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Parametrization of All Stable Unbiased $\mathscr{H}_\infty$ Estimators Based on Model Matching

By

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Abstract

This paper considers the $\mathscr{H}_\infty$ estimation problem by using a model matching technique. For a given possibly unstable plant, the $\mathscr{H}_\infty$ estimation problem is reduced to a standard model matching problem by introducing a class of all stable and unbiased estimators. Based on Nehari's theorem, a necessary and suffcient condition for the existence of a solution of the $\mathscr{H}_\infty$ estimation problem is derived in terms of an $\mathscr{H}_\infty$-type algebraic Riccati equation. The LFT representation of the class of all solutions is also developed.

1. Introduction

Recently considerable attention has been directed to $\mathscr{H}_\infty$ estimation problems [1]–[7]. The estimation with $\mathscr{H}_\infty$ criterion is appropriate when there is significant uncertainty in the spectral density of disturbance. Yaesh and Shaked have developed full-order estimators by using game theoretic approaches [1],[2] and the bounded real condition [3]. An operator theoretic LQ optimization technique in the time domain is employed to derive $\mathscr{H}_\infty$ filters and smoothers by Nagpal and Khargonekar [4]. An $\mathscr{H}_2$ estimator with an $\mathscr{H}_\infty$ error bound is also developed in [5] based on a coupled system of modified Riccati equations. Fernandes *et al.* [6] have presented design techniques for robust estimators based on a parametrization of all stable unbiased estimators and $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_\infty$ optimizations. Recently, Limebeer and Shaked [7] considered a minimax terminal estimation and the $\mathscr{H}_\infty$ filtering problem by employing a game theoretic approach and the duality between estimation and control. In particular, for the infinite-horizon case, they derived a necessary and sufficient condition for the existence of stable estimators for a possibly unstable plant based on the bounded real condition and they parametrized all stable unbiased $\mathscr{H}_\infty$ estimators. It may be noted that although a parametrization of all $\mathscr{H}_\infty$ estimators is given as a solution to OE problem in [8], it cannot be directly applied to an unstable plant

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due to the restriction of internal stability.

In this paper, we derive a state-space description of all solutions to the $\mathscr{H}_\infty$ estimation problem based on a model matching technique. Since the $\mathscr{H}_\infty$ estimation problem is not a standard model matching problem when the plant is unstable, we first reduce the $\mathscr{H}_\infty$ estimation problem to a standard model matching problem by using a parametrization of all stable unbiased estimators [6], [9]. We then derive a necessary and sufficient condition for the existence of a solution based on Nehari's theorem. Although the main result of the paper is also contained in [7], we present a straightforward proof based on a purely frequency domain approach [10]. Finally we develop an LFT (linear fractional transformation) representation of all solutions to the $\mathscr{H}_\infty$ estimation problem.

The notation used in this paper is standard. In particular, $[\cdot]_+$ and $[\cdot]_-$ are the stable and antistable parts of a transfer matrix by partial fraction expansion, respectively. And $G(s)^\sim$ denotes $G(-s)^T$. A transfer matrix $G(s)$ in terms of state-space data is denoted by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI-A)^{-1}B+D$$

An LFT $\mathscr{F}_l(T,Q)$ with $T=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ is defined by

$$\mathscr{F}_l(T,Q) := T_{11}+T_{12}Q(I-T_{22}Q)^{-1}T_{21}$$

2. Problem Formulation

We consider a linear time invariant plant described by

$$\dot{x}=Ax+B_1w+B_2u \tag{2.1}$$

$$z=C_1x \tag{2.2}$$

$$y=C_2x+Dw \tag{2.3}$$

where x,y and z are the state, the measurement and the output to be estimated, respectively. We have two exogenous signals u and w. The signal u is the known control input, while w is the unknown disturbance that has finite energy, i.e. $w\in\mathscr{L}_2$. We assume that (A,B_1) is stabilizable and that (C_2,A) is detectable. Moreover, to simplify the discussion, we assume that $\begin{bmatrix} B_1 \\ D \end{bmatrix} D^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$. This condition ensures that the process disturbance and measurement noise are independent and that the measurement noise is normalized.

Let $T_{est}(s)$ be an estimator transfer matrix from $\begin{bmatrix} u \\ y \end{bmatrix}$ to the estimate $\hat{z}$, i.e. $\hat{z} = T_{est}\begin{bmatrix} u \\ y \end{bmatrix}$. Then the transfer matrix from w to the estimation error $e := z - \hat{z}$ is given as

$$T_{ew} = T_{zw} - T_{est}\begin{bmatrix} 0 \\ T_{yw} \end{bmatrix} \tag{2.4}$$

where $T_{zw}(s) = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & 0 \end{array}\right]$ and $T_{yw}(s) = \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & D \end{array}\right]$ are the transfer matrices from w to z and y, respectively. Then the $\mathscr{H}_\infty$ estimation problem considered in this paper is stated as follows.

$\mathscr{H}_\infty$ Estimation Problem

For a given constant $\gamma > 0$, we wish to establish a necessary and sufficient condition for the existence of a stable unbiased estimator $T_{est}(s)$ that satisfies the $\mathscr{H}_\infty$ error bound $\|T_{ew}\|_\infty < \gamma$. Moreover, if the solvability condition holds, we derive a class of all stable unbiased estimators that satisfy the $\mathscr{H}_\infty$ norm bound $\|T_{ew}\|_\infty \leq \gamma$.

3. Preliminaries

In this section, we summarize some useful results for the class of all stable unbiased estimators, the model matching problem and the $\mathscr{H}_\infty$ norm of a transfer matrix based on [8]–[10].

3.1 The class of all stable unbiased estimators

Definition 1 An estimator is stable if the estimate is generated by a stable proper linear time invariant system subject to the plant inputs and outputs.

Definition 2 An estimator is unbiased if the estimation error decays to zero for any plant inputs in the absence of modeling errors and disturbances.

Lemma 1 Consider the linear time invariant plant of (1), (2) and (3). Suppose that a stable unbiased state estimator is given by

$$\dot{\hat{x}}_0 = A\hat{x}_0 + L(y - C_2\hat{x}_0) + B_2u \tag{3.1}$$

where $\hat{x}_0$ is an estimate of x, and $A_L := A - LC_2$ is a stability matrix. Then a class of all stable unbiased estimators $T_{est}(s)$ is given by

$$T_{est}(s) = [T_{e1}(s)\ T_{e2}(s)] \tag{3.2}$$

$$T_{e1} = \left[\begin{array}{c|c} A-LC_2 & B_2 \\ \hline C_1 & 0 \end{array}\right] - K\left[\begin{array}{c|c} A-LC_2 & B_2 \\ \hline C_2 & 0 \end{array}\right] \tag{3.3}$$

$$T_{e2} = \left[\begin{array}{c|c} A-LC_2 & L \\ \hline C_1 & 0 \end{array}\right] + K\left[\begin{array}{c|c} A-LC_2 & L \\ \hline -C_2 & I \end{array}\right] \tag{3.4}$$

where $K(s)$ is an arbitrary transfer matrix in $\mathscr{RH}_\infty$.
Proof See Appendix A.

3.2 Model matching problem

The model matching problem of the Nehari type is to approximate a given $R(s) \in \mathscr{RL}_\infty$ by $X(s) \in \mathscr{RH}_\infty$. It is well known as Nehari's theorem [10] that $\inf_{X \in \mathscr{RH}_\infty} \|R - X\|_\infty = \|R\|_H$, where $\|\cdot\|_H$ denotes the Hankel norm.

Lemma 2 For a given $R(s) \in \mathscr{RL}_\infty$ such that $\|R\|_H < 1$, define $G(s) = \begin{bmatrix} I & [R]_- \\ 0 & I \end{bmatrix}$ and $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ Then there exists a J-spectral factor $\Pi(s)$ such that

$$G^\sim JG = \Pi^\sim J \Pi, \quad \Pi,\ \Pi^{-1} \in \mathscr{RH}_\infty$$

Furthermore, a class of all $X(s) \in \mathscr{RH}_\infty$ satisfying $\|R - X\|_\infty \leq 1$ is given by

$$X = R - (L_1 Q + L_2)(L_3 Q + L_4)^{-1}$$

where $\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} = G\Pi^{-1}$ and $Q(s)$ is an arbitrary transfer matrix in $\mathscr{RH}_\infty$ such that $\|Q\|_\infty \leq 1$.

Proof A proof is given in [10].

3.3 $\mathscr{H}_\infty$ norm of a transfer matrix

Lemma 3 For a given $G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ we assume that (A, B) is stabilizable and that (C, A) is detectable. Then $G \in \mathscr{RH}_\infty$ and $\|G\|_\infty < 1$ if and only if there exists a non-negative stabilizing solution M of the following algebraic Riccati equation (ARE).

$$AM + MA^T + MC^TCM + BB^T = 0$$

Proof See Lemma 4 in [8].

Lemma 4 Consider an LFT system $\mathscr{F}_l(T, Q)$, where $T(s)=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ and $Q(s)$ is a proper rational transfer matrix. Suppose $TT^{\sim}=I$ and $T_{12}^{-1} \in \mathscr{RH}_\infty$. Then the following are equivalent.

(i) The system is internally stable and $\|\mathscr{F}_l(T, Q)\|_\infty < 1$.

(ii) $Q(s) \in \mathscr{RH}_\infty$ and $\|Q\|_\infty < 1$.

Proof This lemma is dual to Lemma 15 in [8].

4. Solution to $\mathscr{H}_\infty$ Estimation Problem

The main result of this paper is summarized in the following theorem.

《Theorem》

(i) There exists a stable unbiased estimator $T_{est}(s)$ satisfying $\|T_{ew}\|_\infty < \gamma$ if and only if there exists a non-negative stabilizing solution P_∞ to the following ARE.

$$AP_\infty + P_\infty A^T + P_\infty(\gamma^{-2}C_1^TC_1 - C_2^TC_2)P_\infty + B_1B_1^T = 0 \tag{4.1}$$

(ii) The class of all stable unbiased estimators satisfying $\|T_{ew}\|_\infty \le \gamma$ is given by

$$T_{est}(s) = \mathscr{F}_l(G, U) \tag{4.2}$$

$$G(s) = \left[\begin{array}{c|ccc} A_\infty & [B_2 & P_\infty C_2^T] & -\gamma^{-2}P_\infty C_1^T \\ \hline C_1 & [0 & 0\ ] & I \\ C_2 & [0 & -I\ ] & 0 \end{array}\right] \tag{4.3}$$

where $A_\infty = A - P_\infty C_2^T C_2$, and $U(s)$ is an arbitrary transfer matrix in $\mathscr{RH}_\infty$ such that $\|U\|_\infty \le \gamma$. Furthermore, $\|T_{ew}\|_\infty < \gamma$ holds if and only if $\|U\|_\infty < \gamma$.

In the remainder of this section, we give a proof of Theorem based on Nehari's theorem.

4.1 Reduction to model matching problem

Since T_{zw} and T_{yw} may not be stable in general, the standard model matching approach cannot be directly applied to the $\mathscr{H}_\infty$ estimation problem. Thus, we first introduce a class of all estimators stabilizing T_{ew}. Suppose P_2 is the unique non-negative stabilizing solution to the following ARE.

$$AP_2+P_2A^T-P_2C_2^TC_2P_2+B_1B_1^T=0 \tag{4.4}$$

Then the $\mathscr{H}_2$ optimal estimator, or the steady-state Kalman filter, is given by

$$\dot{\hat{x}}_2=A\hat{x}_2+P_2C_2^T(y-C_2\hat{x}_2)+B_2u$$

$$\hat{z}_2=C_1\hat{x}_2$$

where $\hat{x}_2$ and $\hat{z}_2$ are the estimate of x and z, respectively. Define $A_2:=A-P_2C_2^TC_2$. Then from Lemma 1, a class of all stable unbiased estimators is given by

$$T_{est}(s)=[T_{e1}(s)\ T_{e2}(s)] \tag{4.5}$$

$$T_{e1}(s)=\left[\begin{array}{c|c}A_2 & B_2\\ \hline C_1 & 0\end{array}\right]-K\left[\begin{array}{c|c}A_2 & B_2\\ \hline C_2 & 0\end{array}\right] \tag{4.6}$$

$$T_{e2}(s)=\left[\begin{array}{c|c}A_2 & P_2C_2^T\\ \hline C_1 & 0\end{array}\right]+K\left[\begin{array}{c|c}A_2 & P_2C_2^T\\ \hline -C_2 & I\end{array}\right] \tag{4.7}$$

where $K(s)$ is an arbitrary transfer matrix in $\mathscr{RH}_\infty$. Substituting (4.5) into (2.4) and some algebraic manipulations yield

$$T_{ew}(s)=\left[\begin{array}{c|c}A_2 & B_1-P_2C_2^TD\\ \hline C_1 & 0\end{array}\right]-K(s)\left[\begin{array}{c|c}A_2 & B_1-P_2C_2^TD\\ \hline C_2 & D\end{array}\right] \tag{4.8}$$

Since A_2 is a stability matrix, the $\mathscr{H}_\infty$ estimation problem is reduced to a standard model matching problem of finding $\hat{K}=\gamma^{-1}K\in\mathscr{RH}_\infty$ such that

$$\|\gamma^{-1}T_{ew}\|_\infty=\|T_1-\hat{K}T_2\|_\infty<1 \tag{4.9}$$

where

$$C_\gamma=\gamma^{-1}C_1,\ T_1(s)=\left[\begin{array}{c|c}A_2 & B_1-P_2C_2^TD\\ \hline C_\gamma & 0\end{array}\right],\ T_2(s)=\left[\begin{array}{c|c}A_2 & B_1-P_2C_2^TD\\ \hline C_2 & D\end{array}\right]$$

4.2 Proof of Theorem

(Necessity) Since $T_2T_2^\sim=I$, it is easily seen that $EE^\sim=I$ with $E=[T_2^\sim \quad I-T_2^\sim T_2]$, so that

$$\|T_1-\hat{K}T_2\|_\infty=\|(T_1-\hat{K}T_2)E\|_\infty$$
$$=\|[T_1T_2^\sim-\hat{K} \quad T_1(I-T_2^\sim T_2)]\|_\infty$$

Therefore, if there exists $\hat{K}(s)\in\mathscr{RH}_\infty$ that satisfies (4.9), then $\|Y\|_\infty<1$ holds, where $Y=T_1(I-T_2^\sim T_2)$.

Furthermore, if $\|Y\|_\infty<1$, there exists a co-spectral factor $Y_o(s)$ such that

$$I-YY^\sim=Y_oY_o^\sim,\quad Y_o,\ Y_o^{-1}\in\mathscr{RH}_\infty$$

We see from Lemma 8.2 of [10] that the following inequality also holds.

$$\|Y_o^{-1}T_1T_2^\sim-Y_o^{-1}\hat{K}\|_\infty<1 \tag{4.10}$$

We now derive the state-space representations of the conditions of $\|Y\|_\infty<1$ and (4.10). After some algebraic manipulations, we get

$$T_1T_1^\sim=\left[\begin{array}{c|c}A_2 & P_2C_\gamma^T\\ \hline C_\gamma & 0\end{array}\right]+\left[\begin{array}{c|c}A_2 & P_2C_\gamma^T\\ \hline C_\gamma & 0\end{array}\right]^\sim$$

$$T_1T_2^\sim=\left[\begin{array}{c|c}-A_2^T & -C_2^T\\ \hline C_\gamma P_2 & 0\end{array}\right]$$

$$T_1T_2^\sim T_2T_1^\sim=\left[\begin{array}{c|c}A_2 & P_2C_\gamma^T\\ \hline C_\gamma P_2R & 0\end{array}\right]+\left[\begin{array}{c|c}A_2 & P_2C_\gamma^T\\ \hline C_\gamma P_2R & 0\end{array}\right]^\sim$$

where R is a unique non-negative definite solution to the following Lyapunov equation.

$$A_2^TR+RA_2+C_2^TC_2=0 \tag{4.11}$$

Thus,

$$\begin{aligned} YY^{\sim} &= T_1(I-T_2^{\sim}T_2)(I-T_2^{\sim}T_2)^{\sim}T_1^{\sim} \\ &= T_1(I-T_2^{\sim}T_2)T_1^{\sim} = T_1T_1^{\sim} - T_1T_2^{\sim}T_2T_1^{\sim} \\ &= \left[\begin{array}{c|c} A_2 & P_2C_y^T \\ \hline C_y(I-P_2R) & 0 \end{array}\right] + \left[\begin{array}{c|c} A_2 & P_2C_y^T \\ \hline C_y(I-P_2R) & 0 \end{array}\right]^{\sim} \end{aligned} \tag{4.12}$$

In order to derive a condition for $\|Y\|_\infty < 1$, we introduce Lemmas 5 and 6.

Lemma 5 Let P_2 be a non-negative stabilizing solution to (4.4) and R be a non-negative solution to (4.11). Then $I-P_2R>0$ holds.

Proof See Appendix B.

Lemma 6 $\|Y\|_\infty = \left\| \left[\begin{array}{c|c} A_2 & (I-P_2R)^{-1}B_1 \\ \hline C_y(I-P_2R) & 0 \end{array}\right] \right\|_\infty$

Proof See Appendix C.

Therefore, Lemmas 3 and 6 show that $\|Y\|_\infty < 1$ holds if and only if there exists an $M = M^T \geq 0$ satisfying

$$A_2M + MA_2^T + M(I-RP_2)C_y^TC_y(I-P_2R)M + (I-P_2R)^{-1}B_1B_1^T(I-RP_2)^{-1} = 0 \tag{4.13}$$

with $A_M := A_2 + M(I-RP_2)C_y^TC_y(I-P_2R)$ stable. A state-space realization of the co-spectral factor $Y_o(s)$ is then given by

$$Y_o(s) = \left[\begin{array}{c|c} A_2 & M(I-RP_2)C_y^T \\ \hline C_y(I-P_2R) & I \end{array}\right]$$

It therefore follows that

$$\begin{aligned} Y_o^{-1}T_1T_2^{\sim} &= \left[\begin{array}{c|c} A_M & M(I-RP_2)C_y^T \\ \hline C_y(I-P_2R) & I \end{array}\right] \left[\begin{array}{c|c} -A_2^T & -C_2^T \\ \hline C_yP_2 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_M & M(I-RP_2)C_y^TC_yP_2 & 0 \\ 0 & -A_2^T & -C_2^T \\ \hline C_y(I-P_2R) & C_yP_2 & 0 \end{array}\right] \end{aligned} \tag{4.14}$$

We define $S := M - P_2(I-RP_2)^{-1}$. Then S is a non-negative symmetric solution to the following Lyapunov equation.

$$A_2S + SA_2^T + M(I-RP_2)C_y^TC_y(I-P_2R)M = 0 \tag{4.15}$$

Furthermore, we get

$$A_M S + S A_2^T + M(I - RP_2)C_y^T C_y (I - P_2 R)P_2 = 0 \tag{4.16}$$

Applying the basis change $\begin{bmatrix} I & S \\ 0 & I \end{bmatrix}$ to (4.14) and using (4.16) yield

$$Y_o^{-1} T_1 T_2^{\sim} = \left[\begin{array}{c|c} A_M & SC_2^T \\ \hline C_y(I - P_2 R) & 0 \end{array}\right] + \left[\begin{array}{c|c} -A_2^T & -C_2^T \\ \hline C_y(I - P_2 R)M & 0 \end{array}\right]$$

Since A_M is stable and $-A_2^T$ is antistable, we get

$$[Y_o^{-1} T_1 T_2^{\sim}]_+ = \left[\begin{array}{c|c} A_M & SC_2^T \\ \hline C_y(I - P_2 R) & 0 \end{array}\right]$$

$$[Y_o^{-1} T_1 T_2^{\sim}]_- = \left[\begin{array}{c|c} -A_2^T & -C_2^T \\ \hline C_y(I - P_2 R)M & 0 \end{array}\right]$$

Here, the Hankel norm of $Y_o^{-1} T_1 T_2^{\sim}$ can be computed by $\| Y_o^{-1} T_1 T_2^{\sim} \|_H = \rho(L_o L_c)$ [10], where L_o and L_c are defined by

$$-A_2 L_o - L_o A_2^T = M(I - RP_2)C_y^T C_y (I - P_2 R)M$$

$$-A_2^T L_c - L_c A_2 = C_2^T C_2$$

From (4.11) and (4.15), we get $L_o = S$ and $L_c = R$. Therefore, it follows from Nehari's theorem that there exists $Y_o^{-1}\hat{K}$ that satisfies (4.10) if and only if $\rho(SR) < 1$ holds. Hereafter, we assume that there exists a non-negative stabilizing solution M to the ARE of (4.13) and that $\rho(SR) < 1$ holds.

We define $P_\infty := P_2 + (I - SR)^{-1} S$ [11]. Then P_∞ is also non-negative definite since $P_2 \geq 0$ and $S \geq 0$. It is straightforward to show that P_∞ satisfies

$$M(I - RP_2) = (I - SR)P_\infty$$

$$AP_\infty + P_\infty A^T + P_\infty (C_y^T C_y - C_2^T C_2) P_\infty + B_1 B_1^T = 0$$

$$A + P_\infty (C_y^T C_y - C_2^T C_2) = (I - SR)^{-1} A_M (I - SR)$$

Moreover, since A_M is a stability matrix, it follows from the above equations that $A + P_\infty (C_y^T C_y - C_2^T C_2)$ is stable, i.e. P_∞ is a stabilizing solution to the ARE of (4.1).

(Sufficiency) Suppose that there exists a non-negative definite stabilizing solution

P_∞ to the ARE of (4.1) and $\rho(SR)<1$. Then, it is easily seen that a unimodular matrix Π satisfying

$$\begin{bmatrix} I & [Y_o^{-1}T_1T_2^\sim]_- \\ 0 & I \end{bmatrix}^\sim J \begin{bmatrix} I & [Y_o^{-1}T_1T_2^\sim]_- \\ 0 & I \end{bmatrix} = \Pi^\sim J \Pi$$

is given by

$$\Pi = \left[\begin{array}{c|cc} (I-SR)^{-1}A_2(I-SR) & P_\infty C_y^T & (P_\infty - P_2)C_2^T \\ \hline C_y P_\infty R(I-SR) & I & 0 \\ -C_2 & 0 & I \end{array}\right]$$

It therefore follows from Lemma 2 that $\hat{K}(s)\in\mathcal{RH}_\infty$ satisfying $\|Y_o^{-1}T_1T_2^\sim - Y_o^{-1}\hat{K}\|_\infty \leq 1$ is expressed as

$$\hat{K}(s) = T_1T_2^\sim - Y_o(L_1Q+L_2)(L_3Q+L_4)^{-1}$$

$$L_1 = \left[\begin{array}{c|c} -A_2^T & -RP_\infty C_y^T \\ \hline C_y(I-P_2R)M & I \end{array}\right]$$

$$L_2 = \left[\begin{array}{c|c} -A_2^T & -(I-RS)^{-1}C_2^T \\ \hline C_y(I-P_2R)M & 0 \end{array}\right]$$

$$L_3 = \left[\begin{array}{c|c} A_2 & P_\infty C_y^T \\ \hline C_2 & 0 \end{array}\right]$$

$$L_4 = \left[\begin{array}{c|c} A_2 & (P_\infty - P_2)C_2^T \\ \hline C_2 & I \end{array}\right]$$

where $Q(s)$ is an arbitrary transfer matrix in $\mathcal{RH}_\infty$ such that $\|Q\|_\infty \leq 1$. We define

$$A_\infty := A - P_\infty C_2^T C_2, \quad U(s) = \gamma Q(s)$$

$$\bar{G}(s) := \begin{bmatrix} \gamma(T_1T_2^\sim - Y_oL_2L_4^{-1}) & Y_o(L_1 - L_2L_4^{-1}L_3) \\ -L_4^{-1} & -\gamma^{-1}L_4^{-1}L_3 \end{bmatrix}$$

Then the LFT representation of $K(s)$ is given by

$$K(s) = \gamma\hat{K}(s) = \mathcal{F}_l(\bar{G}, U) \tag{4.17}$$

$$\tilde{G}=\left[\begin{array}{c|cc} A_\infty & (P_\infty-P_2)C_2^T & -\gamma^{-2}P_\infty C_1^T \\ \hline C_1 & 0 & I \\ C_2 & -I & 0 \end{array}\right] \tag{4.18}$$

Thus, from (4.6) and (4.7), we obtain

$$T_{e1}=\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_1 & 0 \end{array}\right]-K\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & 0 \end{array}\right]=\mathscr{F}_l\left(\left[\begin{array}{c|cc} A_\infty & B_2 & -\gamma^{-2}P_\infty C_1^T \\ \hline C_1 & 0 & I \\ C_2 & 0 & 0 \end{array}\right], U\right)$$

$$T_{e2}=\left[\begin{array}{c|c} A_2 & P_2C_2^T \\ \hline C_1 & 0 \end{array}\right]+K\left[\begin{array}{c|c} A_2 & P_2C_2^T \\ \hline -C_2 & I \end{array}\right]=\mathscr{F}_l\left(\left[\begin{array}{c|cc} A_\infty & P_\infty C_2^T & -\gamma^{-2}P_\infty C_1^T \\ \hline C_1 & 0 & I \\ C_2 & -I & 0 \end{array}\right], U\right)$$

Substituting the above equations into (4.5) yields (4.2) and (4.3), as was to be shown.

Moreover, from (4.2) and (4.3), $\gamma^{-1}T_{ew}=T_1-\hat{K}T_2$ is expressed as

$$\gamma^{-1}T_{ew}=T_1-\hat{K}T_2=\mathscr{F}_l(T,Q)$$

$$T(s)=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}=\left[\begin{array}{c|cc} A_\infty & B_1-P_\infty C_2^T D & -P_\infty C_\gamma^T \\ \hline C_\gamma & 0 & I \\ C_2 & D & 0 \end{array}\right]$$

where $Q(s)=\gamma U(s)$ and $\|Q\|_\infty\leq 1$. It is straightforward to verify that $TT^\sim=I$ and $T_{12}^{-1}=\left[\begin{array}{c|c} A+P_\infty(C_\gamma^T C_\gamma-C_2^T C_2) & P_\infty C_\gamma^T \\ \hline C_\gamma & I \end{array}\right]$. Since P_∞ is a stabilizing solution to the ARE of (4.1), $T_{12}^{-1}\in\mathscr{RH}_\infty$. Therefore, it follows from Lemma 4 that $\|T_1-\hat{K}T_2\|_\infty<1$ iff $\|Q\|_\infty<1$, namely $\|U\|_\infty<\gamma$. It may be noted that the parametrization of (4.2) and (4.3) does not contain S and R. Thus, a sufficient condition for the existence of $K(s)$ satisfying $\|T_{ew}\|_\infty<\gamma$ is that there exists a non-negative definite solution to the ARE of (4.1). □

Remark The transfer matrix $K(s)$ of (4.8) is very important in that it characterizes the relationship between $\mathscr{H}_\infty$ estimators and $\mathscr{H}_2$ optimal estimator, modifying the estimation error $z-\hat{z}_2$ using the innovation $y-C_2\hat{x}_2$ so that $\|T_{ew}\|_\infty<\gamma$. Comparing (2.4) with (4.8), it is easily seen that $K(s)$ must have the same estimator structure as $T_{est}(s)$ except for the control input term. In fact, we see from (4.17) and (4.18) that $K(s)$ has the same form as $T_{est}(s)$ with the filter gain $(P_\infty - P_2)C_2^T$. Furthermore, P_∞ tends to P_2 and $U(s)$ becomes arbitrary in $\mathscr{RH}_\infty$ as γ tends to infinity. Therefore, if γ tends to infinity, $K(s)$ tends to $-U(s)$ and $T_{est}(s)$ becomes the $\mathscr{H}_2$ optimal estimator with $U(s)=0$.

5. Conclusion

In this paper, we have derived a necessary and suffcient condition for the existence of a solution to the $\mathscr{H}_\infty$ estimation problem in terms of the ARE of (4.1) and developed a class of all solutions based on the Nehari's theorem. The property of the transfer matrix $K(s)$ that characterizes the relationship between $\mathscr{H}_\infty$ estimator and $\mathscr{H}_2$ optimal estimator has been examined.

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Appendix A: Proof of Lemma 1

We give a proof of Lemma 1 based on the technique of [9]. We first define $\hat{z}_0 = C_1\hat{x}_0$ and $v_0 = y - C_2\hat{x}_0$. Then, it follows from (3.1) that

$$\hat{z}_0 = \left[\begin{array}{c|c} A-LC_2 & L \\ \hline C_1 & 0 \end{array}\right] y + \left[\begin{array}{c|c} A-LC_2 & B_2 \\ \hline C_2 & 0 \end{array}\right] u \tag{A.1}$$

$$v_0 = \left[\begin{array}{c|c} A-LC_2 & L \\ \hline -C_2 & I \end{array}\right] y - \left[\begin{array}{c|c} A-LC_2 & B_2 \\ \hline C_2 & 0 \end{array}\right] u \tag{A.2}$$

(Sufficiency) From (3.2)–(3.4), (A.1) and (A.2), we get

$$\hat{z} = T_{est} \begin{bmatrix} y \\ u \end{bmatrix} = \hat{z}_0 + K v_0$$

Here, we assume that there exist no modeling errors and disturbance. Then, since $\hat{x}_0$ is an unbiased estimate of x, v_0 and $z-\hat{z}_0$ tend to zero as time t tends to infinity. Therefore, $z-\hat{z}$ also tends to zero, i.e. $\hat{z}$ is unbiased.

The stability of $T_{est}(s)$ is immediate from the stabilities of $K(s)$ and $A-LC_2$.

(Necessity) We assume that stable estimates $\hat{z}$ and $\hat{z}_0$ are given by

$$\hat{z} = \Gamma u + \Lambda y \tag{A.3}$$

$$\hat{z}_0 = \Gamma_0 u + \Lambda_0 y \tag{A.4}$$

where $\Gamma(s)$ and $\Lambda(s)$ are $\mathscr{RH}_\infty$ matrices and

$$\Gamma_0(s) = \left[\begin{array}{c|c} A-LC_2 & B_2 \\ \hline C_1 & 0 \end{array}\right], \quad \Lambda_0(s) = \left[\begin{array}{c|c} A-LC_2 & L \\ \hline C_1 & 0 \end{array}\right] \tag{A.5}$$

It then follows that

$$\begin{aligned} \hat{z} - \hat{z}_0 &= (\Gamma - \Gamma_0)u + (\Lambda - \Lambda_0)y \\ &= \{(\Gamma - \Gamma_0) + (\Lambda - \Lambda_0)\tilde{N}\tilde{M}^{-1}\} u \end{aligned} \tag{A.6}$$

where $\tilde{N}\tilde{M}^{-1}$ is a right coprime factorization of $T_{yu}(s) = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right]$ Without modeling errors and disturbance, the left hand side of (A.6) is zero due to the unbiasedness of $\hat{z}$ and $\hat{z}_0$. Thus, we get

$$(\Gamma - \Gamma_0)\tilde{M} + (\Lambda - \Lambda_0)\tilde{N} = 0 \tag{A.7}$$

Since $\tilde{N}(s)$ and $\tilde{M}(s)$ are coprime in $\mathscr{RH}_\infty$, $\Gamma(s)$ and $\Lambda(s)$ satisfying (A.7) are expressed as

$$\Gamma = \Gamma_0 + KT_{vou} \tag{A.8}$$

$$\Lambda = \Lambda_0 + KT_{voy} \tag{A.9}$$

where $K(s)$ is an arbitrary transfer matrix in $\mathscr{RH}_\infty$ and

$$T_{vou}(s) = -\left[\begin{array}{c|c} A-LC_2 & B_2 \\ \hline C_2 & 0 \end{array}\right], \quad T_{voy}(s) = \left[\begin{array}{c|c} A-LC_2 & L \\ \hline -C_2 & I \end{array}\right] \tag{A.10}$$

It should be noted that $T_{yu} = T_{voy}^{-1} T_{vou}$ is a left coprime factorization. Substituting (A.8) and (A.9) into (A.3) yields

$$\begin{aligned} \hat{z} &= \Gamma u + \Lambda y \\ &= (\Gamma_0 + KT_{vou})u + (\Lambda_0 + KT_{voy})y \end{aligned} \tag{A.11}$$

Thus

$$T_{est} = [\Lambda_0 + KT_{voy} \quad \Gamma_0 + KT_{vou}] \tag{A.12}$$

Substituting (A.5) and (A.10) into (A.12) yields (3.2)–(3.4). □

Appendix B: Proof of Lemma 5

Without loss of generality, we assume that the matrices A, B_1 and C_2 are of the forms

$$A = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \ B_1 = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \ C_2 = [\bar{C} \ 0]$$

where $(\bar{A}_{11}, \bar{B}_1)$ is stabilizable, $(\bar{C}, \bar{A}_{11})$ is observable and $\bar{A}_{22}$ is a stability matrix. Let $P_2 = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_2^T & \bar{P}_3 \end{bmatrix}$ be a non-negative stabilizing solution to the ARE of (4.4). Then we get

$$\bar{A}_{11}\bar{P}_1 + \bar{P}_1\bar{A}_{11}^T - \bar{P}_1\bar{C}^T\bar{C}\bar{P}_1 + \bar{B}_1\bar{B}_1^T = 0 \tag{B.1}$$

$$(\bar{A}_{11} - \bar{P}_1\bar{C}^T\bar{C})\bar{P}_2 + \bar{P}_2\bar{A}_{22}^T + \bar{B}_1\bar{B}_2^T = 0 \tag{B.2}$$

$$\bar{A}_{22}\bar{P}_3+\bar{P}_3\bar{A}_{22}^T+\bar{A}_{21}\bar{P}_2+\bar{P}_2^T\bar{A}_{21}^T+\bar{B}_2\bar{B}_2^T=0 \tag{B.3}$$

Since P_2 is a non–negative stabilizing solution to (4.4), $\bar{P}_1$ is also a non-negative stabilizing solution to (B.1), i.e. $\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}$ is stable. Moreover, we see from (B.2) and (B.3) that $\bar{P}_2$ and $\bar{P}_3$ are uniquely determined by $\bar{P}_1$.

Similarly, let $R=\begin{bmatrix}\bar{R}_1 & \bar{R}_2\\ \bar{R}_2^T & \bar{R}_3\end{bmatrix}$ be a non-negative solution to the Lyapunov equation of (4.11). Then we get

$$(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C})^T\bar{R}_1+\bar{R}_1(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C})$$
$$+(\bar{A}_{22}-\bar{P}_2^T\bar{C}^T\bar{C})^T\bar{R}_2^T+\bar{R}_2(\bar{A}_{22}-\bar{P}_2^T\bar{C}^T\bar{C})+\bar{C}^T\bar{C}=0 \tag{B.4}$$

$$(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C})\bar{R}_2+\bar{R}_2\bar{A}_{22}^T+(\bar{A}_{21}-\bar{P}_2^T\bar{C}^T\bar{C})^T\bar{R}_3=0 \tag{B.5}$$

$$\bar{A}_{22}^T\bar{R}_3+\bar{R}_3\bar{A}_{22}=0 \tag{B.6}$$

We see from (B.6) that $\bar{R}_3=0$ since $\bar{A}_{22}$ is stable. Since $\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}$ and $\bar{A}_{22}$ are stable, substituting $\bar{R}_3=0$ into (B.5) yields $\bar{R}_2=0$. Thus, (B.4) is reduced to the Lyapunov equation

$$(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C})^T\bar{R}_1+\bar{R}_1(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C})+\bar{C}^T\bar{C}=0 \tag{B.7}$$

It follows from the observability of $(\bar{C},\bar{A}_{11})$ that $\bar{R}_1$ is a unique positive definite solution to (B.7). Therefore, we obtain

$$I-P_2R=\begin{bmatrix}I-\bar{P}_1\bar{R}_1 & 0\\ -\bar{P}_2^T\bar{R}_1 & I\end{bmatrix}$$

It remains to show $I-\bar{P}_1\bar{R}_1>0$. Since $\bar{R}_1>0$ holds, we see from (B.7) that

$$(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C})\bar{R}_1^{-1}+\bar{R}_1^{-1}(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C})^T$$
$$-\bar{R}_1^{-1}\bar{C}^T\bar{C}\bar{R}_1^{-1}=0 \tag{B.8}$$

Since $(\bar{C},\bar{A}_{11})$ is observable, $(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C},\ \bar{R}_1^{-1}\bar{C}^T)$ is stabilizable. Thus, it follows from $\bar{R}_1^{-1}>0$ that $\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C}$ is antistable. Moreover, from (B.1), we get

$$(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C})\bar{P}_1+\bar{P}_1(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C})^T$$

$$-\bar{R}_1^{-1}\bar{C}^T\bar{C}\bar{R}_1^{-1}+(\bar{P}_1-\bar{R}_1^{-1})\bar{C}^T\bar{C}(\bar{P}_1-\bar{R}_1^{-1})+\bar{B}_1\bar{B}_1^T=0 \quad \text{(B.9)}$$

Subtracting (B.8) from (B.9) yields

$$(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C})(\bar{P}_1-\bar{R}_1^{-1})+(\bar{P}_1-\bar{R}_1^{-1})(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C})^T$$
$$+(\bar{P}_1-\bar{R}_1^{-1})\bar{C}^T\bar{C}(\bar{P}_1-\bar{R}_1^{-1})+\bar{B}_1\bar{B}_1^T=0 \quad \text{(B.10)}$$

Since $\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C}$ is antistable and $(\bar{A}_{11}-\bar{P}_1\bar{C}^T\bar{C}+\bar{R}_1^{-1}\bar{C}^T\bar{C},\ [(\bar{P}_1-\bar{R}_1^{-1})\bar{C}^T\ \bar{B}_1])$ is controllable, $\bar{P}_1-\bar{R}_1^{-1}<0$. This implies $I-\bar{P}_1\bar{R}_1>0$. □

Appendix C: Proof of Lemma 6

From Lemma 5, there exists a matrix $V=P_2(I-RP_2)^{-1}$. It follows from (4.4) and (4.11) that V is a non-negative definite solution to the Lyapunov equation.

$$A_2V+VA_2^T+(I-P_2R)^{-1}B_1B_1^T(I-RP_2)^{-1}=0 \quad \text{(C.1)}$$

We see from (4.12) that

$$YY^{\sim}=\left[\begin{array}{c|c} A_2 & P_2C_Y^T \\ \hline C_Y(I-P_2R) & 0\end{array}\right]+\left[\begin{array}{c|c} A_2 & P_2C_Y^T \\ \hline C_Y(I-P_2R) & 0\end{array}\right]^{\sim}$$

$$=\left[\begin{array}{cc|c} A_2 & 0 & P_2C_Y^T \\ 0 & -A_2^T & -(I-RP_2)C_Y^T \\ \hline C_Y(I-P_2R) & C_YP_2 & 0\end{array}\right] \quad \text{(C.2)}$$

Applying the basis change $\begin{bmatrix} I & V \\ 0 & I\end{bmatrix}$ to (C.2) yields

$$YY^{\sim}=\left[\begin{array}{cc|c} A_2 & -(I-P_2R)^{-1}B_1B_1^T(I-RP_2)^{-1} & 0 \\ 0 & -A_2^T & (I-RP_2)C_Y^T \\ \hline C_Y(I-P_2R) & 0 & 0\end{array}\right]$$

$$=\left[\begin{array}{c|c} A_2 & (I-P_2R)^{-1}B_1 \\ \hline C_Y(I-P_2R) & 0\end{array}\right]\left[\begin{array}{c|c} A_2 & (I-P_2R)^{-1}B_1 \\ \hline C_Y(I-P_2R) & 0\end{array}\right]^{\sim}$$

This completes a proof. □