A High-Speed Numerical Analysis Technique for Millimetre Wave Aperture Antennae

by

Masamitsu NAKAJIMA and Paul R. WINNING.

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Abstract

By use of the geometrical theory of optics the radiation characteristics of a WGM antenna have already been investigated and designed. In order to improve the accuracy of the present calculation it was necessary to account for diffraction. Since the conventional method based on Kirchhoff-Huygens principle requires much computational time and memory, a numerical analysis method has been developed to overcome such a difficulty, i.e. the antenna aperture is subdivided into a series of subapertures of small area so that the field amplitude can be assumed uniform with the phase either uniform or linearly changing. The radiation fields from the subapertures are evaluated by use of an appropriate analytical solution. The radiated field pattern of the aperture is subsequently obtained by summing the radiation contributions from each of the subapertures in turn. Results from the most recent WGM calculations will be presented.

1. Introduction

The traditional and orthodox method of computation of the radiation pattern from an aperture antenna is to integrate the field over the area of the aperture based on the Kirchhoff-Huygens principle [1]. Although this method gives accurate results, it takes exceedingly large amounts of computer time, especially when the wavelength concerned is short compared with the aperture diameter.
The present method is presented with the philosophy of substituting some known analytical solution for parts of the numerical computation, thus computational time and memory are largely saved. For this purpose to be realised, the antenna aperture is divided into a number of subapertures with uniform field distribution, whose radiation pattern is expressed by an analytical function such as the Fresnel or Fraunhoffer formulae. [2]

In this case it may be noted that the smaller the area of the subapertures, the more accurate the near-field pattern. This is because the accuracy is determined by the factor D^2/λ, where D is the diameter of the subaperture and λ is the wavelength.

Both these methods have been applied to the case of the Whispering Gallery Mode

Department of Electronics.

(WGM) antenna which is currently used for heating and diagnostic purposes of plasmas in magnetic confinement type fusion reactors [3, 4].

2. Theory of the Subaperture Method

2.1 Formulation by use of the Fresnel Integral

This method involves dividing the aperture into a series of rectangular subapertures (Fig. 1). It is assumed in the first instance that the magnitude of the electric field (and phase) are constant across each subaperture. From subaperture to subaperture the magnitude of the field will vary in some predetermined way defined by the type of waveguide aperture under analysis i.e. cylindrical or rectangular. Appendix A (See Figs. A2 and A3) shows that the radiated field from a rectangular aperture can be expressed in terms of the Fresnel integral

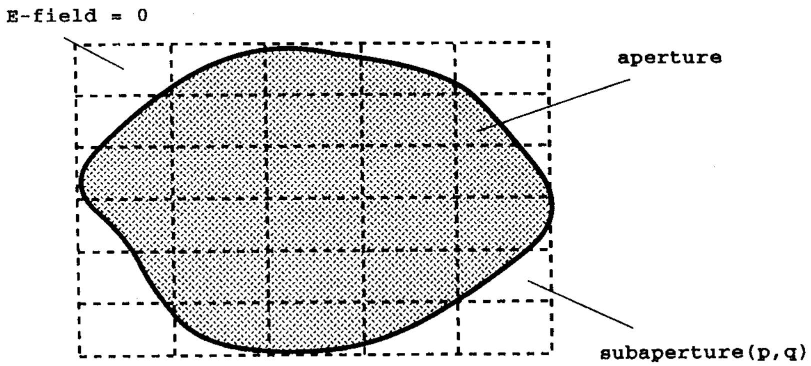


Fig. 1 Division of an arbitrary aperture into a series of rectangular subapertures

$$U(x_i, y_i) = \frac{1}{2j} e^{-jkz_i} F(\alpha, \beta) \tag{1}$$

where

$$F(\alpha, \beta) = \int_{\alpha_1}^{\alpha_2} e^{j\frac{\pi}{2}\varepsilon^2} d\varepsilon \int_{\beta_1}^{\beta_2} e^{j\frac{\pi}{2}\eta^2} d\eta \tag{2}$$

the limits of integration being given in terms of the dimensions of the rectangular aperture

$$\alpha_2 = \sqrt{\frac{2}{\lambda z_i}}[a - f_x \lambda z_i], \quad \alpha_1 = \sqrt{\frac{2}{\lambda z_i}}[(-a) - f_x \lambda z_i],$$

$$\beta_2 = \sqrt{\frac{2}{\lambda z_i}}[b - f_y \lambda z_i], \quad \beta_1 = \sqrt{\frac{2}{\lambda z_i}}[(-b) - f_y \lambda z_i].$$

By use of the Fresnel Cosine and Sine integrals it is possible to express this equation in real and imaginary parts

$$Re[F(\alpha, \beta)] = C(\alpha_{2,1})C(\beta_{2,1}) - S(\alpha_{2,1})S(\beta_{2,1}) \tag{3}$$

$$Im[F(\alpha, \beta)] = C(\alpha_{2,1})S(\beta_{2,1}) + S(\alpha_{2,1})C(\beta_{2,1}) \tag{4}$$

where $C(\alpha_{2,1}) = C(\alpha_2) - C(\alpha_1)$, $S(\alpha_{2,1}) = S(\alpha_2) - S(\alpha_1)$ and similarly for $C(\beta)$ and $S(\beta)$. The Fresnel integral can be used to represent the radiation field for the elemental subaperture (p, q) i.e. pth along the x-axis, qth along the y-axis, in the aperture

$$U_{pq} = \frac{e^{-jkz_i}}{2j}(\mathrm{Re}[F(\alpha, \beta)] + j\mathrm{Im}[F(\alpha, \beta)]) \tag{5}$$

In calculating the field at an observation point $P(x_i, y_i, z_i)$ the average field is taken as the field over each subaperture. The problem of a changing phase distribution across the aperture will be dealt with in a later section. For the present it is assumed that the waveguide aperture cut is in the xy-plane i.e. perpendicular to the direction of wave propagation, a plane of constant phase. To calculate the E-field at the point P in the observation plane sum over all the E-field contributions from each subaperture. This therefore becomes the complete form of the Fresnel subaperture radiation equation.

$$\boldsymbol{E_P} = E_P^x \boldsymbol{x} + E_P^y \boldsymbol{y} + E_P^z \boldsymbol{z} = \sum_{p,q} (E_x(p, q)\boldsymbol{x} + E_y(p, q)\boldsymbol{y}) U_{pq} \tag{6}$$

where $E_x(p, q)$ is the x-component of the uniform subaperture field and $E_y(p, q)$ is the y-component. These values vary with the position of the subaperture in the aperture and the type of aperture E-field distribution under analysis.

Thus to calculate the radiated power from the aperture at the observation point $P(x_i, y_i, z_i)$

$$P(x_i, y_i, z_i) = \frac{1}{2}\epsilon |E^2| \tag{7}$$

$$= \frac{1}{2}\epsilon(|E_P^x|^2 + |E_P^y|^2 + |E_P^z|^2)$$

where $E_P^z = 0$.

2.2 Calculating the Approximated Subaperture *E*-field

To find the average field in each subaperture a simple sampling technique is employed (Fig. 2)

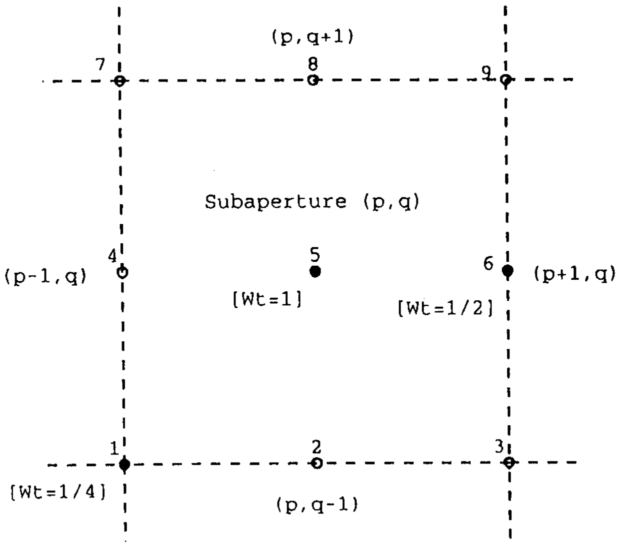


Fig. 2 Field sampling over the subaperture (p, q).

$$E_x(p, q) = \sum_{N=1}^{Ns} \frac{E_x(x_o, y_o) \times W_t}{Ns} \tag{8}$$

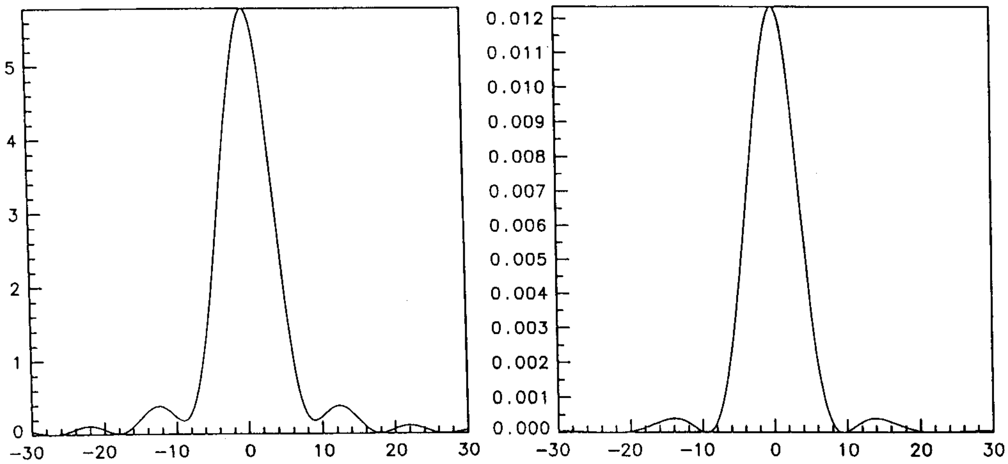
$$E_y(p, q) = \sum_{N=1}^{Ns} \frac{E_y(x_o, y_o) \times W_t}{Ns} \tag{9}$$

where N_s is the total number of sampling points in each subaperture and W_t is the weighting value assigned to each point. If sample point $P_o(x_o, y_o)$ is on the side of the subaperture at the juncture of two subapertures then $W_t = 1/2$. Similarly if it is located at a corner $W_t = 1/4$ and if located inside the boundaries of the subaperture $W_t = 1$. It should also be noted that if the sample point is on the boundary of the aperture and if the field is not equal to zero it is assigned a weighting value of 1.

The validity of this method will be proved by comparing the computational results with the known analytical ones in two cases as follows.

Example 1-The uniformly illuminated single slit

Using the Fresnel integral, as shown above, the radiation field for a uniformly illuminated single slit was calculated at the observation plane $z_i = 30\,\text{mm}$ for an aperture of width $2a = 10\,\text{mm}$ at a wavelength of $\lambda = 3\,\text{mm}$, (Fig. 3(a)). It is useful to compare this with the analytical solution of the same problem (Fig. 3(b)). The most important point that can be taken from these two patterns is that close to the centre of the aperture, on the main lobe, there is good agreement between the two calculations. However as we move further



(a) Fresnel integral (z_i = 30 mm, 2a = 10 mm, λ = 3.0 mm), (b) Analytical solution.

Fig. 3 Comparison of the power profile for a uniformly illuminated slit

from the aperture axis it is found that the disparity between the calculations becomes larger. Such a phenomenon is expected since in the derivation of the Fresnel integral from Huygens equation, see Appendix A, the paraxial condition is assumed i.e. the observation point is limited to be close to the aperture axis.

Example 2-The rectangular waveguide aperture

In this case it is assumed that the field distribution in the aperture is that of the fundamental TE_{10} rectangular waveguide mode. The aperture has length 2a along the x-axis and 2b

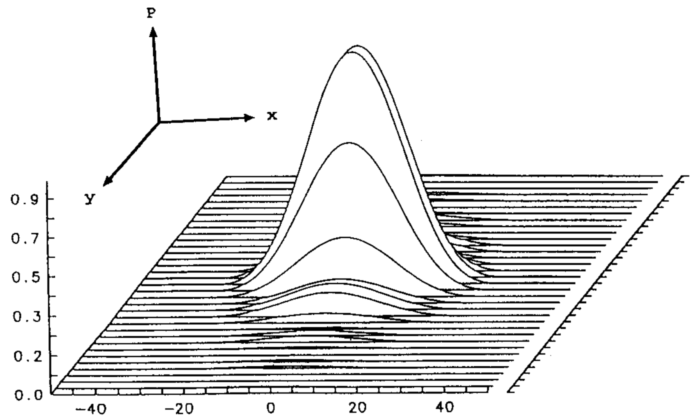


Fig. 4 Calculated field distribution from the TE_{10} mode.
($z_i = 24\lambda$, λ = 3.0 mm, a = 5.0 mm, b = 2a)

along the y-axis. The polarisation of the E-field is along the y-axis. The form of the field is given by the expression

$$E_y = A_o \cos\left(\frac{\pi x_o}{2a}\right) \tag{10}$$

$$E_x = 0 \tag{11}$$

where A_o is an arbitrary constant. Fig. 4 ($z_i = 24\lambda$) shows the radiated field pattern from a rectangular aperture that was divided into 100 subapertures. It is of interest to note that as is expected in the x-direction the side-lobes have been supressed due to the fact that the field in the guide is tapered along the x-axis.

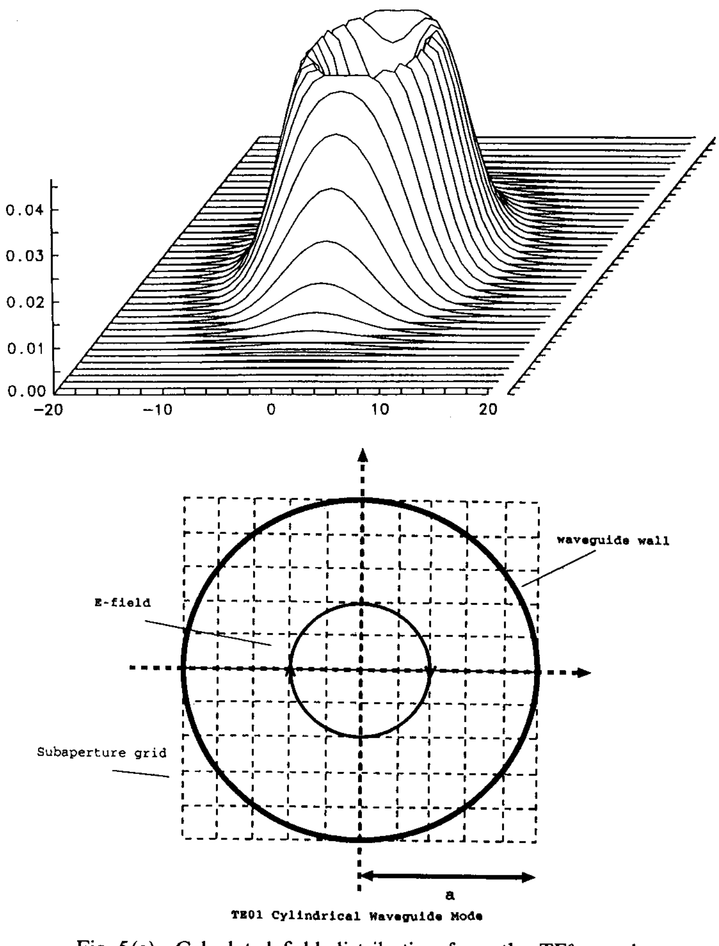


Fig. 5(a) Calculated field distribution from the TE^o_{01} mode.
($z_i = 6\lambda$, $\lambda = 3.0\,\text{mm}$, $a = 5.0\,\text{mm}$)

Example 3-The circular waveguide aperture

The form of the E-field for a TE_{mn} waveguide mode ($E_z = 0$) in cylindrical coordinates (ρ, ψ, z) is

$$E_\rho = j\frac{\omega\mu H_o}{k_c}\left[m\frac{J_m(k_c\rho)}{k_c\rho}\right]\sin(m\psi) \tag{12}$$

$$E_\psi = j\frac{\omega\mu H_o}{k_c}J'_m(k_c\rho)\cos(m\psi) \tag{13}$$

where $J'_m(X)$ is the derivative of the mth order Bessel function, $k_c = \chi'_{mn}/a$ is the cut-off wavenumber, a is the waveguide radius and H_o is an arbitrary constant. Therefore using eqn. (6) it is possible to find the field at the observation point $P(x_i, y_i, z_i)$. Fig. 5 shows the calculated radiation field pattern for the TE_{01} mode. In this case $a = 5.0\,\text{mm}$, $\lambda = 3.0\,\text{mm}$ and $z_i = 6\lambda$. The lack of 'smoothness' in the calculated curve is not due to the Fresnel integral but simply to the spacing between the calculated data points.

3. The Whispering Gallery Mode Antenna

3.1 Introduction

Currently there is great interest in the application of gyrotrons to the area of ECRH (Electron Cyclotron Resonance Heating) of plasmas in magnetic confinement type fusion reactors. Recently the Whispering Gallery Mode (WGM) has been employed for this purpose. The reason is that the WGM has smaller problems of mode competition and through its use higher output powers and frequencies ($\leq 1\,\text{MW}$, $\geq 100\,\text{GHz}$) can be achieved from the gyrotron. This mode is a helically winding mode propagating along a cylindrical waveguide. For ECRH a uniformly polarized gaussian beam is required for efficient heating. Therefore some modification of the output beam is required. This can be accomplished by use of the WGM quasi-optical reflector antennae. This is composed of a helically-cut output waveguide and a cylindrical-parabolic reflector. Such an antenna has been previously investigated and designed by use of geometrical optics (GO). To improve the accuracy of the design and to make a more complete comparison between theory and experiment, diffraction has been taken into account. Although several studies exist to model such a problem, these do not easily lend themselves to the treatment of this case. It was therefore necessary to develop a generalized numerical analysis technique.

3.2 The Whispering Gallery Mode

High-order circular electric TE_{mn} ($m \gg n$, $n \approx 1$ or 2) modes are excited in the whispering gallery mode (WGM) gyrotrons. The rays from these high order rotating waveguide modes

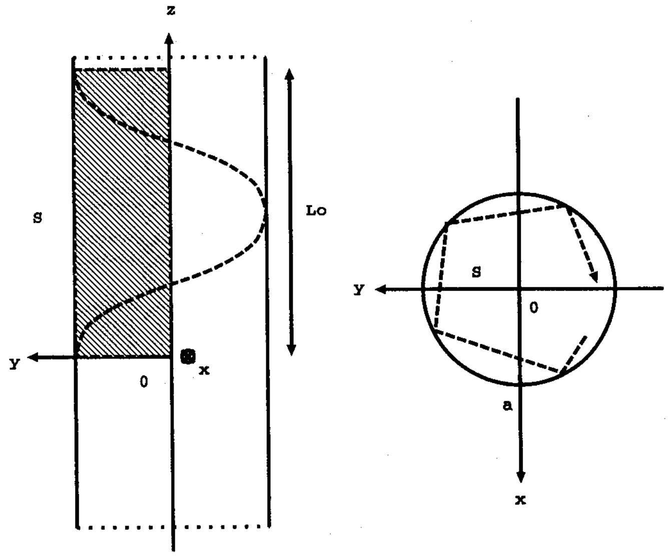


Fig. 6(a) The whispering-gallery-mode aperture S

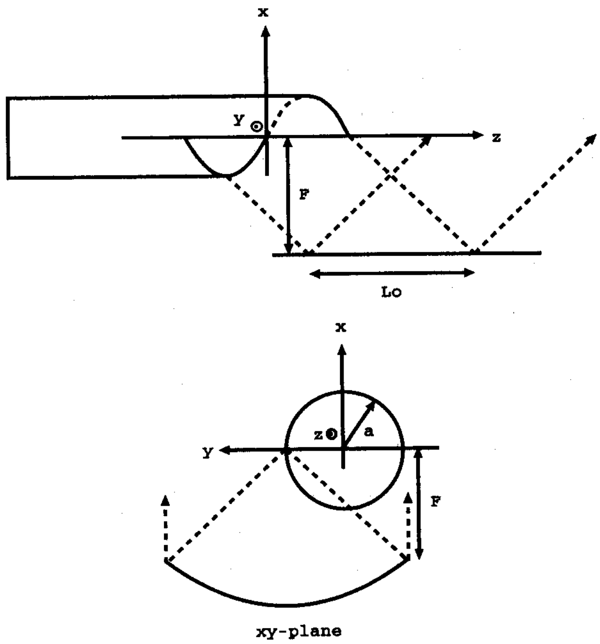


Fig. 6(b) The WGM α-cut Vlasov antenna form a modal caustic which is a circular cylinder of radius ρ_{mn} in the waveguide (Fig. 6(a) and (b))

$$\rho_{mn} = \frac{m}{\chi'_{mn}} a \tag{14}$$

where a is the waveguide radius.

In this case the clockwise rotating wave will be dealt with. The electromagnetic field of the TE_{mn} mode is determined by the longitudinal magnetic field component H_z, which is written in terms of the cylindrical coordinate system (ρ, ψ, z) as

$$H_z^+ = A_{mn}^+ J_m(k_c\rho)e^{-jm\psi}e^{-j\beta z} \tag{15}$$

where $A_{mn}^+ = 2\pi A(-1)^m(k_c/\omega\mu)\exp(j\frac{\pi}{2}m)$, the $^+$ sign indicating rotation in the clockwise direction, and A is an arbitrary constant. The E- and H-field components in the ρ- and ψ-directions are given by

$$E_\psi^+ = j\frac{\omega\mu}{k_c} A_{mn}^+ J'_m(k_c\rho)e^{-jm\psi}e^{-j\beta z} \tag{16}$$

$$E_\rho^+ = -\frac{\omega\mu m}{k_c^2} A_{mn}^+ \frac{J_m(k_c\rho)}{\rho} e^{-jm\psi}e^{-j\beta z} \tag{17}$$

$$H_\psi^+ = -\frac{\beta m}{k_c^2} A_{mn}^+ \frac{J_m(k_c\rho)}{\rho} e^{-jm\psi}e^{-j\beta z} \tag{18}$$

$$H_\rho^+ = j\frac{\beta}{k_c} A_{mn}^+ J'_m(k_c\rho)e^{-jm\psi}e^{-j\beta z} \tag{19}$$

The mode propagates in a circular waveguide winding helically with a pitch angle

$$\theta_B = \arcsin\left(\frac{k_c}{k}\right) \tag{20}$$

The WGM waveguide is terminated with a helical α-cut the field being radiated into space at pitch angle θ_B from the straight edge of the waveguide. In order to analyse the radiation output from the α-cut, the surface S (at $\phi = \pi/2$) is chosen as the radiaing aperture. This is due to the fact that all the rays emanating from the modal caustic pass through this surface. The dimensions of this aperture are L_o along the z-axis and from ρ_{mn} to a along the y-axis.

$$L_o = 2\pi a \cot\theta_B \frac{\sin\theta_W}{\theta_W} \tag{21}$$

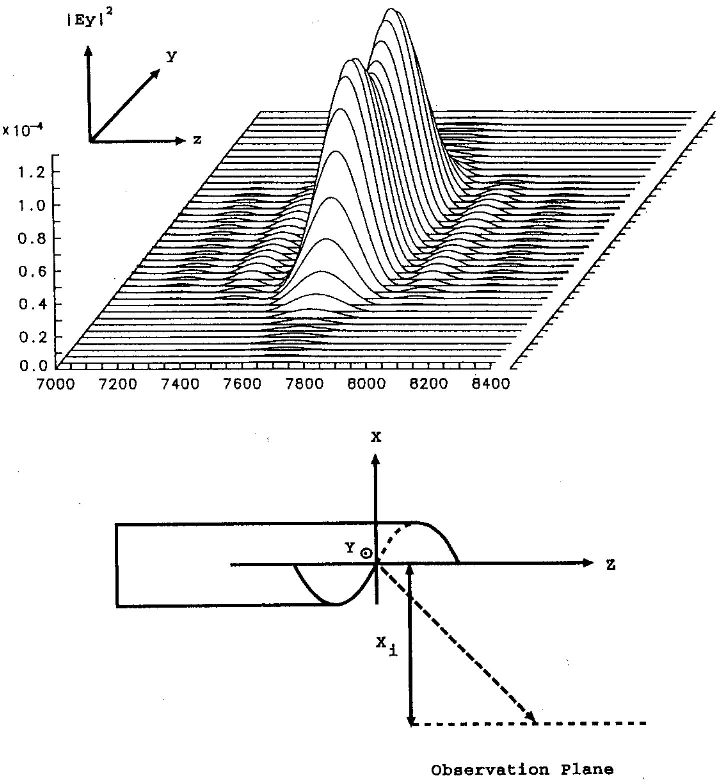


Fig. 7 Calculated field distribution from the $TE^{o}_{15,2}$ mode (Far-field, $x_i = 9.5\,\mathrm{m}$, $f = 140\,\mathrm{GHz}$, $a = 12.8\,\mathrm{mm}$).

$$\theta_W = \arccos \frac{m}{\chi'_{mn}} \tag{22}$$

The radiating aperture is parallel with the propagation direction of the mode. Thus the phase change from subaperture to subaperture would have to be accounted for.

Fig. 7 shows the results of calculation for the $TE_{15,2}$ mode from the WGM α-cut, this being the far-field pattern ($a = 12.8\,\mathrm{mm}$, $f = 140\,\mathrm{GHz}$ and $L_o = 193.0\,\mathrm{mm}$). One of the problems that was found by computation of the WGM was that the position of the beam centre, as expected by geometrical optics, and that observed from calculation differed. It was concluded that this problem arose from the fact that the Fresnel integral could not accurately compute the correct field distribution for an antenna that had a linear change in phase across the aperture plane i.e. the angle of the radiation from the aperture normal was too large.

In order to overcome the problems encountered in applying the Fresnel integral to the case of the WGM it was necessary to interchange the subaperture equation for a more suitable function accounting for the linear phase variation in each subaperture.

4. Asymptotically Evaluated Subaperture Formula

4.1 The Fourier Transform Method

Appendix B it was shows that the radiated field can be expressed in the form

$$\boldsymbol{E}(x, y, z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \boldsymbol{f}(k_x, k_y) e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} dk_x dk_y \tag{23}$$

where $\boldsymbol{k} \cdot \boldsymbol{r} = k_x x + k_y y + k_z z$ and $\boldsymbol{f}(k_x, k_y)$ is still to be found.

This equation states that an arbitrary electric field in the half-space $z \geq 0$ can be represented as a spectrum of plane waves since $\boldsymbol{f}(k_x, k_y)e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$ represents a plane wave with field amplitude $\boldsymbol{f}$ propagating in the direction of the propagation or wavenumber vector $\boldsymbol{k}$. In Appendix B the z-component of the wavenumber vector is defined as $k_z^2 = k_o^2 - k_x^2 - k_y^2$. This means that $|\boldsymbol{k}| = k_o$. For $k_x^2 + k_y^2 > k_o^2$ the propagation constant k_z is imaginary and the plane waves in this part of the spectrum are exponentially decaying or evanescent in the z-direction. These evanescent waves make up the near-zone field in front of the aperture. Only those plane waves that come from the part of the spectrum corresponding to values of $k_x^2 + k_y^2$ inside the circle of radius k_o in the $k_x - k_y$ plane contribute to the radiated field, since only these waves are outward propagating waves.

When $z = 0$ the solution for the x and y components of the electric field must equal the assumed known aperture tangential field. Thus if $\boldsymbol{f}_t$ represents the tangential (x- and y-) components of $\boldsymbol{f}$ then

$$\begin{aligned} \boldsymbol{E}_a(x, y) &= \boldsymbol{E}_{tan}(x, y, 0) \\ &= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \boldsymbol{f}_t(k_x, k_y) e^{-jk_x x - jk_y y} dk_x dk_y \end{aligned} \tag{24}$$

This expression can be recognised as a 2D-Fourier transform and thus

$$\begin{aligned} \boldsymbol{f}_t(k_x, k_y) &= [f_x(k_x, k_y), f_y(k_x, k_y)] \\ &= \iint_{S_a} \boldsymbol{E}_a e^{jk_x x + jk_y y} dx\, dy \end{aligned} \tag{25}$$

so $\boldsymbol{f}_t$ is given in terms of the Fourier transform of the aperture field. From eq. (B9) f_z can be found

$$f_z = -\frac{\boldsymbol{k}_t \cdot \boldsymbol{f}_t}{k_z} = \frac{-k_x f_x - k_y f_y}{\sqrt{(k_o^2 - k_x^2 - k_y^2)}} \tag{26}$$

In this way the vector $\boldsymbol{f}$ can be determined and using Eqn. (23) $\boldsymbol{E}(x, y, z)$ is calculated.

4.2 Asymptotic Evaluation with a Linear Phase Variation

Provided eq. (23) can be evaluated we now have a formal solution for the electric field everywhere in the region $z \geq 0$. In the radiation zone, where r is large compared with λ_o i.e. $k_o r$ is large, this becomes possible. In general we are concerned with the far-zone field therefore it is possible to take the asymptotic value of eq. (23) as r tends to infinity. The result of this asymptotic evaluation is found in the appendix of Chapter 4 of ref. ([7]) and is

$$\boldsymbol{E}(\boldsymbol{r}) \approx \frac{jk_o \cos\theta}{2\pi r} e^{-jk_o r} \boldsymbol{f}(k_o \sin\theta\cos\phi,\ k_o \sin\theta\sin\phi) \tag{27}$$

where θ and ϕ are the spherical coordinate angles, $k_x = k_o \sin\theta\cos\phi$ and $k_y = k_o \sin\theta\sin\phi$. This result shows that the far-zone radiation field, which is the diffraction pattern of the aperture field, is simply related to the Fourier transform of the aperture field by k_x and k_y. In the evaluation of $\boldsymbol{f}_t$ the integrals over x and y are taken over all portions of the $z = 0$ plane on which non-zero values of the tangential electric field exist. If S_a is an opening cut in a perfectly conducting screen, then everywhere outside S_a there will be zero tangential electric field.

The magnetic field in the radiation zone is given by

$$\boldsymbol{H} = Y_o \boldsymbol{a}_r \times \boldsymbol{E} \tag{28}$$

Now assume a linear phase variation for a rectangular subaperture of dimensions (2a × 2b)

$$\boldsymbol{E}_a = E_o \boldsymbol{a}_x e^{-j\beta y} \tag{29}$$

for $|x| \leq$ a, $|y| \leq$ b.

The reason for taking only the x-component of the aperture field and choosing a phase variation in the y-direction will become clear when applying this technique to the case of the WGM antenna. The aperture distribution therefore becomes

$$\boldsymbol{f}_t = E_o \boldsymbol{a}_x \int_{-a}^{a} \int_{-b}^{b} e^{jk_x x + j(k_y - \beta)y}\, dy\, dx \tag{30}$$

Hence if we let $u = k_x a$, $v = k_y b$, $v_o = \beta b$ we obtain

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{jk_o 4abE_o \cos\theta}{2\pi r} e^{-jk_o r} \frac{\sin u}{u} \frac{\sin(v - v_o)}{v - v_o} \boldsymbol{a}_x \tag{31}$$

This therefore becomes the new subaperture equation that accounts for the linear phase variation in the WGM aperture. The factor E_o is the magnitude of the constant subaperture E-field, either $E_x(p, q)$ or $E_y(p, q)$.

As was stated previously the radiation distribution calculation for the WGM helically-cut aperture failed due to the fact that the chosen subaperture equation (Fresnel integral) could not properly account for the linear phase variation along the length of the WGM aperture. However this is inherently accomplished by utilising eq. (31). The results of the calculation are shown in Fig. 8 for the $TE_{15,2}$ WGM at 140 GHz ($\lambda = 2.14$ mm) from the helical α-cut in the yz-plane at $X_i = 50\lambda$. (Compare with Fig. 7)
In this case the calculated beam exit angle corresponds to $\theta_B = 26.4°$ as expected by geometrical optics.

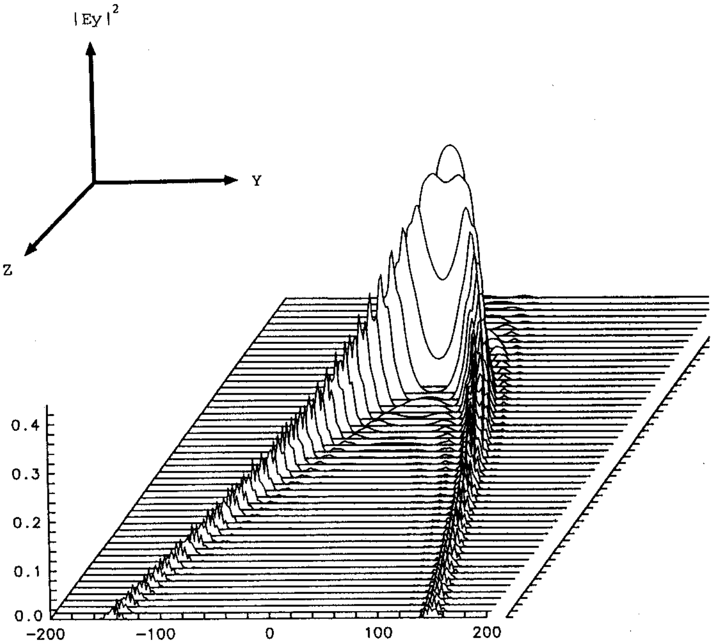


Fig. 8 $TE_{15,2}$ WGM radiator calculation in the YZ-plane ($Xi = 50\ \lambda$)

5. Conclusions

We have proposed a numerical computation method suitable for analysing aperture antennae which have rather complicated field distributions, such as ones used for ECRH. The principle is to divide the aperture into a number of small regularly shaped areas, so that some known functions can be used to obtain analytical radiation patterns from each subaperture. The radiation pattern to be obtained is given by the sum of the contributions of each subaperture. Thus a reduction in computational time is expected, because analytical solutions are used instead of time-consuming numerical integration by a computer.

This technique requires virtually no memory storage for computation, unlike similar computation methods relying upon the use of the Fourier Transform (FT). Also the FT methods, such as the Plane Wave Expansion method, must evaluate the radiation pattern on a plane parallel with the aperture plane. No such restriction exists for the subaperture method.

5.1 Future Plans

Comparison of the results calculated by this technique will be made with experimental results obtained from other institutes. To improve this method further some constraints imposed on the program will be removed.

(1) The next stage will be to improve the accuracy of the near-field calculation.

(2) To apply this technique to the reflector as well as the radiator so that a more accurate WGM antenna calculation can be made.

(3) To aim to reduce further the computational time.

This method would then be applied to more complicated antenna structures and a theory would be established which could be conveniently used to analyse antenna systems in areas of scientific research other than plasma physics applications.

Acknowledgements

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A Derivation of the Fresnel Integral

The mathematical formulation of the Huygens-Fresnel principle for a slit of width Σ in an infinite plane illuminated by a spherical wave is given below

$$U(P) = \int_{\Sigma}\int h(P, P_o)\, U(P_o)\, ds \tag{1}$$

where $U(P_o)$ represents the complex amplitude of either the electric or magnetic field strength on the slit aperture and $U(P)$ is the corresponding field amplitude at the observation point P. The weighting function $h(P, P_o)$ is given explicitly by

$$h(P, P_o) = \frac{1}{j\lambda}\, \frac{\exp(-jkr)}{r} \cos(\boldsymbol{n}, \boldsymbol{r}) \tag{2}$$

where r is the length of the vector $\boldsymbol{r}$ pointing from P to P_o.
Now eq. (1) may be interpreted as implying that the field at an observation point P arises from an infiniy of fictitious 'secondary' point sources located within the aperture (Fig. A1). It should be noted that the amplitude is reduced by the obliquity factor $\cos(\boldsymbol{n}, \boldsymbol{r})$. Also the phase of the secondary source at P_o, on the aperture, leads the phase of the incident wave by 90°.
By using the integral form of the Huygens-Fresnel principle above it is possible to derive

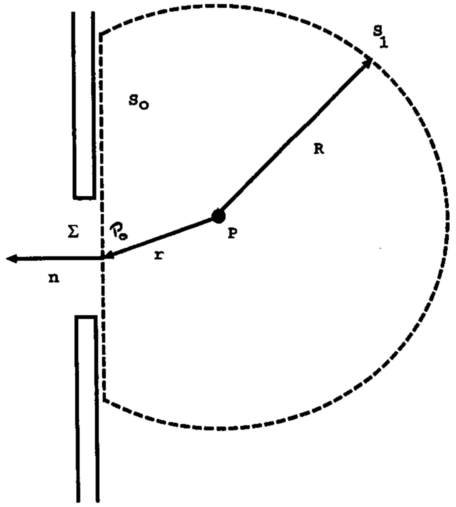

Fig. A1 Kirchhoff formulation of diffraction by a plane screen

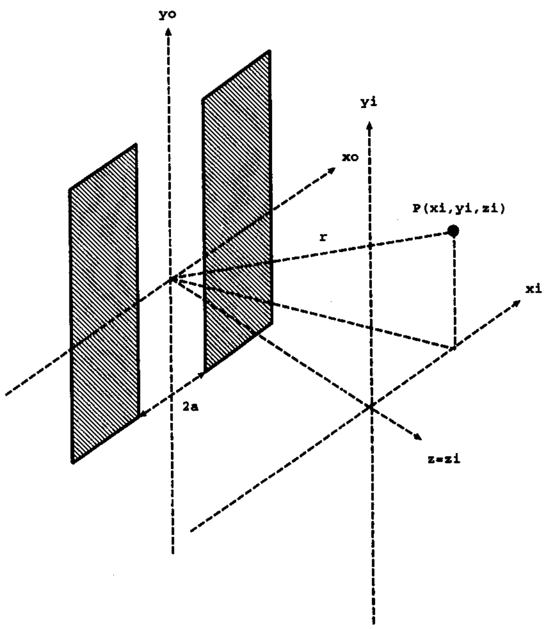

Fig. A2 Coordinate system for an infinite slit a simple solution for the case of a plane wave incident normally upon a slit of dimensions 2a along the x-axis (Fig. A2). [6] The field at the observation point $P(x_i, y_i, z_i)$ on the right hand side of the slit is required. If the analysis is restricted to the situation of the observation point P being far from the aperture and close to the aperture axis then it is possible to set the obliquity factor $\cos(\boldsymbol{n}, \boldsymbol{r}) = 1$. The aperture field amplitude $U(P_o)$ is in this situation restricted to being a function of the x_o-component only and is represented by the function $g(x_o)$. Also the elemental area ds is equivalent to the area $dx_o dy_o$ on the aperture.

Thus the radiation field can be expressed in the integral form of the Huygens-Fresnel principle

$$U(x_i, y_i) = \frac{1}{j\lambda}\int_{-\infty}^{\infty} g(x_o)\int_{-\infty}^{\infty} \frac{e^{-jkr}}{r}\, dy_o\, dx_o \tag{3}$$

where x_o, y_o is the aperture coordinate system, $g(x_o)$ is one component of the amplitude distribution of the electric or magnetic field, $\exp(-jkr/r)$ represents a spherical wave diverging from a point focus on the aperture and $r = \sqrt{[z_i^2 + (x_o - x_i)^2 + (y_o - y_i)^2]}$. The problem in this case is to find an analytical solution of the integral

$$I = \int_{-\infty}^{\infty} \frac{e^{-jkr}}{r}\, dy_o \tag{4}$$

For $y_o = 0$ it can be seen that the projection of r in the xz-plane

$$\rho = \sqrt{z_i^2 + (x_o - x_i)^2} \tag{5}$$

therefore

$$r = \sqrt{\rho^2 + (y_o - y_i)^2} \tag{6}$$

By changing the variable of integration from y_o it becomes possible to analyse the integral I. Thus let

$$\left.\begin{aligned} y_o - y_i &= \rho \sinh(t) \\ \sinh(t) &= \tfrac{e^t - e^{-t}}{2} \end{aligned}\right\} \tag{7}$$

Thus the integral I reduces to the form

$$I = \int_{-\infty}^{\infty} e^{-jk\rho\cosh t}\, dt \tag{8}$$

The form of this integral can be found from the general Hankel function integral

$$H_v^{(l)}(\chi) = \frac{\mp 2je^{\mp v\pi j/2}}{\pi}\int_0^{\infty} e^{\pm j\chi\cosh t}\cosh vt\, dt \tag{9}$$

where $l = 1$ or 2. Choosing ($l = 1$) is equivalent to an argument of $e^{+j\chi \cosh t}$ and ($l = 2$) to $e^{-j\chi \cosh t}$. Thus by choosing $v = 0$ and allowing $\chi = k\rho$ it can be seen that the radiation integral reduces to integration over x_o only

$$U(x_i) = \frac{\pi}{j\lambda} \int_{-\infty}^{\infty} g(x_o) H_0^{(2)}(k\rho)\, dx_o \tag{10}$$

where $H_0^{(2)}(k\rho)$ is the zeroth order Hankel function of the second kind.
As stated earlier the analysis is restricted to the case of

(1) $k\rho \gg 1$ and

(2) $z_i^2 \gg (x_o - x_i)^2$,

the latter condition being referred to as the paraxial condition. Applying these conditions to the Hankel function gives us the one dimensional (1D) form of the Fresnel integral.

$$U(x_i) = \frac{e^{-j(kz_i - \frac{3\pi}{4})}}{\sqrt{z_i \lambda}} \int_{-\infty}^{\infty} g(x_o) e^{-jk(x_o - x_i)^2/2z_i}\, dx_o \tag{11}$$

It is now possible using eq. (11) to formulate an expression for the 2D-Fresnel equation for a uniformly illuminated rectangular aperture of dimensions 2a along the x-axis and 2b along the y-axis.
Thus eq. (11) takes the 2D form

$$U(x_i, y_i) = \frac{j}{2} e^{-jkz_i} \int_{\alpha_1}^{\alpha_2} e^{-j\frac{\pi}{2}\varepsilon^2}\, d\varepsilon \int_{\beta_1}^{\beta_2} e^{-j\frac{\pi}{2}\eta^2}\, d\eta \tag{12}$$

The variables of integration being given by

$$\left.\begin{aligned} \tfrac{2}{\lambda z_i}(x_o - f_x \lambda z_i)^2 &= \varepsilon^2 \\ \tfrac{2}{\lambda z_i}(y_o - f_y \lambda z_i)^2 &= \eta^2 \end{aligned}\right\} \tag{13}$$

where $f_x = x_i/\lambda z_i$, $f_y = y_i/\lambda z_i$.
The limits of integration are

$$\alpha_2 = \sqrt{\frac{2}{\lambda z_i}}[a - f_x \lambda z_i], \quad \alpha_1 = \sqrt{\frac{2}{\lambda z_i}}[(-a) - f_x \lambda z_i],$$

$$\beta_2 = \sqrt{\frac{2}{\lambda z_i}}[b - f_y \lambda z_i], \quad \beta_1 = \sqrt{\frac{2}{\lambda z_i}}[(-b) - f_y \lambda z_i].$$

The Fresnel integral has the form

$$F(\alpha) = \int_0^{\infty} e^{j\frac{\pi}{2}x^2}\, dx \tag{14}$$

where the Fresnel integral is related to the Fresnel cosine and sine integrals by

$$C(\alpha) = \int_0^\alpha \cos\left(\frac{\pi}{2}\varepsilon^2\right) d\varepsilon$$

$$S(\alpha) = \int_0^\alpha \sin\left(\frac{\pi}{2}\varepsilon^2\right) d\varepsilon$$

$$F(\alpha) = C(\alpha) + jS(\alpha)$$

where $C(\alpha)$ is the Fresnel cosine integral and $S(\alpha)$ is the Fresnel sine integral. Thus the radiation field from the rectangular aperture can be expressed as

$$\begin{aligned} U(x_i, y_i) &= \frac{j}{2} e^{-jkz_i}([C(\alpha_2) - C(\alpha_1)] - j[S(\alpha_2) - S(\alpha_1)]) \\ &\quad \times ([C(\beta_2) - C(\beta_1)] - j[S(\beta_2) - S(\beta_1)]) \\ &= \frac{j}{2} e^{-jkz_i} F(\alpha, \beta) \end{aligned} \tag{15}$$

where $F(\alpha, \beta) = \mathrm{Re}[F(\alpha, \beta)] - j\,\mathrm{Im}[F(a, \beta)]$ the real and imaginary parts being represented by

$$Re[F(\alpha, \beta)] = C(\alpha_{2,1})C(\beta_{2,1}) - S(\alpha_{2,1})S(\beta_{2,1}) \tag{16}$$

$$Im[F(\alpha, \beta)] = C(\alpha_{2,1})S(\beta_{2,1}) + S(\alpha_{2,1})C(\beta_{2,1}) \tag{17}$$

A plot of the radiation profile from a uniformly illuminated rectangular aperture as calculated by the Fresnel integral is shown in Fig. A3. This field distribution will subsequently be

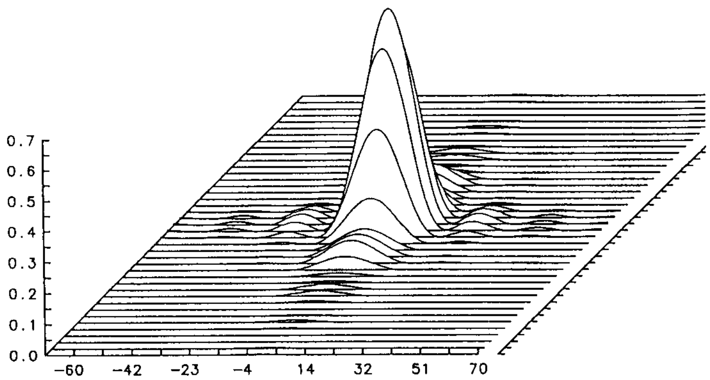

Fig. A3 Calculated field distribution from a uniformly illuminated rectangular aperture.

used as the subaperture radiation profile for calculation of complicated aperture E-field distributions, as detailed in section (2).

B The Plan Wave Expansion Method

This method uses the fact that any electromagnetic wave can be described as a sum of plane waves (Fig. B1) [7]. In order to do this it is first necessary to define the two dimensional Fourier Transform (FT) $F_t(k_x, k_y)$ and inverse FT for the function $f_t(x, y)$

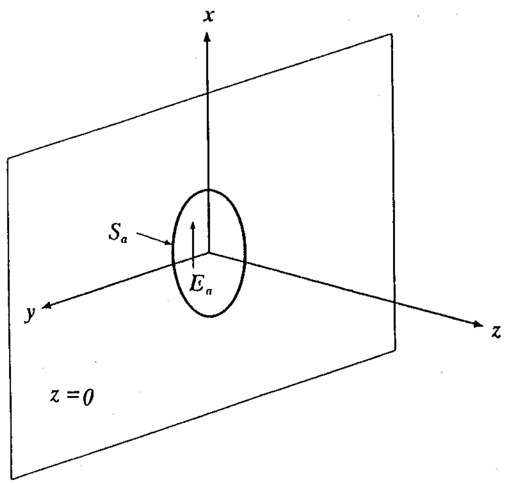

Fig. B1 Coordinate system for Fourier Transform analysis of aperture.

$$F_t(k_x, k_y) = \iint_{-\infty}^{\infty} u(x, y)\, e^{jk_x x + jk_y y}\, dx\, dy$$

$$f_t(x, y) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} U(k_x, k_y) e^{-jk_x x - jk_y y}\, dk_x\, dk_y$$

In the free-space region which contains no electric charge or current

$$\nabla^2 \boldsymbol{E} + k_o^2 \boldsymbol{E} = 0 \tag{1}$$

$$\nabla \cdot \boldsymbol{E} = 0 \tag{2}$$

Separating into component form

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_o^2\right] \boldsymbol{E}(x, y, z) = 0 \tag{3}$$

$$\frac{\partial E_x(x, y, z)}{\partial x} + \frac{\partial E_y(x, y, z)}{\partial y} + \frac{\partial E_z(x, y, z)}{\partial z} = 0 \tag{4}$$

taking the Fourier transform of both of these equations with respect to x and y we obtain

$$\left[\frac{\partial^2}{\partial z^2} + (k_o^2 - k_x^2 - k_y^2)\right] \boldsymbol{E}(k_x, k_y, z) = 0$$

$$k_x E_x(k_x, k_y, z) + k_y E_y(k_x, k_y, z) + j\frac{\partial}{\partial z} E_z(k_x, k_y, z) = 0$$

It is assumed that the tangential components of the electric field on the aperture surface are known and are denoted by $\boldsymbol{E}_a$. $\boldsymbol{E}(k_x, k_y, z)$ is the FT of the electric field wrt. x and y. Let

$$k_z^2 = k_o^2 - k_x^2 - k_y^2 \tag{5}$$

We therefore obtain

$$\frac{\partial^2 E(k_x, ky, z)}{\partial z^2} + k_z^2 E(k_x, ky, z) = 0 \tag{6}$$

which has solution of the form $e^{\pm jk_z z}$. Since the field should consist of waves propagating outwards along the z-axis only the function $e^{-jk_z z}$ will be valid. Thus the general solution is

$$E(k_x, k_y, z) = f(k_x, k_y) e^{-jk_z z} \tag{7}$$

Using eq. (7) in eq. (4) we find that

$$k_x f_x + k_y f_y + k_z f_z = 0 \tag{8}$$

or,

$$\boldsymbol{k} \cdot \boldsymbol{f} = \boldsymbol{0} \tag{9}$$

where $\boldsymbol{k}$ is the vector with components k_x, k_y, k_z.
Using the inverse FT relations the solution for the electric field can be expressed as

$$\boldsymbol{E}(x, y, z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \boldsymbol{E}(k_x, k_y, z) e^{-jk_x x - jk_y y} \, dk_x \, dk_y$$

$$= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \boldsymbol{f}(k_x, k_y) e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \, dk_x \, dk_y \tag{10}$$

where $\boldsymbol{k} \cdot \boldsymbol{r} = k_x x + k_y y + k_z z$.