A Qualitative Analysis on Dynamical Systems: Sign Structure

By

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Abstract

This paper presents several qualitative properties of dynamical systems. These qualitative properties include structures which may have periodic solutions, structures which may have constant solutions, a generalized version of sign stalibity and structures which have solutions whose sign patterns are invariant. These qualitative properties are useful for qualitative analysis of large-scale dynamical systems. We also present a method for classifying a given dynamical system by these qualitative properties.

1. Introduction

After a qualitative model that states the qualitative constraints between variables is obtained by qualitative physics, we can reason the qualitative behavior on the model by:

(i) Qualitative simulation [de Kleer & Bobrow 1984, Kuipers 1986]: propagating qualitative values (signs or some other quantized values) locally through the qualitative model, or

(ii) Qualitative analysis [Weld 1988, Struss 1988, Lee and Kuipers 1988, Ishida 1989, Sacks 1990]: analyzing the qualitative structures of the qualitative model.

In (i), we can envision the qualitative behavior step by step as if the actual system is working through time. However, the step-by-step behavior includes the ambiguity intrinsic to qualitativeness. This feature is critical when trying to see the qualitative behavior of industrial systems, which are large-scale, and hence their simulations easily get stuck with ambiguity. In contrast, in (ii) we cannot get the detailed behavior of the discrete states or modes ordered through time. However, we can get global properties such as stability. Both (i) and (ii) are necessary for making qualitative reasoning sophisticated. They can help with each other. Qualitative analysis, for example, can restrict ambiguities in qualitative simulations.

In this paper, we present a method for this qualitative analysis. We will show

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how a qualitative model can be classified from the viewpoints of these qualitative properties.

We can express or approximate dynamical systems by a linear differential equation:

$$dx/dt = Ax, A \in R^{n \times n} \tag{1.1}$$

In [Ishida 1989] we used the signed matrix A_s[1] to express the qualitative model. In graphical expression of the model, an arc is directed from vertex i to vertex j with the sign $(A_s)_{ij}$. Most of the results of qualitative system theory are obtained for the state-space expression of this linear system.

2. System Theoretic Interpretations of the Graph Coloring

The necessary condition for sign stability is already known:

Theorem [Quirk 1965]

If the qualitative model is sign stable then its graph satisfies:

(1) All the loops must have non-positive signs, and at least one loop must have a negative sign.

(2) All the circuits of length two must have non-positive signs.

(3) There must be no circuit of a leneth greater than three.

If the qualitative model satisfies conditions (1) (2) and (3), it is known not to have divergent solutions. However, the model may have periodic or constant solutions (i.e. Its system matrix may have pure imaginaries or zero as its eigenvalues.). To guarantee that the qualitative model has only stable solutions (i.e. Its system matrix may have eigenvalues with negative real parts.), we must add some constraints to conditions (1) (2) and (3).

The color test is a constraint to see whether or not the qualitative model has periodic solutions.

We now explain how the color test is done [Jefferies 1974] for the graph, and the color test is used to see the qualitative stability.

Definition (color test) [Jefferies 1974]

The graph of a qualitative model is said to pass the color test when we can color every vertex satisfying the following conditions:

(a) Each vertex that has a loop is black.

(b) There must be at least one white vertex in the graph.

(c) Each white vertex is connected to at least one other white vertex.

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[1]Signed matrix A_s of A is a triple value matrix. It can be defined as follows: $(A_s)_{ij} = +, -, 0$ if $(A)_{ij} > 0, < 0, = 0$ respectively.

(d) Each black vertex connected to one white vertex is connected to at least one other white vertex.

In this color test, a black vertex corresponds to non-oscillating elements and a white vertex corresponds to oscillating elements. Thus, we can interpret that passing the color test is having the periodic solution (i.e. the matrix A has pure imaginaries as its eigenvalues.).

Whenever considering the qualitative models satisfying the conditions (1) (2) and (3), we suppose the partition of the system as:

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdots (2.1)$$

where the elements are renumbered so that A_{11} contains all the elements with negative loops and A_{22} those without the negative loops. The observability from elements with negative loops is determined by the observability matrix

$$\begin{bmatrix} A_{12} \\ A_{12}A_{22} \\ \vdots \\ A_{12}A_{22}^k \end{bmatrix} \cdots (2.2)$$

Under conditions (1) (2) and (3),

We can construct Lyapunov function[2] $V(x)$ so that it satisfies:

$$dV(X)/dt = 2\sum_{i=1}^{l} \lambda_i a_{ii} x_i^2 \cdots (2.3)$$

$$(a_{ij} < 0 \text{ for } r = 1 \cdots l,\ a_{ii} = 0 (i = l+1 \cdots n), \lambda_i > 0)$$

Therefore, in order for the model to have an oscillating solution satisfying $V(x) = 0$ by assigning black elements and white elements to zero elements and to oscillating elements, respectively, the coloring must satisfy the color test conditions (a)-(d).

(a) This is because the element with a negative loop cannot be oscillating. (b) This is because elements cannot be oscillating with only one element in linear systems. (c) To cancel the oscillation, an oscillation pair must be connected

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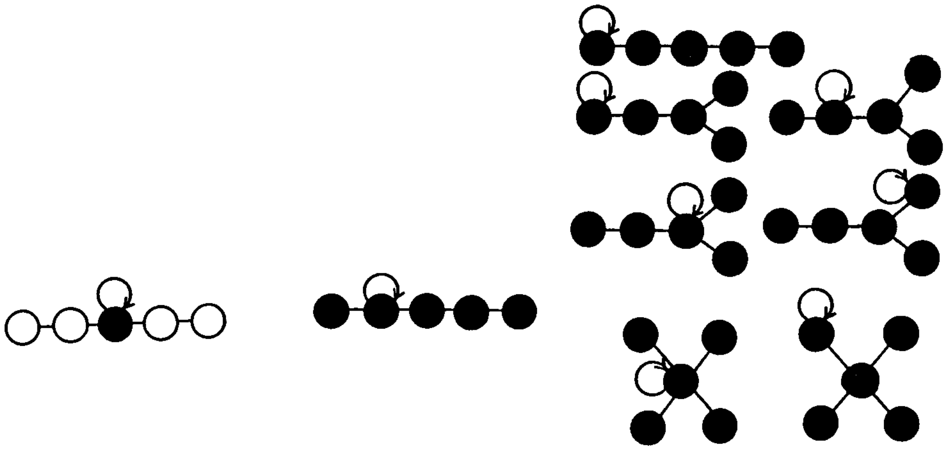
[2]Lyapunov function is a generalized concept of the energy of a system with a quadratic form: $\sum_{i=1}^{n} c_i n_i^2$. Because of its quadratic form,if we can obtain $dV(x)/dt < 0$ for all $x \neq 0$ then we can know the system is stable.

to other oscillation pairs. It is for these reason that there is no model for n = 3,4 that passes the color test.

Example 2.1

Figs. 1 show the qualitative model of one negative loop where the color test is already done.

Pattern 1 does not have a constant solution, but may have a periodic solution. Pattern 2, on the other hand, has a constant solution, but does not have a periodic solution. Therefore, both models have solutions satisfying $x_1 = 0, x_2 \neq 0$. In the next section, we present a systematic way to identify these structures.



Pattern 1 Pattern 2 Pattern 3

Figs. 1 Color Test Done for Several Graphs

3. Qualitative models with specific sign structures

In this section, we present several new qualitative concepts for specifying the modes of solutions. The new concepts include: potentially periodic (the sign structure that may have periodic solutions), potentially constant (the sign structure that may have constant solutions), and sign observable (the signal is always observable from elements with a negative loop for all instances of the sign structure). We suppose the sign structures discussed in this section satisfy conditions (1) (2) and (3) unless otherwise specified. We label theorems PP (PC or SO) if they are related to potentially stable (potentially constant or sign observable), respectively. Further, we label them T, S, or G if they state the graphical property, if they state the system composition rule, or if they state the typical sign structures of the property, respectively. For example, the next theorem PP-T states a graphical test for the potentially periodic structures.

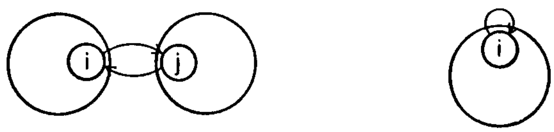
By the arguments of color the relating test and by having periodic solutions in the previous section, the following theorem holds. The theorem states a condition holding for potentially periodic qualitative models.

Theorem PP-T

A qualitative model may have a periodic solution if and only if it passes the color test.

Definition 3.1

We say two systems S_1 and S_2 are connected if two arcs between vertices i and j such that $a_{ij}a_{ji}<0$ are added (Figs. 2 (a)).



(a) system connection (b) adding negative loop

Figs. 2

Using the definition of system connection and that of color test, the next theorem follows:

Theorem PP-S

A qualitative model obtained by connecting two potentially periodic models, is potentially periodic.

Proof

Let S_1 and S_2 be potentially periodic models, then they must have passed the color test. B_i corresponds to black vertices and W_i to white vertices, respectively.

(1) When connecting a vertex in B_1 and a vertex in B_2, the original coloring will pass the color test.

(2) When connecting a vertex in B_1 and a vertex in W_2, the connected system will pass the color test by coloring all the vertices in S_2 black.

(3) When connecting a vertex in W_1 and a vertex in W_2, the original coloring will pass the color test.

These three cases cover all the possible connections. Q.E.D.

By the form of the characteristic equations, the next theorem follows:

Theorem PP-G

By connecting a chain of length three (Figs. 3 (c)) to the potentially periodic models, the connected system is also potentially periodic.

(a) an element with negative loop

(b) a chain of length two

(c) a chain of length three

Figs. 3

Proofs of Theorems PP-G and PC-G

Theorems PP-G and PC-G follow by inspecting the characteristic equations for the connected system. Q.E.D.

We have discussed composition rules to preserve the potential periodicity after the connection of subsystems. Also, we have shown some typical structures that potentially have periodic solutions. With these rules and typical structures, we can know, to some extent, whether or not a given model potentially has a periodic solution. We first identify the structural difference between a given model and the nearest typical models, and then apply the rules. This approach can apply not only to this property (potentially periodic), but to many other qualitative properties. In the following, we briefly show the composition rules and typical structures for other qualitative properties. They include potentially constant, sign observable (section 3), and inertia preservation (section 4) in this order.

Under conditions (1) (2) and (3), there is a graphical method for investigating whether or not the qualitative model has a constant solution [Jefferies 1977].

Definition 3.2 [Anderson 1975]

If we can decompose the graph of the qualitative model to independent circuits of length one (loop) and two (Figs. 3.2 (a) (b)) by removing some arcs, then the qualitative model does not have a constant solution.

With this matching test, the next theorem by [Jefferies 1977] holds.

Theorem PC-T [Jefferies 1977]

If the graph of the qualitative model passes the matching test, then the model does not have a constant solution.

By this theorem PC-T and the definition of the matching test, the next theorem follows:

Theorem PC-S

The qualitative model obtained by connecting two potentially constant models, is potentially constant.

Proof

By the definition of the matching test, when connecting two models that pass the matching test,the connected model also passes the matching test. Q.E.D.

Theorem PC-G

By adding the chain of length two (Fig. 3.2 (b)) to the qualitative model that has (does not have) constant solutions, the resulting system also has (does not have) constant solutions.

With the matching test and color test, the qualitative observable structure is characterized as follows:

Theorem SO-T [Jefferies 1977]

The sign observable structure is such that the graph does not pass the color test, and passes the matching test.

Also for sign observability, we have the composition rules to preserve the property, and typical structures having the property.

We noticed that the sign observable structure is equal to the structure that passes the matching test and that does not pass the color test under the conditions (1) (2) and (3).

Theorem SO-G1

If a qualitative model is sign observable, then the model obtained by connecting any number of one element with a negative loop (Figs. 3.2 (a)) is also sign observable.

Proof (By contraposition)

Suppose a qualitative model obtained by connecting some number of one element with negative loops (Fig. 3.2 (a)) is not sign observable, then we have a solution $x_1=0$, $x_2\neq 0$ for the partition of (2.1). Then the original qualitative model before connecting n negative loops has also a solution of the form: $x'_1=0$, $x_2\neq 0$ where the dimension of x'_1 is reduced by the number of connected elements (1). Thus, the original model turns out to be not gisn observable. Q.E.D.

By the almost similar arguments to the theorem SO-G1, the next theorem holds.

Theorem SO-G2

If a qualitative model is sign observable, then the model obtained by adding a negative loop (Figs. 3.1 (b)) is also sign observable.

The next theorem follows from theorems SO-T, PC-S, PP-S.

Theorem SO-S

The qualitative model obtained by connecting two qualitative models, both of which are not sign observable, is not sign observable.

In the following discussion, we will consider the chain structure with only one negative loop.

Theorem SO-G3

A qualitative model of the chain structure with one negative loop at the end is sign observable.

Proof

In the row vector $A_2A_{22}^j$ of the observability matrix (2.2), the element which are reachable in j+1 steps from the element with negative loop is not zero. Thus, $A_{12}A_{22}^j(j=0,\cdots n-2)$ are all linearly independent. Q.E.D.

Theorem SO-G4

A qualitative model consisting of an element with a single loop, and more than two subsystems of the same structure connected to the element, is not sign observable.

4. Generalization of qualitative stability

Concepts of sign stability and potential stability are generalized with this inertia[3].

Definition 4.1

$I(p,n,i)$ is a class of qualitative models, all of whose instances have the same inertia (p,n,i). And $P(p,n,i)$ is a class of qualitative models, at least one of whose instances has the same inertia (p,n,i).

Sign stable and potentially stable qualitative models are considered as $I(n,0,0)$ and $P(n,0,0)$ respectively.

We present some sufficient conditions for a qualitative model to be inertia preserving. The following theorem is obtained by the Ostrowski-Schneider's theorem [Ostrowski 1962].

Theorem IP-T[4]

A qualitative model belongs to $I(q,p,0)$ if a sign stable sign structure with all negative loops can be transformed to this sign structure by making all the signs of the arcs from q elements of the sign stable sign structure opposite.

Theorem IP-G1

If the graph of a qualitative model is a circuit of length n then the qualitative model is inertia preserving.

Proof

This theorem directly follows from that the characteristic equation of the qualitative model is:

$$\lambda^n = a_{i_1}a_{i_2i_3}\cdots a_{i_ni_1}$$

Theorem IP-G2

If the graph of a qualitative model has no loop and no circuit of a length greater than two, and if every sign of a circuit of length two is negative, then the

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[3]Inertia of a matrix $A \in R^{n\times n}$ indicated by In(A) is defined as triple (p,n,i) of three integers where p the number of eigenvalues of A with positive real part, n with negative real part, and i with zero real part.

qualitative model is of the inertia preserving class $I(0,0,n)$.

Proof

As we have done in the previous section, we can classify the qualitative models by this inertia preserving class.

The next theorem is a rule to identify the inertia preserving class. Identifying the inertia preserving class is done in the following two steps: (1) Find the nearest known class. (2) Analyze the difference between the given sign structure and the nearest known structure.

Theorem IP-S

A qualitative model belongs to $P(p,q,0)$ if the model obtained by the deletion of arcs belongs to $I(p,q,0)$.

Proof

This theorem follows from that the eigenvalues of matrix A of a given model change continuously when the value of the elements of matrix A changes continuously. Since the property that the eigenvalues have inertia $(p,q,0)$ holds in an open set area of the parameter space (in this case, the parameters are the elements of matrix A), the property holds in the neighborhood of the point of parameters corresponding to $I(p,q,0)$.

The above theorem cannot be applied to the inertia preserving qualitative model $P(p,q,i)$ $(i \neq 0)$, because i will change even if the change of elements are sufficiently small.

Theorem 4.1

If a qualitative model belongs to both $P(p,q,0)$ and $P(p-1,q+1,0)$ then it also belongs to $P(p-1,q,1)$.

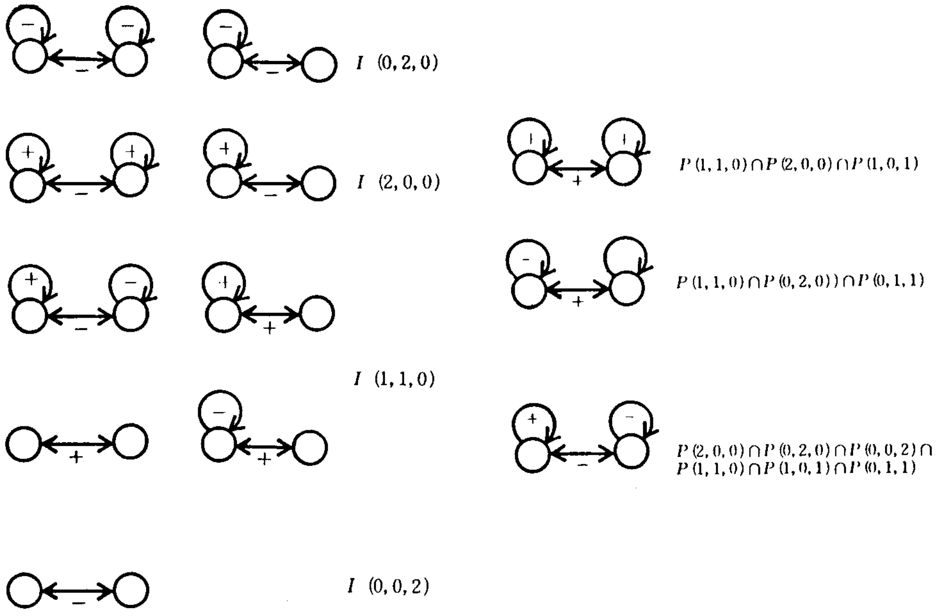
Proof

If a qualitative model belongs to both $P(p,q,0)$ then this means that the sign structure allows one eigenvalue with a negative real part to continuously change to that with a positive real part. Hence, the sign structure must also allow to have the eigenvalve with zero real parts. Likewise, it also holds by the same argument that if a qualitative model belongs to both $P(p,q,0)$ and $P(p+1,q-1,0)$ then it also belongs to $P(p,q-1,1)$. Q.E.D.

Example 4.1

Figs. 4 show how the qualitative models of two elements can be classified from the viewpoint of inertia preservation. In this full classification of models with two elements, $I(0,2,0)$ and $I(0,0,2)$ were given as known types, since their graphical conditions are already known. $I(2,0,0)$ and $I(1,1,0)$ are identified by changing some signs of $I(0,2,0)$ by theorem IP-T. Some classes of $P(p,q,n)$ are identified by determining whether it has $I(p,q,n)$ as its subgraph by theorem IP-S. Other classes of $P(p,q,n)$ are identified by theorem 4.1.

A mass-spring system with friction falls in $I(0,2,0)$. This means the system is always stable. In contrast, a mass-spring system without friction falls in $I(0,0,2)$. This means the system always has periodic solutions.



Figs. 4 Qualitative Classification of Models with Two Elements

5. Qualitative Models Having Solutions with Invariant Sign Patterns

We have definied *invariant sign pattern* in [Ishida 1989] as follows:

Definition (invariant sign pattern)

We call a sign pattern x_s *invariant sign pattern* of a qualitative model if the model stays at the sign pattern x_s all the time, once it attains the state.

We then discussed the graphical condition for a qualitative model to have an invariant sign pattern. The class of qualitative model, which has an invariant sign pattern, is related to the sign unstable class. In this section, we present a variation of invariant sign pattern which plays an important role in characterizing the potentially stable class.

Definition 5.1 (invariant sign pattern with finite time)

We say a qualitative model has an *invariant sign pattern with finite time* x_s if the model stays at the sign pattern x_s all the time $[0,\infty)$, once it attains the state, but converges to a sign pattern different from x_s when $t=\infty$.

The qualitative model $dx/dt = -ax(a>0)$ is an example having an *invariant sign patern with finite time.* In fact, $(x_s, dx/dt_s) = (+,-)$ or $=(-,+)$ are invariant in $t \in [0,\infty)$, however they converge to (0,0) in $t=\infty$.

We have shown in [Ishida 1989] that the qualitative model $dx/dt = Ax$ has an invariant sign pattern when the sign equation $x_s = Ax_s$ has a solution. We also have shown that the solution itself is an invariant sign pattern. In a similar manner to this result, an invariant sign pattern with finite time is obtained by solving a sign equation.

Theorem 5.1

A qualitative model $dx/dt = Ax$ has an *invariant sign pattern with finite time* if the sign equation $x_s = -Ax_s$ has a solution. The solution is an invariant sign pattern with finite time.

With this theorem, the graphical condition for a qualitative model to have an invariant sign pattern with finite time is obtained as follows:

Theorem 5.2

A strongly connected qualitative model $dx/dt = Ax$ has an *invariant sign pattern with finite time* if

(1) All the circuits have a negative sign, and

(2) All the reconvergent fanout paths[4] between two nodes have the same sign.

Proof

This theorem can be obtained by analyzing the graphical conditions under which the sign equation $x_s = -Ax_s$ has a solution. Q.E.D.

As stated before, this *invariant sign pattern with finite time* is related to the potentially stable class. By the definition of *invariant sign pattern with finite time,* the next theorem is almost immediately provable from the definitions of stability and *invariant sign pattern with finite time.*

Theorem 5.3 If the qualitative model has an *invariant sign pattern with finite time* then the model is potentially stable.

6. Conclusion

We studied the qualitative properties of sign stability (and its generalized class of $I(p,q,n)$) and potential stability (and its generalized class of $P(p,q,n)$), which can be used extensively to analyze the global properties of a given qualitative model. Since graphical conditions for some of these classes have not yet been identified mathe-

[4]Reconvergent fanout paths are paths that share the initial and terminal nodes.

matically, we present a systematic way to classify the qualitative models from the viewpoints of these qualitative properties. The classification method is carried out in two steps. General knowledge such that addition and deletion of interactions to $I(p,q,0)$ or $P(p,q,0)$ results in $P(p,q,0)$.

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