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The best constant of discrete Sobolev inequalities corresponding to braced grids – deformability and rigidity

By

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Abstract

Discrete Sobolev inequalities corresponding to planar graphs of braced grids are obtained. The best constants of the inequalities are found explicitly, together with their application to the deformability and rigidity of grids.

§ 1. Introduction

The best constants of Sobolev inequalities

$$\|u\|_{L^p(\mathbb{R})} \leqq C \|\nabla u\|_{L^q(\mathbb{R})}, \quad u \in W^{1,p}(\mathbb{R}) \tag{1.1}$$

have been found expilicitly in the cases where p and q take certain values [1, 2]. In our studies, we have investigated a discrete version of Sobolev inequalities on graphs such as n-sided polygons [3], Platon's regular polyhedra [4], truncated polyhedra [5] and C60 Fullerene [6, 7].

Sobolev inequality and its discrete version have many applications in the field of engineering as well as partial differential equation theory. It is shown, for example in [6, 7], that as the best constant is smaller, the corresponding graph is more rigid. In this paper, we derive a discrete Sobolev inequality and find its best constant on grids with braces. The obtained results are expected to have an application to civil and architectual engineering.

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In Figure 1, two 4×3 grids, in which six braces are added to as many squares among twelve, are presented. We assume that a brace does not bend or stretch. The left grid is not deformable, whereas the central one is deformable, that is, it can be deformed into the right form by applying a suitable force.

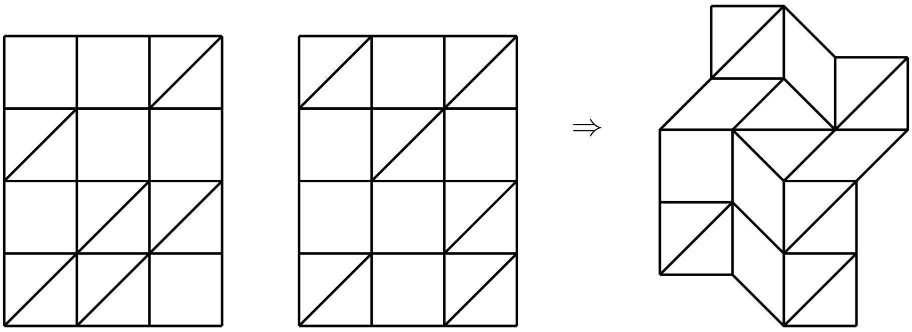


Figure 1. Two 4×3 grids with 6 diagonal braces.

Concerning the deformability of a given braced grid, there is a pioneering work [8, 9] which gives a graph-theoretical framework. In detail, we consider the so-called brace graph, which is a bipartite with vertices $\{r_0, \cdots, r_{m-1}\}$ for the rows and vertices $\{c_0, \cdots, c_{n-1}\}$ for the columns and an edge between r_i and c_j if there is a brace in square (i, j). Then a given braced grid is deformable if and only if the corresponding bipartite is not connected. In Figure 2, for example, the left brace graph is connected, whereas the right brace graph is divided into two connect components $\{r_1, c_1\}$ and $\{r_0, r_2, r_3, c_0, c_2\}$.

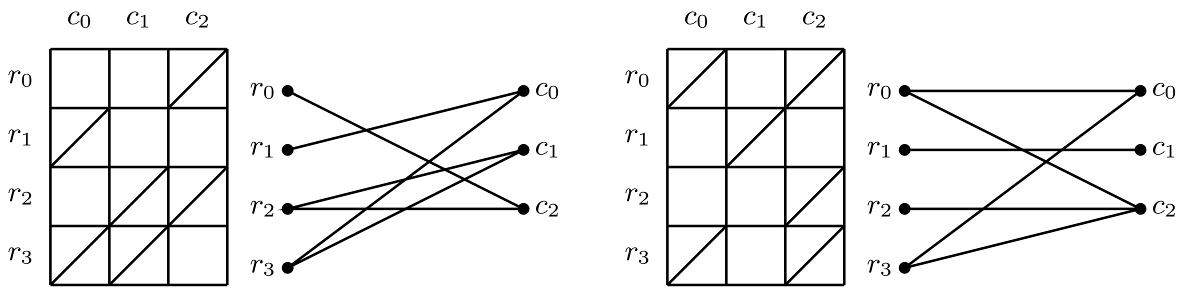
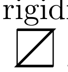


Figure 2. Braced grids and connectivity of their corresponding brace graphs.

The purpose of this paper is to derive discrete Sobolev inequalities corresponding to the braced grids and find their best constants. In §2, we introduce a discrete Laplacian matrix and find the best constant of discrete Sobolev inequality. While investigation of the brace graph gives us information of deformability, the discrete Sobolev inequality gives us that of rigidity. In §3, we focus our attention on the orientation of brace, N-type  and Z-type  .

§ 2. Discrete Laplacian matrix and the best constant of discrete Sobolev inequality on braced grids

We start with the planar graph expressions of the braced grids stated in the previous section.

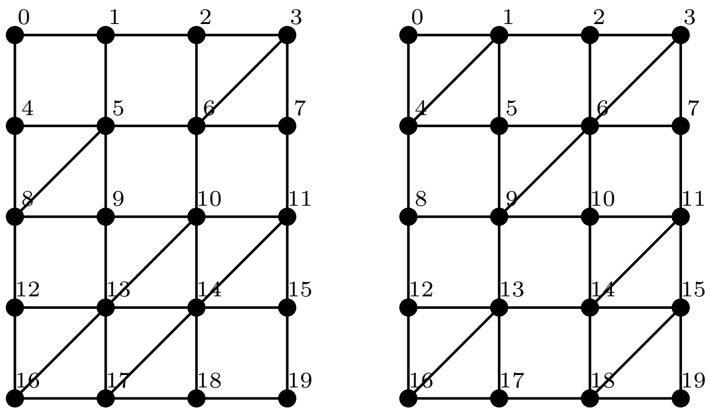


Figure 3. Planar graph expressions of braced grids.

Let m be a number of vertices and the set e be a set of (i, j) where vertices i and j are connected with an edge. For example we have $m = 20$ in the above grids. Let the variables $u(i)$ $(0 \leqq i \leqq m-1)$ be displacements of the i-th vertex from its equilibrium state and introduce a vector $\boldsymbol{u} = {}^t(u(0), \cdots, u(m-1))$. If the force $\boldsymbol{f} = {}^t(f(0), \cdots, f(m-1))$ is acted on the grid, $\boldsymbol{u}$ satisfies the equation

$$A\boldsymbol{u} = \boldsymbol{f} \tag{2.1}$$

where A is a discrete Laplacian matrix defined by

$$A_{ij} = \begin{cases} \text{the number of edges from } i\text{-th vertex} & (i = j) \\ -1 & (i,j),(j,i) \in e \\ 0 & \text{otherwise} \end{cases}$$

We note that A is singular and has an eigenvalue $\lambda_0 = 0$ with an eigenvector $\mathbf{1} = {}^t(1, \cdots, 1)$. However, if we impose the solvability condition

$${}^t\mathbf{1}\boldsymbol{f} = f(0) + \cdots + f(m-1) = 0$$

and orthogonality condition

$${}^t\mathbf{1}\boldsymbol{u} = u(0) + \cdots + u(m-1) = 0,$$

the equation (2.1) possesses a unique solution

$$\boldsymbol{u} = G_* \boldsymbol{f}.$$

The matrix G_*, which we call the Green matrix hereafter, is a Moore-Penrose generalized inverse matrix of A defined by

$$AG_* = G_*A = I - E_0, \quad G_*E_0 = E_0G_* = O,$$

where I and O are $m \times m$ identity and zero matrices, respectively and $E_0 = \frac{1}{m}\mathbf{1}\,{}^t\mathbf{1}$ is a projection matrix.

We introduce a subspace $\mathbb{R}_0^m \subset \mathbb{R}^m$ defined by

$$\mathbb{R}_0^m := \{\boldsymbol{u} \in \mathbb{R}^m \mid {}^t\mathbf{1}\boldsymbol{u} = u(0) + \cdots + u(m-1) = 0\},$$

and a sesquilinear form $(\cdot,\cdot)_A$ defined by

$$(\boldsymbol{u},\boldsymbol{v})_A = (A\boldsymbol{u},\boldsymbol{v}) = {}^t\boldsymbol{v}A\boldsymbol{u}, \quad \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}_0^m.$$

We have the following two theorems.

Theorem 2.1. For any $\boldsymbol{u} \in \mathbb{R}_0^m$, we have the following reproducing relation,

$$u(j) = (\boldsymbol{u}, G_*\boldsymbol{\delta}_j)_A, \quad \boldsymbol{\delta}_j = {}^t(\cdots,\delta_{ij},\cdots) \quad (0 \leqq j \leqq m-1). \tag{2.2}$$

Theorem 2.2. There exists a positive constant C such that for any $\boldsymbol{u} \in \mathbb{R}_0^m$, the following discrete Sobolev inequality,

$$\left(\max_{0\leqq j\leqq m-1} |u(j)|\right)^2 \leqq C \sum_{(i,j)\in e} |u(i)-u(j)|^2 = C\,{}^t\boldsymbol{u}A\boldsymbol{u} \tag{2.3}$$

holds. Among such C, the best (least) constant C_0 is equal to the maximum of diagonal elements of a Green matrix G_*,

$$C_0 = \max_{0\leqq j\leqq m-1} (G_*)_{jj}. \tag{2.4}$$

If the maximun of $(G_*)_{jj}$ is attained at $j = j_0$, the best vector $\boldsymbol{u}_0$, which attains equality in the inequality (2.3) with $C = C_0$, is given by

$$\boldsymbol{u}_0 = k\,G\boldsymbol{\delta}_{j_0}, \qquad k \in \mathbb{C},$$

that is, $\boldsymbol{u}_0$ is parallel to the j_0-th column vector of G_*.

The proofs of the above two theorems are essentially the same as [6, 7], so we omit them.

The meaning of the discrete Sobolev inequality (2.3) is that the maximum of the displacement $u(i)$ is estimated from above by constant multiple of the energy norm. Hence, if the best constant C_0 is smaller, the grid is more rigid.

Concerning the left planar graph in Figure 3, the best constant C_0 is calculated as

$$C_0 = (G_*)_{00} = \frac{1943856757}{2474370560} = 0.785596.$$

As for the right one, we have

$$C_0 = (G_*)_{00} = \frac{2654718621}{3356864200} = 0.790833.$$

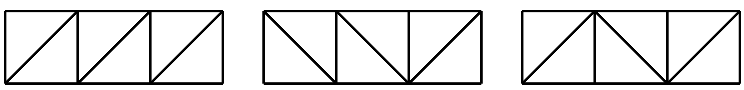
The best constant on the left undeformable grid is smaller than that on the right deformable one, which reflects the fact that the left grid is more rigid than the right one.

§ 3. Rigidity of grids with different oriented braces

As was shown in [8, 9], deformability of a given braced grid is determined by investigating the connectivity of corresponding bipartite. In this section, we change the orientation of brace. That is, we consider two types of braces as  and , which we call N-type brace and Z-type brace, hereafter.

Since changing the orientation of a brace does not influence the connectivity of its brace graph, the deformability itself does not change. However, by changing the orientation we have a different discrete Laplacian matrix A. Therefore, the Green matrix G_* and the best constant $C_0 = \max_{0 \leqq j \leqq m-1} (G_*)_{jj}$ is considered to change. By investigating the best constants of discrete Sobolev inequalities on the same braced grids with different orientation of braces, we can estimate their rigidity.

We start with the simplest $1 \times n$ braced grids ($n = 3, 4, 5$). It is easily seen that a brace should be added to every square for the purpose of keeping the grid undeformable. Introducing a discrete Laplacian matrix A and investigating a Green matrix G_*, we have the following list of best constants.



0.591097 0.579161 0.539373

Figure 4. The best constant of 1×3 braced grids.

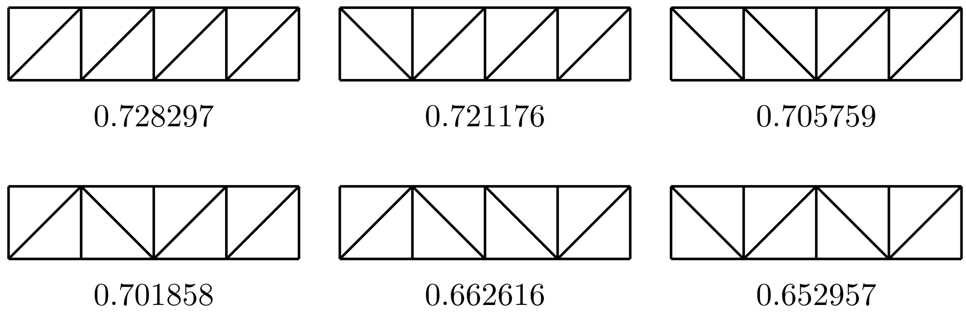


Figure 5. The best constants of 1×4 braced grids.

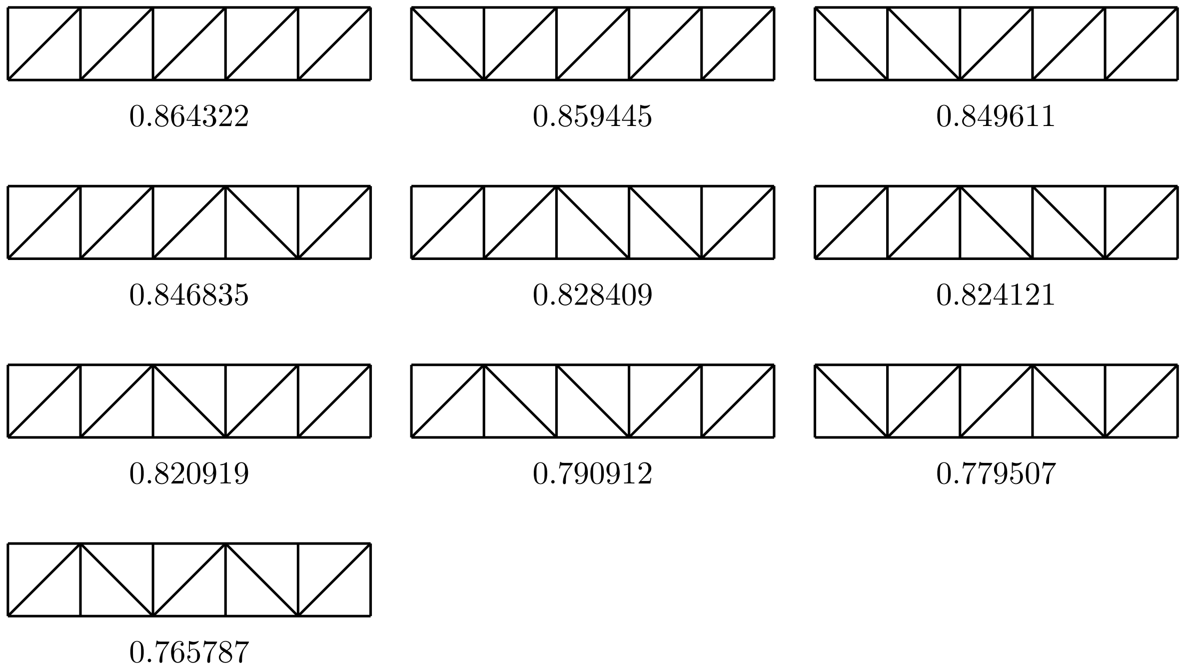


Figure 6. The best constants of 1×5 braced grids.

From the observation of Figures 4–6, we can expect that the following conjecture holds.

Conjecture 3.1. Among the $1 \times n$ braced grids with a N-type or Z-type brace in each square,

1. a grid in which all braces are arranged in the same N-type or Z-types possesses the largest best constant, which means the grid is the least rigid,
2. a grid in which all braces alternate between N-type and Z-type possesses the smallest best constant, which means the grid is the most rigid.

Figure 7 shows the best constant on 2×3 braced grid [10], from which we can find the most and the least rigid types of brace orientations.

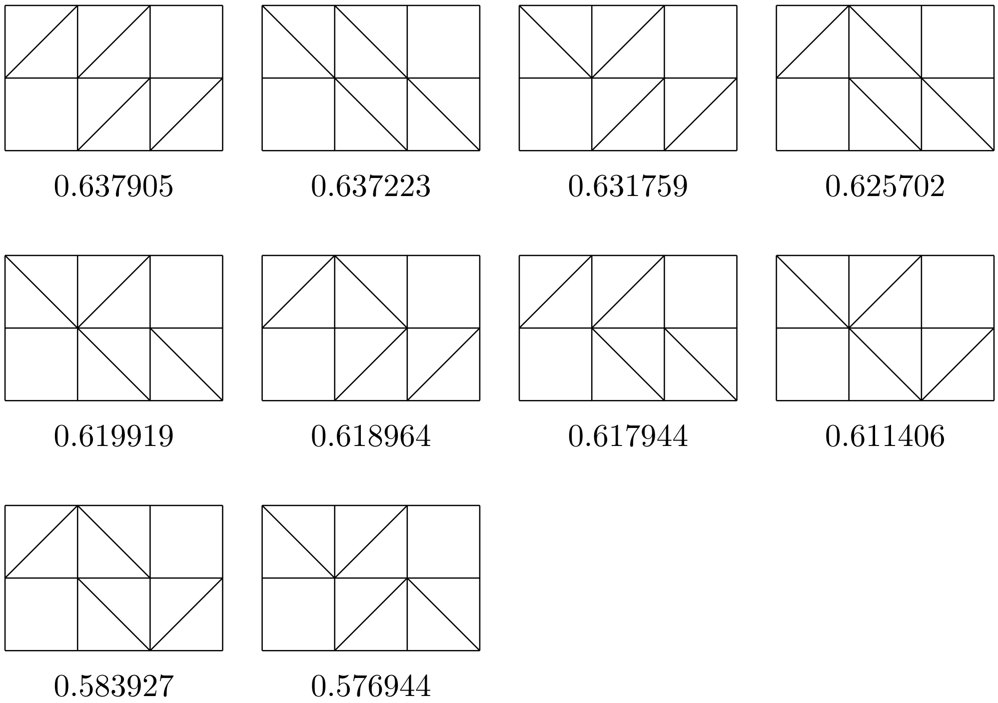


Figure 7. The best constants of 2×3 braced grids.

§ 4. Concluding Remarks

We derived the discrete Sobolev inequalities and their best constants of planar graphs representing the braced grids. Although an establishment of a general theory concerning the best constants on braced grids remains to be an open problem, by comparing the best constants of braced grids with N-type or Z-type brace in each square, we can estimate the rigidity of grids quantitively.

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