Stability Robustness Conditions for Control Systems with Bounded and Unbounded Nonlinear Gain Elements

By

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Abstract

Taking the stability robustness into account in every aspect is a current trend in control theory. However, major recent efforts have been focused on the stability robustness of linear control systems, and so, results for nonlinear systems seem to be few. In this respect, we will make several attempts to cope with stability problems for control systems having bounded or unbounded nonlinear gain elements. We consider two tpyes of Lyapunov functions: L_2 type functions, i.e., quadratic functions and L_1 Type ones. The former is used to show the stability of systems with bounded gain elements and the latter for systems with unbounded gain characteristics. The existence of such functions assures robust stability for these nonlinear systems against perturbations in the nonlinear gain elements.

I Introduction

Taking the robustness property into account is becoming an indispensable factor in developing control theories in these recent years. The central issue in the robustness problems is, of course, the stability property. Pertaining to the stability robustness problems, there are currently two sorts of problem formulations: robust stability against structured uncertainties and that against structured ones. The former deals with systems whose uncertainties are known as perturbations of parameters. The latter considers systems whose uncertainties are estimated only in size. In general, efforts so far in this field have been focused mainly on linear systems and so, results for nonlinear systems seem to be few. Looking back at the history of the control theory, however, we find an established stability robustness theory for nonlinear systems, that is, absolute stability

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problems. They were extensively studied in the 60's and 70's by researchers in the US and USSR, and are now regarded as settled problems.

This paper recasts a class of the problems with a viewpoint of recent theoretical development on stability robustness issues. Considered are the nonlinear systems with bounded nonlinear gain elements and those with elements whose nonlinear characteristics are unbounded but lie in the first and the third quadrants on the input-output plane. We take two types of functions, each belonging to L_1 and L_2, as candidates of the Lyapunov functions. The existence of these functions assures robust stability for the nonlinear systems in question. In the second section, we consider the stability conditions for systems with nonlinear bounded gain elements. In section 3, an existence condition of a Lyapunov function for systems with unbounded gain characteristics is given. Some concluding remarks are included in section 4. We will employ the following symbol conventions. For a matrix or a vector, (′) denotes the transpose. For $X=X'\in R^{n\times n}$, $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimum and maximum eigenvalues of X, $\lambda_i(X)$, $i=1, \cdots, n$, respectively. I is used to denote the identity matrix as usual. $X>0$ and $X<0$ represent the positive definiteness and negative definiteness of $X=X'$, respectively.

II Stability of Systems with Bounded Gain Elements

We consider a control system described by

$$dx/dt=Ax+F(x, t)x, \tag{1}$$

where $x=(x_1, \cdots, x_n)'\in R^n$, $A\in R^{n\times n}$ is a stable matrix and $F(x, t)=(f_{ij}(x, t)) : R^n\times R \to R^{n\times n}$ represents a nonlinear gain matrix satisfying,

$$c_{ij}\leqslant f_{ij}(x, t)\leqslant d_{ij}, \quad i, j=1, \cdots, \text{n}. \tag{2}$$

Matrices $C:=(c_{ij})$ and $D:=(d_{ij})$ give bounds for the nonlinear gain entries. In what follows, we investigate the asymptotic stability of the system (1) under the nonlinear gain perturbations (2).

From the assumption there exists a positive definite solution matrix $P=(p_{ij})$ to the Lyapunov matrix equation,

$$A'P+PA+I=0. \tag{3}$$

Using this matrix, we define:

$$F'P+PF:=Q=(q_{ij}). \tag{4}$$

where the arguments x and t in each entry of the matrix $F(x, t)$ are dropped merely for simplicity of notations. The entry q_{ij} is calculated by

$$q_{ij}=\sum_{k=1}^{n}p_{ik}f_{kj}+\sum_{k=1}^{n}f_{ki}p_{kj}. \tag{5}$$

We furthermore define a symmetric matrix $S=(s_{ij})$ by

$$s_{ij}=\sum_{k\in X1}p_{ik}c_{kj}+\sum_{k\in X2}p_{ik}d_{kj}+\sum_{k\in X3}c_{ki}p_{kj}+\sum_{k\in X4}d_{ki}p_{kj}. \tag{6}$$

$i \geqslant j,\ i, j=1, \cdots, n.$

where $X1(X2)$ and $X3(X4)$ are subsets of a natural number up to n such that $p_{ik} \geqslant 0$ $(p_{ik}<0)$ and $p_{kj} \geqslant 0 (p_{kj}<0)$, respectively. These sets depend upon a specific index i or j, but this dependence is not explicitly written for the sake of simplicity. In quite the same manner, a symmetric matrix $T=(t_{ij})$ is defined by

$$t_{ij}=\sum_{k\in X1} p_{ik}d_{kj}+\sum_{k\in X2} p_{ik}c_{kj}+\sum_{k\in X3} d_{ki}p_{kj}+\sum_{k\in X4} c_{ki}p_{kj}. \tag{7}$$

The sets Xi, $i=1, 2, 3, 4$ are defined similarly. By these definitions we obtain the upper and lower bounds for all the entries of matrix Q as

$$s_{ij} \leqslant q_{ij} \leqslant t_{ij}, \quad i \geqq j, \quad i, j=1, \cdots, n. \tag{8}$$

Any matrix whose entries satisfy the above types of inequalities is called an interval matrix. That is, interval matrices are those whose entries lie in the given intervals. Thus, the matrix Q is a symmetric interval matrix. It is known that any interval matrix can be expressed as a convex combination of its extreme matrices. In this case, Q can be written as

$$Q=\sum_{k=1}^{l} r^k Q_k, \quad r_k \geqslant 0, \quad \sum_{k=1}^{l} r_k=1. \tag{9}$$

Here, $Q^k=(q^k{}_{ij})$ is a matrix defind by

$$q^k{}_{ij}=s_{ij} \text{ or } t_{ij}, \quad i, j=1, \cdots, n. \tag{10}$$

Note that the number, l, of Q^k is given by $l=2^h (h:=n^2)$. We are now ready to state the first main result of this section.

Theorem 1

A sufficient condition for the system (1) to be stable for any gain perturbations (2) is

$$I-Q^k>0, \quad k=1, \cdots, l. \tag{11}$$

For the proof of this theorem, we provide an auxiliary result.

Lemma 1 (See reference 1))

Let $X=X' \in R^{n\times n}$ and $Y=Y' \in R^{n\times n}$. For a linear combination of these two matrices $X+kY$, $k \in R$, we have

$$\lambda_{\min}(X)+\lambda_{\min}(kY) \leqslant \lambda_i(X+kY) \leqslant \lambda_{\max}(X)+\lambda_{\max}(kY) \tag{12}$$

$i=1, \cdots, n.$

Proof of Theorem 1 : we take a quadratic function $x'Px$ as a candidate of a Lyapunov function using the solution matrix P of the Lyapunov matrix equation (3). Then, the negativity of the time derivative of this function along system solutions requires

$$(A+F)'P+P(A+F)<0. \tag{13}$$

In virtue of (3) and (4), this can be rewritten as

$$I-Q>0. \tag{14}$$

Since Q has an expression as in (9), (14) leads to

$$I-\sum_{k=1}^{l} r_k Q^k>0, \quad r_k \geqslant 0, \quad \sum_{k=1}^{l} r_k=1. \tag{15}$$

This is nothing but the condition that all the eigenvlues of any convex combination of symmetric matrices have a modulus less than unity. Repeated use of Lemma 1 yields

$$\lambda_{\max}(\sum_k r_k Q^k) \leqslant \sum_k r_k \lambda_{\max}(Q^k). \tag{16}$$

Considering this inequality and the third equality of (15), we see that

$$\lambda_{\max}(Q^k) < 1, \quad k=1, \cdots, l \tag{17}$$

assure (15). The above relations are equivalent to (11). Conversely, it is obvious that the inqualities (17) are necessary conditions for (15). In this way, the condition (11) is necessary and sufficient for (14). This completes the proof.

Q. E. D.

A drawback of this theorem is the fact that as n increases l grows rapidly and becomes a prohibitive number for a large n. We show next that under certain circumstances the positive definiteness of only a single matrix needs to be checked.

Let us assume in (2) $c_{ij}=0$, $i, j=1, \cdots, n$, namely

$$0 \leqslant f_{ij}(x, t) \leqslant d_{ij}, \quad i, j=1, \cdots, n. \tag{18}$$

We furthermore assume that there exists a positive definite symmetric matrix W such that the solution matrix $P=(p_{ij})>0$ to the Lyapunov matrix equation,

$$A'P+PA+W=0, \tag{19}$$

is a nonnegative matrix,

$$p_{ij} \geqslant 0, \quad i, j=1, \cdots, n. \tag{20}$$

Under these settings, we can obtain the following result.

Theorem 2

The system (1) with nonlinear gains (18) is stable, if for a matrix W such that (19) and (20) are met, the inequality,

$$\lambda_{\min}(W) - \lambda_{\max}(T) > 0, \tag{21}$$

is satisfied.

Proof : Under the assumption, S is a null matrix and both Q and T are nonnegative matrices. As in the proof of the previous theorem, a sufficient condition for the stability of the system is given by

$$W-Q>0. \tag{22}$$

This condition is met, if

$$\lambda_{\min}(W) - \lambda_{\max}(Q) > 0. \tag{23}$$

Since Q is a nonnegative matrix, the Perron-Frobenius Theorem is applicable (see reference 2)). The theorem asserts that the spectral radius of a nonnegative matrix increases with respect to its entry. As Q is symmetric, the spectral radius of Q is none other than $\lambda_{\max}(Q)$. This confirms us that the condition (21) is a necessary and sufficient one for (23) to hold for any Q, completing the proof.

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Thus, we have obtained a scalar sufficient condition (21) under the settings of this theorem. This can be further improved with an additional assumption as follows.

Corollary 1

In addition to the assumptions of Theorem 2, we assume that all the offdiagonal entries of W are nonpositive. Then the system (1) is stable, if

$$W-T>0. \tag{24}$$

Proof: By the assumptions, the offdiagonal entries of $W-Q$ are all nonpositive. In this case, the condition (22) is equivalently paraphrazed as the one that $W-Q$ is an M-matrix (for details of M-matrices, refer to 3)). Because of the property of this class of matrices, (22) is satisfied if and only if the same condition is met for the severest case, viz. Q=T. This is what is claimed in (24).

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As to the existence of W satisfying (19) and (20), see reference 4). It is shown there that such matrices actually exist without any additional restrictions other than the stability of A. Thus, this assumption would not be so restrictive.

III Stability of Systems with Unbounded Gain Elements

We have so far considered a control system with bounded nonlinear gain elements. In several occasions, however, systems may have nonlinearities whose input-output relations are given by gain characteristics which fall within the first and third guadrants, allowing infinite gains. In this section, we will devise a method to cover this case. A major difference between the result here and the previous one consists of the type of Lyapunov functions employed to show stability. In section 2, a class of L_2 function, a quadratic form, is used, while in this section an existence condition for an L_1 type Lyapunov function is addressed.

Let us consider a system described by

$$dx/dt=Ax+F(x,t)x, \tag{25}$$

where $F(x, t)=f_{ii}(x, t))$ is a diagonal matrix whose entry $f_{ii}(x, t): R^n\times[t_0, \infty)\rightarrow R^n$ is a continuous and continuously differentiable function satisfying

$$x_if_{ii}(x, t)\geqslant 0,\quad i=1, \cdots, n,\quad t\geqslant t_0. \tag{26}$$

Comparing the above system description with the one in the previous section, we see that $F(x, t)$ is restricted to a diagonal matrix with nonlinear entries while infinite gains are introduced. For this system, we define a matrix $A1=(a^1{}_{ij})$,

$$a^1{}_{ij}=\begin{cases}-a_{ii}, & i=j\\ -|a_{ij}|, & i\neq j.\end{cases} \tag{27}$$

Then, we have a robust stability criterion for the system (25).

Theorem 3

A sufficient condition for the system (25) to be stable for any nonlinear gain elements with (26) is that the matrix $A1$ is an M-matrix, that is, all the leading principal minors of $A1$ are positive.

Proof : This follows from the results reported in 5). The condition of this theorem is a necessary and sufficient one for the existence of a Lyapunov function of the form,

$$\sum_{i=1}^{n} d_i|x_i|, \quad d_i>0, \tag{28}$$

for the system (25). This ensures stability of the system.

Q. E. D

It should be noted that the negativity of all the diagonal entries of A is required. The condition of this theorem implies that these negative diagonal entries dominate the offdiagonal ones in a certain sense. It would be interesting to note that the existence condition can be pharased in terms of the entries of the system matrix directly.

IV Concluding Remarks

Several stability robustness conditions are derived for control systems including nonlinear gain elements. If reported in the early stages of modern control theory, these results would have been categorized into the absolute stability problems. They can, indeed, cope with possible perturbations in nonlinear characteristics. Two classes of systems are considered : a system with bounded nonlinear gain elements and one with unbounded gain elements. The L_2 Lyapunov function approach is used for the former case, while in the latter the existence of an L_1 function is assured. An interesting open problem is what condition should be imposed on the system matrix for the existence of an L_2 Lyapunov function in the latter problem formulation.

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