end{array}\right] \left[\begin{array}{cc|cc} A_H & HC & 0 & HL \\ 0 & A & G & 0 \\ \hline -F & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|cc} A_F & -BF & 0 & 0 & 0 \\ 0 & A_H & HC & 0 & HL \\ 0 & 0 & A & G & 0 \\ \hline Q^{1/2} & 0 & 0 & 0 & 0 \\ R^{1/2}F & -R^{1/2}F & 0 & 0 & 0 \end{array}\right] \tag{A6}$$

Introducing the basis change

$$T=\begin{bmatrix} I & 0 & -I \\ 0 & I & -I \\ 0 & 0 & I \end{bmatrix}$$

into (A 6) gives

$$\tilde{T}_1 = \left[\begin{array}{ccc|cc} A_F & -BF & 0 & G & 0 \\ 0 & A_H & 0 & G & HL \\ 0 & 0 & A & G & 0 \\ \hline Q^{1/2} & 0 & -Q^{1/2} & 0 & 0 \\ R^{1/2}F & -R^{1/2}F & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} A_F & -BF & G & 0 \\ 0 & A_H & G & HL \\ \hline Q^{1/2} & 0 & 0 & 0 \\ R^{1/2}F & -R^{1/2}F & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} A & G & 0 \\ \hline Q^{1/2} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{A7}$$

However, the second term of the r. h. s. of (A 7) equals P_{11} from (4.5). Thus we have (4.15 a), since $T_1(s)=\tilde{T}_1(s)+P_{11}(s)$.

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