Integral Operators in Crack Problems and their Perturbations in the Direction of the Crack Extension

by

Kohji OHTSUKA* and Naoshi NISHIMURA**

(Received September 14, 1990)

Abstract

The perturbation, in the direction of the crack extension, of the elastostatic hypersingular integral operator for crack problems is calculated rigorously. A variational form including this perturbation is written in terms of the energy release rate. The obtained result is applied to Griffith's crack extension.

1. Introduction

The equilibrium of a linear elastic body with a crack is described by a boundary value problem on a domain $\Omega \subset R^3$. Here, the undeformed shape Ω is expressed in the form of $\Omega = G \backslash C$, where $G \subset R^3$ is a domain with a smooth boundary Γ, and C a surface with a smooth boundary. The undeformed shape C of the crack lies on the smooth boundary S of a domain G_0 such that $\bar{G}_0 \subset G$. This body is fixed on a part of Γ named Γ_D whose closure $\bar{\Gamma}_D$ is a surface with a boundary, and the surface force is given on the remainder $\Gamma_N = \Gamma \backslash \bar{\Gamma}_D$. Also, the body force is neglected. Then the displacement vector $\vec{u}$ is given as the minimizer of the potential energy functional

$$\mathcal{E}_\Omega(\vec{v}; \vec{g}) = \frac{1}{2} a_\Omega(\vec{v}, \vec{v}) - \int_{\Gamma_N} g \cdot \vec{v} \, dS \tag{1.1}$$

defined over the space

$$V(\Omega) = \{\vec{v} \in H^1(\Omega)^3 \,|\, \vec{v} = \vec{0} \text{ on } \Gamma_D\}$$

where $\vec{g}$ is the density of the surface force, and

$$a_\Omega(\vec{v}, \vec{v}) = \int_\Omega \sigma(\vec{v}) \varepsilon(\vec{v}) \, dx.$$

* Department of Liberal Arts, Hiroshima-Denki Institute of Technology
** Department of Civil Engineering, Kyoto University

In this equation, $\varepsilon(\vec{v})$ is the small strain tensor given by,

$$\varepsilon(\vec{v}) = (\varepsilon_{ij}(\vec{v})), \quad \varepsilon_{ij}(\vec{v}) = (D_i v_j + D_j v_i)/2,$$

and the stress tensor $\sigma(\vec{v})$ is expressed by Hooke's law:

$$\sigma(\vec{v}) = (\sigma_{ij}(\vec{v})), \quad \sigma_{ij}(\vec{v}) = C_{ijkl}\varepsilon_{kl}(\vec{v}).$$

Here, the elements C_{ijkl} of Hooke's tensor are constants such that

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{klij}.$$

The space $H^1(\Omega)$ consists of square integrable functions on Ω whose first distributional derivatives are also square integrable, that is,

$$\int_\Omega |f(x)|^2 dx + \sum_{j=1}^{3}\int_\Omega |D_j f(x)|^2 dx < \infty \text{ if } f \in H^1(\Omega).$$

and $\vec{f}=(f_1, f_2, f_3) \in H^1(\Omega)^3$ if $f_i \in H^1(\Omega)$, $i=1, 2, 3$. Here, $D_j f = \partial f/\partial x_j$, and the summation convention is used throughout this paper.

A hypersingular operator H appears in the boundary integral equation derived from the crack problem stated just above; this integral equation has been obtained in references 1), 3) and 9). In this paper, we shall consider a smooth crack extension $\mathcal{C}(t)$ and calculate the derivative $\delta H_{\{\mathcal{C}(t)\}}$ of H with respect to $\mathcal{C}(t)$. The variational form including the kernel $\delta H_{\{\mathcal{C}(t)\}}$ is then expressed in terms of the energy release rate. Our result has a certain similarity to the anti-plane result in reference 4), which has been obtained in connection with crack shape determination problems. Finally, the obtained result is applied to a simple two-dimensional example, in which the variational form is expressed by stress intensity factors.

To simplify the statements and the proofs, we omit the functional spaces for surface forces, for the domain of H and for the domain taken by the limit of $\delta H_{\{\mathcal{C}(t)\}}$, etc. For a mathematical proof, refer to reference 7).

2. Hypersingular operator

We first introduce the fundamental solution U_{ij}, $i, j=1, 2, 3$, defined by

$$-D_k \sigma_{ik}(\vec{U}_j) = \delta(x)\delta_{ij}, \quad \vec{U}_j = (U_{j1}, U_{j2}, U_{j3}).$$

Setting $\Sigma_{ik;j}(x) = \sigma_{ik}(\vec{U}_j)(x)$ and $T(\vec{n})(x) = (T_{ij}(\vec{n}))(x) = (-\Sigma_{ik;j}(x)n_k)$, we get

$$\begin{aligned}\sigma_{ij}(\vec{u})(x) = &\int_\Gamma \Sigma_{ij;m}(x-y)\sigma_{ml}(\vec{u}(y))n_l(y)dS_y \\ &-\int_\Gamma C_{ijkl}D_{x_l}T_{mk}(\vec{n}(y))(x-y)u_m(y)dS_y \qquad (2.1)\\ &+\int_S C_{ijkl}D_{x_l}T_{mk}(\vec{n}(y))(x-y)[u_m](y)dS_y\end{aligned}$$

for $x \in \Omega$, where $[f]=f^+-f^-$ is the difference between the limit f^- from the inside of G_0 and the limit f^+ from the outside, and $\vec{n}(y)$ the unit outward normal at $y \in \partial G_0$ or at $y \in \Gamma$. There exists a number $\varepsilon_0 > 0$ such that the map

$$F_0\colon (x, t) \to x + t\vec{n}(x), \quad \vec{n}(x)\colon \text{the unit outward normal to } \partial G_0$$

is one-to-one and smooth from $S \times (-\varepsilon_0, \varepsilon_0)$ into G. By $U(S)$, we denote its image by F_0, that is,

$$U(S) = F_0(S \times (-\varepsilon_0, \varepsilon_0))\,. \tag{2.2}$$

Taking limits in (2.1), we have for each $x_0 \in S$,

$$\lim_{\varepsilon \to +0} \sigma(\vec{u})(x_0 + \varepsilon\vec{n})\vec{n} = \lim_{\varepsilon \to -0} \sigma(\vec{u})(x_0 + \varepsilon\vec{n})\vec{n}\,.$$

This result, when $\sigma(\vec{u})$ is smooth up to S, is known as the Liapunov-Tauber theorem in elasticity (See reference 2), p. 319.). But, in fracture mechanics, the singularities of $\sigma(\vec{u})$ at the edge $\partial C \subset S$ of the crack are essential and the coefficients of this singular term are called stress intensity factors. Using the result in Becache, Nedelec and Nishimura[1], we can show that

$$\begin{aligned} \sigma_{ij}(\vec{u}(x))n_j = & \int_\Gamma T_{im}(\vec{n}_x)(x-y)\sigma_{ml}(\vec{u})n_l(y)\, dS_y \\ & -\int_\Gamma C_{ijkl}n_j(x)D_{x_l}T_{mk}(\vec{n}_y)(x-y)u_m(y)dS_y \\ & +\int_S C_{ijkl}n_j(x)D_{x_l}T_{mk}(\vec{n}_y)(x-y)[u_m](y)dS_y \end{aligned}$$

for $x \in S$, in the distributional sense (refer to reference 7) for the details). For all smooth functions $\vec{\psi}$ defined on S, we consider the following variational formula:

$$\begin{aligned} \langle \sigma(\vec{u})\vec{n}, \vec{\psi} \rangle_S = & \int_S \vec{\psi}(x)dS_x \int_\Gamma T(\vec{n}_x)(x-y)\sigma(\vec{u})\vec{n}_y\, dS_y \\ & -\int_S \vec{\psi}(x)dS_x \int_\Gamma \sigma_x(T(\vec{n}_y)(x-y)\vec{u}(y))\vec{n}_x dS_y + \langle H[\vec{u}], \vec{\psi} \rangle_S\,, \end{aligned} \tag{2.3}$$

where

$$H[\vec{u}](x) = \int_S \sigma_x(T(\vec{n}_y)(x-y))\vec{n}_x[\vec{u}](y)dS_y \tag{2.4}$$

in the distributional sense, and

$$\langle \vec{f}, \vec{g} \rangle_S = \int_S \vec{f} \cdot \vec{g}\, dS$$

if the inner product $\vec{f} \cdot \vec{g}$ of two functions $\vec{f}$ and $\vec{g}$ defined on S is integrable. Otherwise

$\langle \cdot, \cdot \rangle_S$ means the duality in functional analysis. The integral on the right hand side of (2.4) is hypersinglar, since

$$\sigma_x(T(\vec{n}_y)(x-y))\vec{n}_x \sim \frac{1}{|x-y|^3} \text{ as } |x-y| \to 0 .$$

3. Crack extension

We next consider the following class of crack extensions $\{\mathcal{C}(t)\}$ of $\mathcal{C}$.

DEFINITION 3.1. For $T>0$, a family $\{\mathcal{C}(t)\}_{t\in[0,T]}$ of surfaces of R^3 with a boundary is called a *smooth crack extension* of $\mathcal{C}$ if it satisfies the following conditions (3.1–3):

- $\mathcal{C}(t) \subset S$ for all $t \in [0, T]$. (3.1)
- $\mathcal{C}(0) = \mathcal{C} \subset \mathcal{C}(t) \subset \mathcal{C}(t')$ if $0<t<t'<T$. (3.2)
- For each $t \in [0, T]$, there exists a C^∞-diffeomorphism

$$\phi_t \colon \partial\mathcal{C} \to \partial\mathcal{C}(t) \tag{3.3}$$

such that the map $\phi_t \colon \partial\mathcal{C} \times [0, T] \to S$ is of class C^∞.

We now set $\Omega(t) = G \backslash \mathcal{C}(t)$, and consider the following problem: For the same surface force $\vec{g}$ as in (1.1), find the displacement vector $\vec{u}(t)$, that is given as the minimizer of the potential energy functional

$$\mathcal{E}_{\Omega(t)}(\vec{v}; \vec{g}) = \frac{1}{2}\int_{\Omega(t)} \sigma(\vec{v})\varepsilon(\vec{v})dx - \int_{\Gamma_N} \vec{g}\cdot\vec{v}\, dS$$

over the space

$$V(\Omega(t)) = \{\vec{v} \in H^1(\Omega(t))^3 \,|\, \vec{v} = \vec{0} \text{ on } \Gamma_D\} .$$

The energy release rate $\mathcal{G}(\vec{g}, \{\mathcal{C}(t)\})$ by the crack extension $\{\mathcal{C}(t)\}$ with respect to the parameter t is written as

$$\mathcal{G}(\vec{g}, \{\mathcal{C}(t)\}) = \lim_{\tau \to +0} \frac{1}{\tau} \{\mathcal{E}_\Omega(\vec{u}; \vec{g}) - \mathcal{E}_{\Omega(\tau)}(\vec{u}(\tau); \vec{g})\} .$$

To describe the crack extension more precisely, we introduce a curvilinear coordinate system $(\mathcal{U}(\partial\mathcal{C}), (y_1, y_2, y_3))$ in a region $U(S)$ defined in (2.2) as follows (see reference 5)).

$$y_1(x) = x \text{ whenever } x \text{ is in } \partial\mathcal{C}. \tag{3.4}$$

$$\begin{aligned} \mathcal{U}(\partial\mathcal{C}) \cap S &= \{x \in U(S) \,|\, -1 < y_2(x) < 1, y_3(x) = 0\}, \\ \mathcal{U}(\partial\mathcal{C}) \cap \mathcal{C} &= \{x \in U(S) \,|\, -1 < y_2(x) < 0, y_3(x) = 0\} . \end{aligned} \tag{3.5}$$

With the inverse F of the mapping $x \to (y_1(x), y_2(x), y_3(x))$ from $\mathcal{U}(\partial C)$ onto $\partial C \times (-1, 1) \times (-\varepsilon_0, \varepsilon_0)$, we rewrite (3.5) as

$$\begin{aligned} \mathcal{U}(\partial C) \cap S &= F(\partial C \times (-1, 1) \times \{0\}) , \\ \mathcal{U}(\partial C) \cap C &= F(\partial C \times (-1, 0) \times \{0\}) . \end{aligned} \tag{3.5}$$

By the use of this local coordinate, the edge $\partial C(t)$ of the crack $C(t)$ is written as in the following statement: There exists a family of smooth functions $h(x, t)$ defined on $\partial C \times [0, T]$ such that

$$\partial C(t) = \{x \in U(S) \mid y_1(x) \in \partial C, y_2(x) = h(y_1(x), t), y_3(x) = 0\} .$$

Here, $h(x, 0) = 0$. For a proof, refer to reference 5).

Now consider a map Φ_t which plays a basic role in our calculation. For the construction of Φ_t, we take a smooth function $\beta \geq 0$ such that

$$\text{supp } \beta \subset \mathcal{U}(\partial C) \text{ and } \beta = 1 \text{ on } Q , \tag{3.6}$$

where Q is an open neighborhood of ∂C in R^3 such that

$$\partial C(t) \subset Q \text{ for any } t \in [0, T] \text{ and } \bar{Q} \subset \mathcal{U}(\partial C) .$$

We now put

$$\Phi_t(x) = \begin{cases} F(y_1(x), y_2(x) - \beta(x) h(y_1(x), t), y_3(x)), & \text{for } x \in \mathcal{U}(\partial C), \\ x, & \text{for } x \in R^3 - \mathcal{U}(\partial C) . \end{cases} \tag{3.7}$$

Our construction of Φ_t yields the following

LEMMA 3.2. *There exists a positive number $t_0 \leq T$ such that the family of maps $\{\Phi_t\}_{t \in [0, t_0]}$ satisfies the following*:

The map Φ_t: $R^3 \to R^3$ is a C^∞-diffeomorphism for each $t \in [0, t_0]$. (3.8)

$\Phi_t(C(t)) = C$ for all $t \in [0, t_0]$. (3.9)

The map Φ^3: $R^3 \times [0, t_0] \to R^3$ is smooth. (3.10)

If we set $\vec{u}_t^*(x) = \vec{u}(\Phi_t(x))$ in $\Omega(t)$, then the displacement vector $\vec{u}$ is transformed to the function $\vec{u}_t^*(x)$ defined on $\Omega(t)$. Irwin's formula is written as follows:

THEOREM 3.3.[6] *Under the same surface force $\vec{g}$ on Γ_N as introduced in* (1.1), *the energy release rate with respect to t is written as*

$$\mathcal{G}(\vec{g}; \{C(t)\}) = \lim_{\tau \to +0} \frac{1}{2\tau} \int_{C(\tau) \setminus C} \sigma_{ij}(\vec{u}) n_j [(\vec{u}_\tau^*)_i] \, dS. \tag{3.11}$$

REMARK: Since $\sigma(\vec{u})\vec{n} = \vec{0}$ on C, $[\vec{u}_\tau^*] = \vec{0}$ on $S \setminus C(\tau)$ and $[\vec{u}] = \vec{0}$ on $S \setminus C$, we can rewrite (3.11) as

$$\mathcal{G}(\vec{g};\{C(t)\}) = \lim_{\tau\to+0}\frac{1}{2\tau}\int_S \sigma_{ij}(\vec{u})n_j[(\vec{u}_\tau^*)_i-\vec{u}_i]\,dS\,. \tag{3.11'}$$

Combining (2.3) and (3.11′), we get

$$\mathcal{G}(g;\{C(t)\}) = \lim_{\tau\to+0}\frac{1}{2\tau}\left\{\int_S [\vec{u}_\tau^*-\vec{u}](x)\cdot\vec{\eta}(x)dS_x+\langle H[\vec{u}],\,[\vec{u}_\tau^*-\vec{u}]\rangle_S\right\},$$

$$\vec{\eta}(x) = \int_\Gamma \{T(\vec{n}(x))(x-y)\sigma(\vec{u})\vec{n}(y)-\sigma_x(T(\vec{n}_y)(x-y)\vec{u}(y))\vec{n}(x)\}\,dS_y\,.$$

Since $d\Phi_t/dt\,|_{t=0}=\beta\vec{Z}$, with

$$\vec{Z}(x) = \frac{d}{dt}F(y_1(x),\,y_2(x)-h(y_1(x),\,t),\,y_3(x))\,|_{t=0},\quad \forall x\in\mathcal{U}(\partial C),$$

we obtain

$$\lim_{\tau\to+0}\frac{1}{\tau}\int_S [\vec{u}_\tau^*-\vec{u}](x)\cdot\vec{\eta}(x)dS_x = -\int_S [\vec{u}]\cdot(\beta(\vec{Z}\cdot\nabla)\vec{\eta}(x)+\vec{\eta}(x)\mathrm{div}(\beta\vec{Z}))dS_x.$$

Here, we note that $\eta(x)$ is differentiable with respect to x in the usual sense. For any $\varepsilon>0$, however, we can take β and T in (3.6) so that

$$\int_S [\vec{u}]\cdot(\beta(\vec{Z}\cdot\nabla)\vec{\eta}(x)+\vec{\eta}(x)\mathrm{div}(\beta\vec{Z}))dS_x<\varepsilon$$

holds. Hence we obtain

$$\lim_{\tau\to+0}\frac{1}{\tau}\langle H[\vec{u}],\,[\vec{u}_\tau^*-\vec{u}]\rangle_S = 2\mathcal{G}(\vec{g};\,\{C(t)\}). \tag{3.12}$$

We now introduce $\Psi_t(x)=\Phi_t^{-1}(x)$ for $x\in R^3$ and $(\Psi_t^*H[\vec{u}])(x)=H[\vec{u}](\Psi_t(x))$. A change of variables yields,

$$\langle H[u],\,[u_t^*]\rangle_S = \langle \Psi_t^*H[\vec{u}]\det(\nabla\Psi_t)\,|\,{}^t(\nabla\Psi_t)\vec{n}\,|,\,[\vec{u}]\rangle_S. \tag{3.13}$$

On the other hand, we have

$$\lim_{\tau\to+0}\frac{1}{\tau}(\det(\nabla\Psi_\tau)\,|\,{}^t(\nabla\Psi_\tau)\vec{n}\,|-1) = \mathrm{div}_\Gamma\,\vec{X},\ \vec{X}=\frac{d}{dt}\Psi_t\,|_{t=0}. \tag{3.14}$$

Combining (3.12)–(3.14), we now have

$$\lim_{\tau\to+0}\frac{1}{\tau}\langle \Psi_\tau^*H[\vec{u}]-H[\vec{u}],\,[\vec{u}]\rangle_S = 2\mathcal{G}(g;\,\{C(t)\})-\langle H[\vec{u}]\mathrm{div}_\Gamma\,\vec{X},\,[\vec{u}]\rangle_S. \tag{3.15}$$

Since $H[\vec{u}]=\vec{0}$ on C and $[\vec{u}]=\vec{0}$ on $S\backslash C$, the last term on the right hand side of (3.15) vanishes.

The main result of this paper is the following

THEOREM 3.4. *Let $\vec{g}_a$ and $\vec{g}_b$ be surface forces given on Γ_N and $\vec{u}_a$, $\vec{u}_b$ the displacement vectors, respectively. If we set*

$$\delta\mathcal{Q}(\vec{g}_a, g_b; \{C(t)\}) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\mathcal{Q}(\vec{g}_a+\varepsilon\vec{g}_b; \{C(t)\})-\mathcal{Q}(\vec{g}_a; \{C(t)\})]$$

then it follows

$$\langle\delta H_{\{C(t)\}}[\vec{u}_a], [\vec{u}_b]\rangle_S = \delta\mathcal{Q}(\vec{g}_a, \vec{g}_b; \{C(t)\}),$$

where

$$\delta H_{\{C(t)\}}[\vec{u}_a] = \lim_{\tau\to +0}\frac{1}{\tau}(\Psi_\tau^* H[\vec{u}_a]-H[\vec{u}_a])$$

in the distributional sense.

PROOF: From (3.15), we get

$$\begin{aligned}\mathcal{Q}(\vec{g}_a+\varepsilon\vec{g}_b; \{C(t)\})-\mathcal{Q}(\vec{g}_a; \{C(t)\})\\ &= \frac{1}{2}\langle\delta H_{\{C(t)\}}[\vec{u}_a+\varepsilon\vec{u}_b], [\vec{u}_a+\varepsilon\vec{u}_b]\rangle_S\\ &-\frac{1}{2}\langle\delta H_{\{C(t)\}}[\vec{u}_a], [\vec{u}_a]\rangle_S.\end{aligned}$$

Since (for a proof refer to reference 1))

$$\langle H[\vec{u}_a], [\vec{u}_b]\rangle_S = \langle H[\vec{u}_b], [\vec{u}_a]\rangle_S,$$

we have

$$\begin{aligned}\langle H[\vec{u}_a+\varepsilon\vec{u}_b], [\vec{u}_a+\varepsilon\vec{u}_b]\rangle_S &= \langle H[\vec{u}_a], [\vec{u}_a]\rangle_S\\ &+2\varepsilon\langle H[\vec{u}_a], [\vec{u}_b]\rangle_S+\varepsilon^2\langle H[\vec{u}_b], [\vec{u}_b]\rangle_S.\end{aligned}$$

Hence we use (3.12) and (3.15) to get

$$\frac{1}{\varepsilon}\{\mathcal{Q}(\vec{g}_a+\varepsilon\vec{g}_b; \{C(t)\})-\mathcal{Q}(\vec{g}_a; \{C(t)\})\} = \langle\delta H_{\{C(t)\}}[\vec{u}_a], [\vec{u}_b]\rangle_S+O(\varepsilon).$$

We complete the proof of the theorem by letting $\varepsilon\to 0$.

4. Two-dimensional case and Griffith's crack extension

In-plane elastic deformations are described by a partial differential equation on

$\Omega \subset R^2$ similar to the one discussed in Section two. In this case the undeformed shape C of the crack is a curve in R^2. Results analogous to those given in the preceding section hold in two-dimensional cases also.

We now consider a simple crack extension called Griffith's crack extension. The crack has the form

$$C = \{(x_1, 0) \in R^2 \mid -l \leq x_1 \leq l\}$$

initially, and subsequently extends in a way described by

$$C(t) = \{(x_1, 0) \in R^2 \mid -l \leq x_1 \leq l+t\}.$$

Let G be a domain in R^2 such that $C(t) \subset G$, for all $0 \leq t \leq T$, and let the set $\{x \mid |x-p| \leq 4T\}$ be included in G, where $p=(l, 0)$. Also, let $\beta(x) \geq 0$ be a smooth function defined on R^2 such that $\beta(x)=1$ if $|x-p| \leq 2T$, and $\beta(x)=0$ if $|x-p| \geq 3T$. In this case, we have the well known formula by Irwin (see reference 8)):

$$\mathcal{G}(\vec{g};\, \{C(t)\}) = \frac{1-\nu^2}{E}(K_I^2(\vec{g})+K_{II}^2(g)) \,,$$

where $K_I(\vec{g})$, $K_{II}(\vec{g})$ are the stress intensity factors under the surface force $\vec{g}$, E is Young's modulus and ν is Poisson's ratio. Since the stress intensity factors are linear with respect to loads, we have, for example,

$$K_I(k\vec{g}_a+l\vec{g}_b) = kK_I(g_a)+lK_I(\vec{g}_b) \,.$$

Therefore we obtain

$$\delta\mathcal{G}(g_a, g_b;\, \{C(t)\}) = \frac{2(1-\nu^2)}{E}(K_I(\vec{g}_a)K_I(\vec{g}_b)+K_{II}(\vec{g}_a)K_{II}(\vec{g}_b)) \,.$$

On other hand, the map Φ_t given in (3.7) is written as

$$\Phi_t(x) = (x_1-\beta(x)t,\, x_2) \,.$$

The hypersingular operator $H[\vec{u}]$ is expressed as

$$(H[\vec{u}])_i(x_1, 0) = -\int_{-\infty}^{\infty} C_{i2kl} D_{x_l} \Sigma_{m2;k}((x_1, 0), (y_1, 0))[u_m](y_1, 0)dy_1$$

in the distributional sense, and formally

$$\begin{aligned}\delta H_{\{C(t)\}}[\vec{u}](x_1, 0) &= \lim_{\tau\to+0} \frac{1}{\tau}(\Psi_\tau^* H[\vec{u}](x_1, 0)-H[\vec{u}](x_1, 0)) \\ &= \lim_{\tau\to+0} \frac{1}{\tau}(H[\vec{u}](x_1+\tau, 0)-H[\vec{u}](x_1, 0))\beta(x_1, 0) \,.\end{aligned}$$

From Theorem 3.4, we derive the following formula:

$$\langle \delta H_{\{C(t)\}}[\vec{u}_a](\cdot, 0), [\vec{u}_b](\cdot, 0) \rangle_{R^1}$$
$$= \frac{2(1-\nu^2)}{E} (K_I(\vec{g}_a) K_I(\vec{g}_b) + K_{II}(\vec{g}_a) K_{II}(\vec{g}_b)) .$$

5. Conclusion

This paper has shown a mathematically rigorous way of computing the perturbation, in the direction of the crack extension, of the elastostatic hypersingular integral operator for crack problems. The analysis in this paper is expected to justify the engineering computation carried out in reference 4). Our Theorem 3.4, together with similar perturbation results in the normal direction of a crack, is expected to serve as a mathematical foundation for inverse problems related to the detection of cracks in engineering materials.

References

1) Becache, E., Nedelec, J. C., Nishimura, N.: Regularization in 3D for anisotropic elastodynamic crack and obstacle problems, Internal report n°205 of Centre de Mathématiques appliquées, Ecole Polytechnique, France, 1989.
2) Kupradze, V.D.: Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity, North-Holland, Amsterdam, 1979.
3) Nishimura, N. and Kobayashi, S.: A regularized boundary integral equation method for elastodynamic crack problems, Comp. Mech., 4, 319–328, 1989.
4) Nishimura, N.: Regularised BIEs in crack shape determination problems, In; M. Tanaka et al. (eds.), Proc. BEM12, 2, 425–434, Comp. Mech. Publ., Southampton 1990.
5) Ohtsuka, K.: Generalized *J*-integral and three-dimensional fracture mechanics I, Hiroshima Math. J., 11, 21–52, 1981.
6) Ohtsuka, K.: Irwin's formula in three-dimensional fracture mechanics, in preparation
7) Ohtsuka, K.: Hypersingular operator in three-dimensional crack problems and its perturbations along crack extensions, in preparation.
8) Rice, J.R.: Mathematical analysis in the mechanics of fracture, In; H. Liebowitz (ed.), Fracture—An Advanced Treatise, 191–311, Academic Press, New York and London, 1968.
9) Stephan, E.P.: A boundary integral equation method for three-dimensional crack problems in elasticity, Math. Meth. Appl. Sci., 8, 609–623, 1986.