Mem. Fac. Eng., Kyoto Univ., Vol. 55, No. 3 (1993)

On the H_∞-Wellposed Cauchy Problem for Some Schödinger type Equations

By

Shigeo TARAMA

(Received April 7, 1993)

Abstract

We consider the H_∞-wellposedness of the Cauchy problem for the operator:

$$\frac{\partial}{\partial t}u(t, x)+\frac{i}{2}\Delta u(t, x)+\sum_{k=1}^{n} a_k(x)\frac{\partial}{\partial x_k}u(t, x)+b(x)u(t, x)=f(t, x)$$

when an initial value on the plane $t=0$. *We show some sufficient conditions on the imaginary parts of the coefficients* $a_k(x)$ for the wellposedness.

I. Introduction

Let L be the Schrödinger type operator given by

$$Lu(t, x)=\frac{\partial}{\partial t}u(t, x)+\frac{i}{2}\Delta u(t, x)+\sum_{k=1}^{n} a_k(x)\frac{\partial}{\partial x_k}u(t, x)+b(x)u(t, x) \tag{L}$$

where Δ is Laplacian, that is, $\Delta u(t, x)=\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}u(t, x)$ and the coefficients $a_k(x)$ and $b(x)$ belong to the space $B^\infty(R^n)$ consisting of all smooth functions on R^n which are bounded with their derivatives of any order.

Let T be an arbitarily fixed positive number. We consider the Cauchy problem for L: for the given $g(x)\in H_\infty$ and $f(t, x)\in C([0, T], H_\infty)$ find a solution $u(t, x)\in C^1([0, T], H_\infty)$ of

$$\begin{cases} Lu(t, x)=f(t, x) & \text{on } [0, T]\times R^n \\ u(0, x)=g(x) & \text{on } R^n. \end{cases} \tag{C}$$

Dept. of Applied Mathematics and Physics Faculty of Enginering, Kyoto University 606–01 Kyoto Japan

Here $H_\infty=\bigcap_{k=0}^{\infty} H_{(k)}(R^n)$ where $H_{(k)}(R^n)$ is the space of all $u(x)\in L^2(R^n)$ such that $\frac{\partial^\alpha}{\partial x^\alpha}u(x)\in L^2(R^n)$ when $|\alpha|\le k$ with a norm $\|u(x)\|_k^2=\sum_{|\alpha|\le k}\|\frac{\partial^\alpha}{\partial x^\alpha} u(x)\|_{L^2}^2$ and $C([0, T], H_\infty)$ is the space of all H_∞-valued continuous functions on $[0, T]$. Here we use the following notations: $\alpha=(\alpha_1, \alpha_2,\cdots,\alpha_n)$, $|\alpha|=\sum_{k=1,\cdots,n} \alpha_k$ and $\frac{\partial^\alpha}{\partial x^\alpha}=\frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$.

We say that the problem (C) is H_∞-wellposed if for any $f(t, x)$ and $g(x)$ there exists one and only one solution $u(t, x)$ that is an H_∞-valued C^1 function on $[0, T]$. Thanks to Banach's closed graph theorem the wellposedness implies that the mapping $H_\infty\times C([0, T], H_\infty)\ni(g(x), f(t, x))\mapsto u(t, x)\in C^1([0, T], H_\infty)$ is continuous. $H_{(k)}$- and S-wellposedness are defined similarly.

W. Ichinose [3] shows the following necessary condition for the H_∞-wellposedness (see also J. Takeuchi [6]): there exists a constant K such that for any x and $\xi\in R^n$ we have

$$\text{(N)} \qquad \left|\int_0^1 \sum_{k=1}^n a_k^I(x+\xi t)\xi_k\, dt\right| \le K \log(|\xi|+2),$$

where $a_k^I(x)$ is the imginary part of $a_k(x)$ and its real part is denoted by $a_k^R(x)$.

We suppose that 1-form $\sum_{k=1}^n a_k^I(x)dx_k$ is closed, that is to say,

$$\text{(1.1)} \qquad \frac{\partial}{\partial x_k}a_l^I(x)-\frac{\partial}{\partial x_l}a_k^I(x)=0$$

for any $k,l=1,2,\cdots n$. Then the function on R^n defined by

$$\text{(1.2)} \qquad \mathrm{F(x)}=\sum_{k=1}^n \int_0^1 a_k^I(tx)x_k\, dt$$

satisfies $\frac{\partial}{\partial x_k}F(x)=a_k^I(x)$. If (N) is satisfied, we have

$$\text{(1.3)} \qquad |F(x)-F(y)|\le K \log\, (|\mathrm{x}-\mathrm{y}|+2).$$

Since $\frac{\partial}{\partial x_k}F(x)=a_k^I(x)\in B^\infty(R^n)$, (1.3) implies that the multiplication by $e^{\frac{1}{2}F(x)}$

is an isomorphism in Schwartz space S (see [2] or [5] for the definition of Schwartz space).

On the other hand we define the operator $\tilde{L}$ by

$$\begin{aligned}\tilde{L} &= e^{\frac{1}{2}F(x)}Le^{-\frac{1}{2}F(x)} \\ &= \frac{\partial}{\partial t}u(t, x) + \frac{i}{2}\Delta u(t, x) + \sum_{k=1}^{n} a_k^R(x)\frac{\partial}{\partial x_k}u(t, x) + \tilde{b}(x)u(t, x)\end{aligned}$$

with $\tilde{b}(x) \in B^\infty$. Then, because the coefficients $a_k^R(x)$ of $\frac{\partial}{\partial x_k}u(t, x)$ are real-valued, the Cauchy problem for $\tilde{L}$

$$(\tilde{C}) \qquad \begin{cases}\tilde{L}u(t, x) = h(t, x) & \text{on } ([0, T] \times R^n) \\ u(t, x) \;\, = k(x) & \text{on } R^n\end{cases}$$

is S-wellposed and $H_{(l)}$-wellposed for any l. (See for example R. Dautray and J. L. Lions [1] for $H_{(l)}$-wellposedness and M. Tsutsumi [8, Lemma 3.1] or Appendix of this article for S-wellposedness.)

Hence we see that under (N) and (1.1) the Cauchy problem (C) is S-wellposed. In this article we show

Theorem. *If the conditions* (N) *and* (1.1) *are satisfied, the Cauchy problem* (C) *is* H_∞*-wellposed.*

We prove the Theorem in the next section. The idea of proof is identical to that of S. Tarama [7]. For any function of R^n, $\int f(x)dx$ means $\int_{R^n} f(x)dx$. We denote by C or $C*$ suffixed by some letter $*$ an arbitary constant which may be different line by line.

2. Proof of Theorem. In the following we assume that (N) and (1.1) are satisfied.

First of all, we decompose the data $g(x)$ and the right hand side of the equation $f(t, x)$ to the sum of functions in S or in $C([0, T], S)$ in the following way. We choose $\phi(x) \in C^\infty(R^n)$ satisfying $\phi(x) \geq 0$, $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi(x)dx = 1$. Then we have

$$g(x) = \int \phi(x-y)g(x)dy$$

and

$$f(t, x)=\int\phi(x-y)f(t, x)dy.$$

Since $g(x)\in H_\infty$ [*resp.* $f(t, x)\in C([0, T], H_\infty)$] and $\phi(x-y)$ vanishes for $|x-y|\geq 1$, we see that $\phi(x-y)g(x)\in S$ [resp. $\phi(x-y)f(t, x)\in C([0, T], S)$] and $\phi(x-y)g(x)$ [resp. $\phi(x-y)f(t, x)$] is an S-valued [resp. $C([0, T], S)$-valued] continuous function of $y\in R^n$.

Since the problem (C) is S-wellposed under conditions (N) and (1.1), we have a solution $u_y(t, x)\in C^1([0, T], S)$ of

$$(\mathrm{C}_y)\qquad \begin{cases} Lu_y(t, x)=\phi(x-y)f(t, x) & \text{on } ([0, T]\times R^n) \\ u(0, x) \quad =\phi(x-y)g(x) & \text{on } R^n \end{cases}.$$

The S-wellposedness implies that $u_y(t, x)$ is a $C^1([0, T], S)$-valued continuous function of $y\in R^n$.

We will show that the function $u(t, x)$ defined by

$$u(t, x)=\int u_y(t, x)dy$$

is a solution of the problem (C).

First we remark that, for any integer $l\geq 0$,

$$C^{-1}\|g(\cdot)\|_l^2\leq\int \|\phi(\cdot-y)g(\cdot)\|_l^2dy\leq C\|g(\cdot)\|_l^2 \tag{2.1}$$

and

$$C^{-1}\int_0^t \|f(s,\cdot)\|_l^2ds\leq\int\int_0^t\|\phi(\cdot-y)f(s,\cdot)\|_l^2dsdy\leq C\int_0^t\|f(s,\cdot)\|_l^2ds. \tag{2.2}$$

Indeed, noting $\int\phi(x-y)dy=1$, we have

$$\begin{aligned}\|g(\cdot)\|_{L^2}^2&=\int\left(\int \phi(x-y)g(x)dy\right)\left(\int\phi(x-z)\overline{g(x)}\ dz\right)dx\\ &=\int dw\int dy\int \phi(x-y)g(x)\phi(x-y-w)\overline{g(x)}\ dx,\end{aligned}$$

noting $\phi(x)=0$ for $|x|\geq 1$

$$\int_{|w|\leq 2} dw\int dy\int |\phi(x-y)g(x)||\phi(x-y-w)g(x)|dx$$

from which, using Schwarz inequality, we can draw the left side inequality of

(2.1) for $l=0$.
On the other hand, Fubini's Theorem implies a right side inequality of (2.1) for $l=0$. Similarly (2.1) for any $l\geq 0$ and (2.2) can be shown.

Lemma 1. *There exists an integer $N\geq 0$ such that we have, for any integer $l\geq 0$,* $z\in R^n$ and $t\in[0,T]$,

$$(2.3)\qquad \|\phi(\cdot-z)u_y(t,\cdot)\|_l\leq C_l<z-y>^{-n-1}\Big(\|\phi(\cdot-y)g(t,\cdot)\|_{l+N}+\int_0^t \|\phi(\cdot-y)f(s,\cdot)\|_{l+N}\,ds\Big)$$

where the constant C_l is independent of z, y and t.

For the proof of Lemma 1, we use the following lemma, whose proof, which is sketched in the appendix of this note, is similar to that of Proposition 7 of S. Tarama [7] (see also T. Kato [4, Section 8]).

Lemma 2. *For the solution $v(t,x)\in C^1([0,T],S)$ of the problem ($\tilde{C}$) with $k(x)\in S$ and $h(t,x)\in C([0,T],S)$ we have the following: for any integers N and $l\geq 0$, $y\in R^n$ and $t\in[0,T]$*

$$(2.4)\qquad \|<\cdot-y>^N v(t,\cdot)\|_l\leq C_{l,N}\Big(\|<\cdot-y>^N k(\cdot)\|_{l+N}+\int_0^t \|<\cdot-y>^N h(s,\cdot)\|_{l+N}\,ds\Big),$$

where the constant $C_{l,N}$ is independent of y.

Proof of Lemma 1. In this proof C or $C_{l,N}$ denotes a constant which is independent of $y\in R^n$.

We note that the function $v(t.x)=\exp(\frac{1}{2}F(x))u_y(t,x)$ is a solution of the problem ($\tilde{C}$) with

$$k(x)=\exp(\frac{1}{2}F(x))\phi(x-y)g(x)$$

and

$$h(t,x)=\exp(\frac{1}{2}F(x))\phi(x-y)f(t,x).$$

For any integer $l \geq 0$,

$$\|\phi(\cdot - z)u_y(t,\cdot)\|_l = \|\phi(\cdot - z)\exp(-\frac{1}{2}F(\cdot))v(t,\cdot)\|_l,$$

since $\phi(x)=0$ for $|x|\geq 1$

$$\leq C\ \exp(-\frac{1}{2}F(z))\|\phi(\cdot - z)v(t,\cdot)\|_l$$

$$\leq C\ \exp(-\frac{1}{2}F(z)) <z-y>^{-N}\| <\cdot - y>^N \phi(\cdot - z)v(t,\cdot)\|_l$$

$$\leq C\ \exp(-\frac{1}{2}F(z)) <z-y>^{-N}\| <\cdot - y>^N v(t,\cdot)\|_l,$$

from (2.4)

$$\leq C_{l,N}\ \exp(-\frac{1}{2}F(z)) <z-y>^{-N} \times$$

$$(\| <\cdot - y>^N\ \exp(\frac{1}{2}F(\cdot))\phi(\cdot - y)g(\cdot)\|_{l+N} + \int_0^t \| <\cdot - y>^N$$

$$\exp(\frac{1}{2}F(\cdot))\phi(\cdot - y)f(s,\cdot)\|_{l+N}\ ds),$$

since $\phi(x)=0$ for $|x|\geq 1$,

$$\leq C_{l,N}\ \exp(\frac{1}{2}(-F(z)+F(y))) <z-y>^{-N} \times$$

$$\left(\|\phi(\cdot - y)g(\cdot)\|_{l+N} + \int_0^t \|\phi(\cdot - y)f(s,\cdot)\|_{l+N}\ ds\right),$$

since we have, from (1.3), $\exp(\frac{1}{2}(-F(z)+F(y))) \leq C <z-y>^{\frac{1}{2}K}$, by taking $N \geq \frac{1}{2}K+n+1$,

$$\leq C_{l,N} <z-y>^{-n-1}(\|\phi(\cdot - y)g(\cdot)\|_{l+N} + \int_0^t \|\phi(\cdot - y)f(s,\cdot)\|_{l+N}\ ds). \qquad \square$$

As we remarked above, $u_y(t,x)$ is a $C^1([0, T], S)$-valued continuous function. Thus for any $r\geq 0$

$$\int_{|y|\leq r} u_y(t, x)dy \in C^1([0, T], S).$$

Since, for any $r\geq 0$, $\int_{|z|\leq r+1} \phi(x-z)dz=1$ on $|x|\leq r$,

$$u_y(t,x)=\int_{|z|\leq r+1} \phi(x-z)u_y(t, x)dz \text{ on } |x|\leq r$$

and

$$\|u_y(t,\cdot)\|_{H_{(l)}(\{x\in R^n;|x|<r\})} \leq \int_{|z|\leq r+1} \|\phi(\cdot-z)u_y(t,\cdot)\|_l \, dz.$$

Schwarz inequality and (2.3) of Lemma 1 imply that

$$\int\int_{|z|\leq r+1} \|\phi(\cdot-z)u_y(t,\cdot)\|_l \, dz \, dy$$
$$\leq C \int_{|z|\leq r+1}\left(\int <z-y>^{-2n-2} dy\right)^{\frac{1}{2}} \times$$
$$\left(\int\|\phi(\cdot-y)g(\cdot)\|_{l+N}^2 dy + t\int_0^t \int \|\phi(\cdot-y)f(s,\cdot)\|_{l+N}^2 \, dy \, ds\right)^{\frac{1}{2}},$$

from (2.1) and (2.2)

$$\leq C\left(\|g(\cdot)\|_{l+N}^2 + t\int_0^t \|f(s,\cdot)\|_{l+N}^2 \, ds\right)^{\frac{1}{2}}.$$

Hence

$$u(t, x)=\int u_y(t, x)dy \in C([0, T], H_{(l),loc}(R^n)).$$

Since $u_y(t, x)$ is a solution of the problem (C_y), we see that

$$\frac{\partial}{\partial t}u(t, x) \in C([0, T], H_{(l-2),loc}(R^n))$$

and $u(t, x)$ is a solution of the problem (C), where we used

$$\int u_y(0, x)dy = \int \phi(x-y)g(x)dy = g(x)$$

and

$$\int \phi(x-y)f(t, x)dy = f(t, x).$$

Further, it follows from (2.3) and Hausdorff-Young inequality, since $\int <x>^{-n-1} dx < +\infty$, that

$$\int \left(\int \|\phi(\cdot - z)u_y(t,\cdot)\|_l \, dy \right)^2 dz \le C_l \left(\int \|\phi(\cdot - y)g(\cdot)\|_{l+N}^2 \, dy + t\int_0^t \int \|\phi(\cdot - y)f(s,\cdot)\|_{l+N}^2 \, dy \, ds \right),$$

from (2.1) and (2.2)

$$\le C_l \left(\|g(\cdot)\|_{l+N}^2 + \int_0^t \|f(s,\cdot)\|_{l+N}^2 \, ds \right)$$

and for any $r \ge 0$

$$\int_{|z| \ge r} \left(\int \|\phi(\cdot - z)u_y(t,\cdot)\|_l^2 \, dy \right)^2 dz \le$$

$$C_l \int_{|z| \ge r} \left(\int <z-y>^{-n-1} \|\phi(\cdot - y)g(\cdot)\|_{l+N}^2 \, dy + \int t \int_0^t <z-y>^{-n-1} \|\phi(\cdot - y)f(s,\cdot)\|_{l+N}^2 \, ds \, dy \right) dz.$$

Thus we see that

$$u(t, x) = \lim_{r \to +\infty} \int_{|z| \le r} \int \phi(x-z)u_y(t, x)dy \, dz \in C([0, T], H_{(l)})$$

and

$$\|u(t,\cdot)\|_l^2 \le C_l \left(\|g(\cdot)\|_{l+N}^2 + \int_0^t \|f(s,\cdot)\|_{l+N}^2 \, ds \right). \tag{2.5}$$

The above arguments are valid for any integer $l \ge 0$ and $u(t,x)$ satisfies

$Lu(t, x)=f(t, x)$. Thus $u(t, x)$ is a solution of the problem (C) belonging to $C^1([0, T], H_\infty)$.

Concerning the uniqueness of solutions, we remark first that the following Cauchy problem (C*) for the formal adjoint L^* of L:

$$L^*u(t,x)=$$
$$-\frac{\partial}{\partial t}u(t, x)-i\Delta u(t, x)-\sum_{k=1}^{n}\bar{a}_j(x)\frac{\partial}{\partial x_j}u(t, x)+\left(-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\bar{a}_j(x)+\bar{b}(x)\right)u(t,x)$$

satisfies (N) and (1.1).

Thus the following backward Cauchy problem (C*) for L^*:

$$\text{(C*)}\qquad \begin{cases} L^*u(t,x)=f(t,x) & \text{on } [0,T]\times R^n \\ u(T,x)=g(x) & \text{on } R^n \end{cases}$$

is also S-wellposed, from which we see the uniqueness of solutions for the problem (C), (See for example S. Mizohata [5, Proof of Theorem 4.2]). Hence the problem (C) is H_∞-wellposed. The proof of Theorem is completed.

Appendix. In this appendix we sketch the proof of Lemma 2. We consider only the operator L whose coefficients $a_j(x)$ are real valued.

Lemma 2 results from the following lemma.

Lemma A. *For any integers $N\geq 0$ and $u(t, x)\in C^1([0, T], S)$ we have*

$$\text{(A.1)}\qquad \sum_{j=0}^{N}\| <x>^j\, u(t, x)\|_{N-j}$$
$$C\left(\sum_{j=0}^{N}\| <x>^j u(0, x)\|_{N-j}+\sum_{j=0}^{N}\int_0^t \| <x>^j\, Lu(s, x)\|_{N-j}\, ds\right),$$

where the constant C depends only on T and the translation invariant norm of the coefficients, i.e.

$$\sum_{|\alpha|\leq M}\left(\sum_{j=0}^{n}\sup_{x\in R^n}\left|\frac{\partial^\alpha}{\partial x^\alpha}a_j(x)\right|+\sup_{x\in R^n}\left|\frac{\partial^\alpha}{\partial x^\alpha}b(x)\right|\right) \textit{ with } M=\max\{N,1\}.$$

Let L_y be defined by

$$L_y u(t, x)=\frac{\partial}{\partial t}u(t, x)+\frac{i}{2}\Delta u(t, x)+\sum_{k=1}^{n}a_k(x+y)\frac{\partial}{\partial x_k}u(t, x)+b(x+y)u(t, x).$$

Then the inequality (A.1) for L_y is valid with the constant C which is independent of $y \in R^n$, from which we draw Lemma 2.

Proof of Lemma A. We remark that

$$\|u(t, x)\| \leq e^{Ct}\left(\|u(0, x)\| + \int_0^t \|Lu(s, x)\| ds\right), \tag{A.2}$$

where $\|\cdot\|$ is a L^2-norm and $C = \sum_{j=1,\cdots,n} \sup_{x \in R^n} |\frac{\partial}{\partial x_j} a_j(x)| + \sup_{x \in R^n} |b(x)|$.

Let $Op(N)$ be a linear space generated by all the operators $x^\alpha \frac{\partial^\beta}{\partial x^\beta}$ with $|\alpha| + |\beta| \leq N$. Then we see that, for any $T \in Op(N)$, the commutator $[\Delta, T] = \Delta T - T\Delta$ belongs to $Op(N)$ and that $[a_j(x)\frac{\partial^j}{\partial x_j}, T]$ and $[b(x), T]$ can be written by a linear combination of products of some element in $Op(N)$ and the derivative, whose order is at most N, of $a_j(x)$ or $b(x)$. Hence for any α and β satisfying $|\alpha| + |\beta| \leq N$,

$$\begin{aligned} L x^\alpha \frac{\partial^\beta}{\partial x^\beta} u(t,x) &= x^\alpha \frac{\partial^\beta}{\partial x^\beta} Lu(t,x) \\ &+ \sum_{|\gamma| + |\delta| \leq N} C_{\alpha.\beta,\gamma,\delta}(x) x^\gamma \frac{\partial^\delta}{\partial x^\delta} u(t, x) \end{aligned} \tag{A.3}$$

where $C_{\alpha.\beta,\gamma,\delta}(x)$ can be written by the linear combination of the derivatives of $a_j(x)$ or $b(x)$ of order at most N.

We see Lemma A from (A.2) and (A.3). □

Noting that, for any $j,k = 1, \cdots, n$, $\left[\frac{\partial}{\partial x_j}, \frac{x_k}{<\varepsilon x>}\right] = k_{j,k}(\varepsilon x)$ with some $k_{j,k}(x) \in B^\infty(R^n)$, we can show, by using an argument similar to the proof of Lemma A, the following estimates, which with Lemma A imply the S-wellposedness of the problem ($\tilde{\mathrm{C}}$): for any integer N and $0 < \varepsilon \leq 1$

$$\begin{aligned} &\sum_{j=0}^N \|(\frac{<x>}{<\varepsilon x>})^j u(t,x)\|_{N-j} \\ &\leq \\ &C\left(\sum_{j=0}^N \|(\frac{<x>}{<\varepsilon x>})^j u(0,x)\|_{N-j} + \sum_{j=0}^N \int_0^t \|(\frac{<x>}{<\varepsilon x>})^j Lu(s,x)\|_{N-j}\, ds\right), \end{aligned} \tag{A.1}$$

where the constant C is independent of ε.

References

[1] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Sciences and Technology*, vol. 5 Evolution Problem 1, Springer-Verlag, Berlin, 1992.
[2] L. Hömander, *The Analysis of Linear Partial Differential Operators* I, 2nd ed., Springer-Verlag, Berlin, 1990.
[3] W. Ichinose, *Some remarks on the Cauchy problem for Schrödinger type equations*, Osaka J. Math. **21** (1984), 565–581.
[4] T. Kato, *On the Cauchy Problem for the (Generalized) Korteweg-de Vries Equation*, Studies in Appl. Math. Ad. in Math. Suppl. Stud..
[5] S. Mizohata, *The theory of partial differential equations*, Cambridge University Press, 1973.
[6] J. Takeuchi, *A necessary condition for H^∞-wellposedness of the Cauchy problem for linear partial differential operators of Schrödinger type*, J. Math. Kyoto Univ. **25** (1985), 459–472.
[7] S. Tarama, *On the wellposed Cauchy problem for some dispersive equations* (to appear).
[8] M. Tsutsumi, *Weight Sobolev Spaces and Rapidly Decreasing Solutions of Some Nonlinear Dispersive Wave Equations*, J. Diff. Equations **42** (1981), 260–281.