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A q-analogue of multiple zeta values and its application to number theory

By

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§ 1. Introduction

We call a tuple of positive integers an *index*. We define the *weight* and the *depth* of an index $\boldsymbol{k} = (k_1, \ldots, k_r)$ by $\mathrm{wt}(\boldsymbol{k}) = k_1 + \cdots + k_r$ and $\mathrm{dep}(\boldsymbol{k}) = r$, respectively. An index $(k_1, \ldots, k_r)$ is said to be *admissible* if $k_1 \geq 2$.

For an admissible index $\boldsymbol{k} = (k_1, \ldots, k_r)$, the *multiple zeta value (MZV)* is a positive real number defined by

$$\zeta(\boldsymbol{k}) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

Note that, the MZV of depth one is a special value of the Riemann zeta function $\zeta(s) = \sum_{m>0} m^{-s}$.

An important problem is to study a structure of the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ spanned by MZVs. Although there are many interesting conjectures and important results, we mention only one conjecture here. We denote by $I_0(k)$ the set of admissible indices of weight k and consider the $\mathbb{Q}$-vector spaces

$$\mathcal{Z}_k = \sum_{\boldsymbol{k} \in I_0(k)} \mathbb{Q}\,\zeta(\boldsymbol{k}), \qquad \mathcal{Z} = \sum_{k \geq 0} \mathcal{Z}_k. \tag{1.1}$$

Here we regard the empty index $\emptyset$ as an admissible one of weight zero and set $\zeta(\emptyset) = 1$. Then $\mathcal{Z}_0 = \mathbb{Q}$, $\mathcal{Z}_1 = 0$, $\mathcal{Z}_2 = \mathbb{Q}\zeta(2)$, $\mathcal{Z}_3 = \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(2,1)$, etc.. The following conjecture about the dimension of $\mathcal{Z}_k$ is widely known:

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Dimension conjecture. (Zagier [30]) *Define the sequence* $\{d_k\}_{k\ge 0}$ *by*

$$d_0 = 1, \quad d_1 = 0, \quad d_2 = 1, \quad d_k = d_{k-2} + d_{k-3}\,(k \ge 3). \tag{1.2}$$

Then it holds that

$$\dim_{\mathbb{Q}} \mathcal{Z}_k \stackrel{?}{=} d_k \tag{1.3}$$

for all $k \ge 0$*.*

It is also believed that

$$\mathcal{Z} \stackrel{?}{=} \bigoplus_{k\ge 0} \mathcal{Z}_k, \tag{1.4}$$

that is, there are no linear relations among MZVs with different weights.

According to the Dimension conjecture, the dimension of $\mathcal{Z}_3$ should be one, and it is correct since $\zeta(3)$ is irrational and, as we will see in Section 2, it holds that $\zeta(3) = \zeta(2,1)$. In general it is proved that the sequence $\{d_k\}$ gives an upper bound of the dimension:

Theorem 1.1 (Goncharov [7], Terasoma [26], Deligne-Goncharov [5])**.** *It holds that* $\dim_{\mathbb{Q}} \mathcal{Z}_k \le d_k$ *for all* $k \ge 0$*.*

In this review article, we introduce a q-analogue of MZVs and discuss its properties and application to number theory. Roughly speaking, 'q-analogue' of a mathematical object is a deformation with one parameter, denoted by q, which recovers the original object in the limit as $q \to 1$. Recently various q-analogue models of MZVs have been studied. Among them we deal with the one called the Bradley-Zhao model in this paper (see Definition 2.2). In Section 2, we see that some relations among MZVs also hold for the q-analogue model (up to some correction terms). In Section 3, we explain a relation of a truncated version of the q-analogue model with q being a root of unity to a conjecture in number theory due to Kaneko and Zagier [14].

§ 2. A q-analogue of multiple zeta value

§ 2.1. Definition

Suppose that $0 < q < 1$. For an integer n the *q-integer* $[n]$ is defined by

$$[n] = \frac{1-q^n}{1-q}.$$

Note that $[n] \to n$ in the limit $q \uparrow 1$. Then a naive q-analogue of MZV would be the multiple sum $\sum_{m_1 > \cdots > m_r > 0} [m_1]^{-k_1} \cdots [m_r]^{-k_r}$. However, it never converges because

$\sup_{m>0}[m] = (1-q)^{-1} < +\infty$. Hence we need to put some decreasing term. There are many choices of such term and various q-analogue models have been proposed. In this article we consider the model which originates from the following discovery due to Kaneko, Kurokawa and Wakayama:

Theorem 2.1 (Kaneko-Kurokawa-Wakayama [13]). *Set*

$$f_q(s,t) = \sum_{m>0} \frac{q^{mt}}{[m]^s} = \frac{q^t}{[1]^s} + \frac{q^{2t}}{[2]^s} + \frac{q^{3t}}{[3]^s} + \cdots .$$

Then, for any $s \in \mathbb{C} \setminus \{1\}$, we have

$$\lim_{q \uparrow 1} f_q(s, s-1) = \zeta(s).$$

The point is that the power t of q^m in the numerator should be shifted by one from the power s of $[m]$ in the denominator so that the limit of $f_q(s,t)$ recovers the Riemann zeta function globally on $\mathbb{C} \setminus \{1\}$. Motivated by this result we introduce the following q-analogue of MZV, which is called the Bradley-Zhao model:

Definition 2.2 (Bradley [3], Zhao [32]). For an admissible index $\boldsymbol{k} = (k_1, \ldots, k_r)$, we define

$$\zeta_q(\boldsymbol{k}) = \sum_{m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1 + \cdots + (k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}. \tag{2.1}$$

Hereafter we call $\zeta_q(\boldsymbol{k})$ a q-analogue of MZV (qMZV for short). In [32] Zhao proved that $\zeta_q(\boldsymbol{k})$ converges to $\zeta(\boldsymbol{k})$ in the limit $q \uparrow 1$. As mentioned above, there are other q-analogue models of MZV. See Zhao's book [33] for properties and relations among them.

§ 2.2. First non-trivial relation

A good point of the Bradley-Zhao model (2.1) is that it satisfies many of the same relations as MZVs. The simplest example is the following relation due to Euler:

Proposition 2.3. *It holds that $\zeta(3) = \zeta(2,1)$.*

Proof. We calculate

$$I = \sum_{m \geq n > 0} \frac{1}{m^2 n}$$

in two ways. First we have

$$I = \left(\sum_{m>n>0} + \sum_{m=n>0} \right) \frac{1}{m^2 n} = \zeta(2,1) + \zeta(3).$$

Second we rewrite

$$I=\sum_{m>0}\frac{1}{m^2}\sum_{n=1}^{m}\frac{1}{n}=\sum_{m>0}\frac{1}{m^2}\sum_{n>0}\left(\frac{1}{n}-\frac{1}{m+n}\right)$$

and use the identity

$$\frac{1}{m^2}\left(\frac{1}{n}-\frac{1}{m+n}\right)=\frac{1}{(m+n)^2n}+\frac{1}{(m+n)^2m}.$$

Thus we obtain

$$I=\sum_{m>0}\sum_{n>0}\left(\frac{1}{(m+n)^2n}+\frac{1}{(m+n)^2m}\right)=\sum_{l>n>0}\frac{1}{l^2n}+\sum_{l>m>0}\frac{1}{l^2m}=2\,\zeta(2,1).$$

Therefore, $I=\zeta(2,1)+\zeta(3)=2\zeta(2,1)$, which implies the desired relation. □

Proposition 2.3 holds also for the qMZV in the same form:

Proposition 2.4. *It holds that* $\zeta_q(3)=\zeta_q(2,1)$.

Proof. In the q-analogue case, we start from

$$I=\sum_{m\ge n>0}\frac{q^m}{[m]^2}\frac{q^n}{[n]}$$

We have

$$I=\left(\sum_{m>n>0}+\sum_{m=n>0}\right)\frac{q^m}{[m]^2}\frac{q^n}{[n]}=\sum_{m>n>0}\frac{q^m}{[m]^2}\frac{q^n}{[n]}+\zeta_q(3). \tag{2.2}$$

Note that the first term of the right-hand side is not the qMZV $\zeta_q(2,1)$. However it will be canceled as follows. We rewrite I in the other way

$$I=\sum_{m>0}\frac{q^m}{[m]^2}\sum_{n=1}^{m}\frac{q^n}{[n]}=\sum_{m>0}\frac{q^m}{[m]^2}\sum_{n>0}\left(\frac{q^n}{[n]}-\frac{q^{m+n}}{[m+n]}\right)$$

and use the identity

$$\frac{q^m}{[m]^2}\left(\frac{q^n}{[n]}-\frac{q^{m+n}}{[m+n]}\right)=\frac{q^{m+n}}{[m+n]^2}\frac{1}{[n]}+\frac{q^{m+n}}{[m+n]^2}\frac{q^m}{[m]}.$$

Then we obtain

$$\begin{aligned}I&=\sum_{m,n>0}\left(\frac{q^{m+n}}{[m+n]^2}\frac{1}{[n]}+\frac{q^{m+n}}{[m+n]^2}\frac{q^m}{[m]}\right)=\sum_{l>n>0}\frac{q^l}{[l]^2}\frac{1}{[n]}+\sum_{l>m>0}\frac{q^l}{[l]^2}\frac{q^m}{[m]}\\&=\zeta_q(2,1)+\sum_{l>m>0}\frac{q^l}{[l]^2}\frac{q^m}{[m]}.\end{aligned} \tag{2.3}$$

Comparing (2.2) and (2.3), we obtain the desired equality. □

§ 2.3. Sum formula, Duality and Ohno's relation

The relation due to Euler (Proposition 2.3) has two generalizations.

First, starting from the infinite sum $\sum_{m\geq n>0} m^{-k}n^{-1}$, we obtain the relation $\sum_{j=2}^{k}\zeta(j,k+1-j)=\zeta(k+1)$ for $k\geq 2$ in the same way as the proof of Proposition 2.3. Note that the left-hand side is the sum over the set of admissible indices of weight $k+1$ and depth two. In general, the following relation holds, which is called the sum formula:

Theorem 2.5 (Granville [6], Zagier [31])**.** *For positive integers k and r with $k>r$, we denote by $I_0(k,r)$ the set of admissible indices of weight k and depth r. Then it holds that*

$$\sum_{\boldsymbol{k}\in I_0(k,r)}\zeta(\boldsymbol{k})=\zeta(k).$$

Second generalization is called duality of MZV. Any admissible index is written in the form

$$\boldsymbol{k}=(a_1+1,\underbrace{1,\ldots,1}_{b_1-1},\cdots,a_s+1,\underbrace{1,\ldots,1}_{b_s-1})$$

with positive integers $a_1,\ldots,a_s$ and $b_1,\ldots,b_s$. Then the dual index $\boldsymbol{k}^\dagger$ of $\boldsymbol{k}$ is defined by

$$\boldsymbol{k}^\dagger=(b_s+1,\underbrace{1,\ldots,1}_{a_s-1},\cdots,b_1+1,\underbrace{1,\ldots,1}_{a_1-1}).$$

For example, if $\boldsymbol{k}=(2,1)$, we have $s=1,a_1=1,b_1=2$, and $\boldsymbol{k}^\dagger=(3)$. Hence Proposition 2.3 states that $\zeta(\boldsymbol{k})=\zeta(\boldsymbol{k}^\dagger)$ for $\boldsymbol{k}=(2,1)$. The fact is that it holds in general (see, e.g., [12, 30] for the proof):

Theorem 2.6. *For any admissible index $\boldsymbol{k}$, it holds that $\zeta(\boldsymbol{k})=\zeta(\boldsymbol{k}^\dagger)$.*

The sum formula and the duality are generalized to a large family of linear relations by Ohno.

Theorem 2.7 (Ohno [18])**.** *Let $\boldsymbol{k}$ be an admissible index and r be its depth. We denote the depth of the dual index $\boldsymbol{k}^\dagger$ by r'. Then, for any $m\geq 0$, it holds that*

$$\sum_{\substack{\boldsymbol{e}\in(\mathbb{Z}_{\geq 0})^r\\ \mathrm{wt}(\boldsymbol{e})=m}}\zeta(\boldsymbol{k}+\boldsymbol{e})=\sum_{\substack{\boldsymbol{e}'\in(\mathbb{Z}_{\geq 0})^{r'}\\ \mathrm{wt}(\boldsymbol{e}')=m}}\zeta(\boldsymbol{k}^\dagger+\boldsymbol{e}'). \tag{2.4}$$

The case where $m=0$ of Ohno's relation is the duality. Setting $\boldsymbol{k}=(r+1)$ and $m=k-r-1$, we recover the sum formula.

A remarkable fact is that the qMZV (2.1) also satisfies Ohno's relation in the same form:

Theorem 2.8 (Bradley [3]). *The relation* (2.4) *with MZV* $\zeta(\cdot)$ *replaced by qMZV* $\zeta_q(\cdot)$ *holds.*

Theorem 2.8 is proved by using the generating function of the both sides of the relation. Recently, another simple and interesting proof is given by Seki and Yamamoto [23]. Their technique can be applied to various series identities. See [22] for examples and details.

Remark. In [15] Kawashima proves a larger family of relations among MZVs which contains Ohno's relation. A q-analogue of Kawashima's relation is obtained in [24].

§ 2.4. Ohno-Zagier relation

The sum formula describes the value of MZVs of fixed weight and depth. Ohno and Zagier gives a formula for more refined sum of MZVs in terms of generating function. We define the *height* of an index $\boldsymbol{k}=(k_1,\dots,k_r)$ by $\mathrm{ht}(\boldsymbol{k})=|\{j\,|\,k_j\ge 2\}|$.

Theorem 2.9 (Ohno-Zagier [19]). *Denote by* $I_0(k,r,s)$ *the set of admissible indices of weight* k*, depth* r *and height* s*. Then it holds that*

$$1+(z-xy)\sum_{\substack{r\ge s\ge 1\\ k\ge r+s}}\left(\sum_{\boldsymbol{k}\in I_0(k,r,s)}\zeta(\boldsymbol{k})\right)x^{k-r-s}y^{r-s}z^{s-1} = \exp\left(\sum_{n\ge 2}\frac{\zeta(n)}{n}(x^n+y^n-\alpha^n-\beta^n)\right), \tag{2.5}$$

where $\alpha+\beta=x+y$ *and* $\alpha\beta=z$.

Remark. By setting $z=xy$ in (2.5), we reproduce the sum formula.

Here we give a sketch of the proof of Theorem 2.9. For an index $\boldsymbol{k}=(k_1,\dots,k_r)$, we define the *multiple polylogarithm* with one variable

$$L(\boldsymbol{k};t)=\sum_{m_1>m_2>\cdots>m_r>0}\frac{t^{m_1}}{m_1^{k_1}m_2^{k_2}\cdots m_r^{k_r}}.$$

Note that, if $\boldsymbol{k}$ is admissible, we have $L(\boldsymbol{k};1)=\zeta(\boldsymbol{k})$. The function $L(\boldsymbol{k};t)$ satisfies the following recurrence relations:

$$\frac{d}{dt}L(\boldsymbol{k};t) = \begin{cases} \dfrac{1}{t}\,L(k_1-1,k_2,\dots,k_r;t) & (k_1\geq 2), \\ \dfrac{1}{1-t}\,L(k_2,\dots,k_r;t) & (k_1=1). \end{cases} \tag{2.6}$$

Now consider the function

$$\Phi_0(t) = \sum_{\substack{r\geq s\geq 1\\ k\geq r+s}} \left(\sum_{\boldsymbol{k}\in I_0(k,r,s)} L(\boldsymbol{k};t) \right) x^{k-r-s}y^{r-s}z^{s-1}.$$

Using (2.6), we see that Φ_0 is the unique solution of the differential equation

$$t(1-t)\frac{d\Phi_0^2}{dt^2} + ((1-x)(1-t)-yt)\frac{d\Phi_0}{dt} + (xy-z)\Phi_0(t) = 1$$

which satisfies $\Phi_0(0)=0$. The solution is given in terms of the Gauss hypergeometric function ${}_2F_1$:

$$\Phi_0(t) = \frac{1}{xy-z}(1 - {}_2F_1(\alpha-x,\beta-x,1-x;t)).$$

Setting $t=1$ and using Gauss's formula

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

and the relation

$$\Gamma(1-s) = \exp\,(\gamma s + \sum_{n\geq 2}\frac{\zeta(n)}{n}s^n),$$

where γ is Euler's constant, we obtain (2.5).

The above calculation works also for qMZV, and we obtain the following q-analogue of the Ohno-Zagier relation:

Theorem 2.10 (Okuda-Takeyama [20]). *Denote by $I_0(k,r,s)$ the set of admissible indices of weight k, depth r and height s. Then it holds that*

$$1 + (z-xy) \sum_{\substack{r\geq s\geq 1\\ k\geq r+s}} \left(\sum_{\boldsymbol{k}\in I_0(k,r,s)} \zeta_q(\boldsymbol{k}) \right) x^{k-r-s}y^{r-s}z^{s-1} = \exp\left(\sum_{n\geq 2}\zeta_q(n) \sum_{m\geq 0}\frac{(q-1)^m}{m+n}(x^{m+n}+y^{m+n}-\alpha^{m+n}-\beta^{m+n}) \right), \tag{2.7}$$

where

$$\alpha+\beta = x+y+(1-q)(xy-z), \qquad \alpha\beta = z.$$

Although we omit the details of the proof of Theorem 2.10, we only mention that it is proved by making use of the basic hypergeometric function

$$ {}_2\phi_1(a,b,c;t)=\sum_{n\ge 0}\prod_{j=0}^{n-1}\frac{(1-aq^j)(1-bq^j)}{(1-q^{j+1})(1-cq^j)}t^n $$

and Heine's formula

$$ {}_2\phi_1(a,b,c;c/ab)=\prod_{j=0}^{\infty}\frac{(1-q^jc/a)(1-q^jc/b)}{(1-q^jc)(1-q^jc/ab)}. $$

See [20] for the details.

Note that, contrary to Ohno's relation, the identity (2.7) for qMZV is not completely the same as (2.5) for MZVs and has correction terms with $1-q$. When we regard $1-q$ as a factor with weight one, we see that the relations among qMZVs obtained by expanding the both sides of (2.7) are homogeneous with respect to their weights. Such correction terms often appear in relations among qMZVs. For example, we see that

$$ \begin{aligned} \zeta(3)\zeta(2)&=\sum_{m>0}\frac{1}{m^3}\sum_{n>0}\frac{1}{n^2}=\left(\sum_{m>n>0}+\sum_{m=n>0}+\sum_{n>m>0}\right)\frac{1}{m^3}\frac{1}{n^2}\\ &=\zeta(3,2)+\zeta(5)+\zeta(2,3). \end{aligned} \tag{2.8} $$

Let us proceed the same calculation for qMZV. We find that

$$ \begin{aligned} \zeta_q(3)\zeta_q(2)&=\sum_{m>0}\frac{q^{2m}}{[m]^3}\sum_{n>0}\frac{q^n}{[n]^2}=\left(\sum_{m>n>0}+\sum_{m=n>0}+\sum_{n>m>0}\right)\frac{q^{2m}}{[m]^3}\frac{q^n}{[n]^2}\\ &=\zeta_q(3,2)+\sum_{m>0}\frac{q^{3m}}{[m]^5}+\zeta_q(2,3). \end{aligned} $$

Note that the second term in the right-hand side is not equal to $\zeta_q(5)$. However, since

$$ \begin{aligned} \sum_{m>0}\frac{q^{3m}}{[m]^5}&=\sum_{m>0}\frac{q^{3m}(1-q^m)+q^{4m}}{[m]^5}=(1-q)\sum_{m>0}\frac{q^{3m}}{[m]^4}+\sum_{m>0}\frac{q^{4m}}{[m]^5}\\ &=(1-q)\zeta_q(4)+\zeta_q(5), \end{aligned} $$

it holds that

$$ \zeta_q(3)\zeta_q(2)=\zeta_q(3,2)+\zeta_q(5)+\zeta_q(2,3)+(1-q)\zeta_q(4). $$

If we count the weight of $1-q$ by one, the weight of the correction term $(1-q)\zeta_q(4)$ is five and the above relation becomes homogeneous.

Remark. Any product of MZVs can be expanded into a $\mathbb{Q}$-linear combination of MZVs in the same way as indicated in (2.8). Hence the vector space $\mathcal{Z}$ defined by (1.1) forms a $\mathbb{Q}$-algebra.

§ 3. Kaneko-Zagier conjecture and Finite multiple harmonic q-sum

In the rest of this article, we illustrate an application of a truncated version of the qMZV to the Kaneko-Zagier conjecture, which states that there exists one-to-one correspondence between two variants of MZVs called finite multiple zeta values and symmetric multiple zeta values.

§ 3.1. Finite multiple zeta value

For $m \geq 1$ and an index $\boldsymbol{k} = (k_1, \dots, k_r)$, we define the *finite multiple harmonic sum* $H_m(\boldsymbol{k})$ by

$$H_m(\boldsymbol{k}) = \sum_{m \geq m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

We set $H_m(\emptyset) = 1$ for the empty index $\emptyset$.

For a prime number p we denote the finite field $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{F}_p$. Consider the quotient

$$\mathcal{A} = \prod_{p:\text{prime}} \mathbb{F}_p \Big/ \bigoplus_{p:\text{prime}} \mathbb{F}_p.$$

Any element of $\mathcal{A}$ is represented by a collection $(a_p)_p$ of elements in $\mathbb{F}_p$, and two elements $(a_p)_p$ and $(b_p)_p$ of $\mathcal{A}$ are equal if and only if $a_p = b_p$ except for finite primes p. We endow $\mathcal{A}$ with the $\mathbb{Q}$-algebra structure by diagonal multiplication.

Definition 3.1. For an index $\boldsymbol{k}$, we define the *finite multiple zeta value (FMZV)* $\zeta_{\mathcal{A}}(\boldsymbol{k})$ as an element of $\mathcal{A}$ by

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) = \left(H_{p-1}(\boldsymbol{k}) \bmod p\right)_p.$$

We set $\mathcal{Z}_{\mathcal{A}} = \sum_{\boldsymbol{k}} \mathbb{Q}\, \zeta_{\mathcal{A}}(\boldsymbol{k})$, which is the $\mathbb{Q}$-subalgebra of $\mathcal{A}$ generated by FMZVs.

Example 3.2.

1. We see that $\zeta_{\mathcal{A}}(k) = 0$ for any $k \geq 1$ by using a primitive root modulo p.

2. For an index of depth two, we have the following formula [8, 34]:

$$\zeta_{\mathcal{A}}(k_1, k_2) = (-1)^{k_1} \binom{k_1 + k_2}{k_1} Z(k_1 + k_2),$$

where $Z(k) \in \mathcal{A}\,(k \geq 2)$ is defined by

$$Z(k) = \left(\frac{B_{p-k}}{k} \bmod p \right)_p .$$

Here B_n denotes the Bernoulli number.

§ 3.2. Symmetric multiple zeta value

In order to define symmetric multiple zeta values, we consider asymptotics of the multiple harmonic sum $H_m(\boldsymbol{k})$ as $m \to +\infty$.

For any admissible index $\boldsymbol{k}$, it is known that

$$H_m(\boldsymbol{k}) = \zeta(\boldsymbol{k}) + O\left(\frac{(\log m)^{J(\boldsymbol{k})}}{m} \right) \tag{3.1}$$

for some $J(\boldsymbol{k}) > 0$. To see what happens if $\boldsymbol{k}$ is not admissible, we calculate some examples. First we have

$$H_m(1) = \sum_{n=1}^{m} \frac{1}{n} = \log m + \gamma + O\left(\frac{1}{m} \right),$$

where γ is Euler's constant. Next let us consider the asymptotics of $H_m(1,1)$. Using the identity

$$\begin{aligned} (H_m(1))^2 &= \sum_{m \geq n_1, n_2 > 0} \frac{1}{n_1 n_2} = \left(\sum_{m \geq n_1 > n_2 > 0} + \sum_{m \geq n_1 = n_2 > 0} + \sum_{m \geq n_2 > n_1 > 0} \right) \frac{1}{n_1 n_2} \\ &= 2H_m(1,1) + H_m(2) \end{aligned}$$

and (3.1) with $\boldsymbol{k} = (2)$, we obtain

$$H_m(1,1) = -\frac{\zeta(2)}{2} + \frac{1}{2}(\log m + \gamma)^2 + O\left(\frac{(\log m)^J}{m} \right)$$

for some $J > 0$. Note that the right-hand side is a polynomial of $\log m + \gamma$ whose coefficients belong to $\mathcal{Z}$. In general, the following theorem holds.

Theorem 3.3 (Ihara-Kaneko-Zagier [9]). *For any index $\boldsymbol{k}$ there exists a unique polynomial $\zeta^*(\boldsymbol{k}; T) \in \mathcal{Z}[T]$ such that*

$$H_m(\boldsymbol{k}) = \zeta^*(\boldsymbol{k}; \gamma + \log m) + O\left(\frac{(\log m)^{J(\boldsymbol{k})}}{m} \right) \qquad (m \to +\infty) \tag{3.2}$$

for some $J(\boldsymbol{k}) > 0$.

Note that $\zeta^*(\boldsymbol{k};T) = \zeta(\boldsymbol{k})$ if $\boldsymbol{k}$ is admissible. Using the polynomial $\zeta^*(\boldsymbol{k};T)$ we define the regularized multiple zeta value:

Definition 3.4. For an index $\boldsymbol{k}$, not necessarily admissible, we define the *regularized multiple zeta value* $\zeta^*(\boldsymbol{k})$ by

$$\zeta^*(\boldsymbol{k}) = \zeta^*(\boldsymbol{k};0).$$

For the empty index we set $\zeta^*(\emptyset) = 1$.

Example 3.5.

1. For any admissible index $\boldsymbol{k}$, it holds that $\zeta^*(\boldsymbol{k}) = \zeta(\boldsymbol{k})$.

2. As seen above, we have $\zeta^*(1;T) = T$ and $\zeta^*(1,1;T) = -\zeta(2)/2 + T^2/2$. Hence $\zeta^*(1) = 0$ and $\zeta^*(1,1) = -\zeta(2)/2$.

Now we define the symmetric multiple zeta value:

Definition 3.6. For an index $\boldsymbol{k} = (k_1, \ldots, k_r)$, we set

$$\zeta^*_{\mathcal{S}}(\boldsymbol{k}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta^*(k_a, k_{a-1}, \ldots, k_1)\, \zeta^*(k_{a+1}, k_{a+2}, \ldots, k_r).$$

Then the *symmetric multiple zeta value (SMZV)* $\zeta_{\mathcal{S}}(\boldsymbol{k})$ is defined as an element of the quotient $\mathbb{Q}$-algebra $\mathcal{Z}/\zeta(2)\mathcal{Z}$ by

$$\zeta_{\mathcal{S}}(\boldsymbol{k}) = \zeta^*_{\mathcal{S}}(\boldsymbol{k}) \bmod \zeta(2)\mathcal{Z}.$$

Example 3.7.

1. In the case of depth one, we see that $\zeta^*_{\mathcal{S}}(k) = (1 + (-1)^k)\zeta^*(k)$. Hence, if k is odd, $\zeta^*_{\mathcal{S}}(k) = 0$. If k is even, $\zeta^*_{\mathcal{S}}(k) = 2\zeta(k)$ is a rational multiple of $\pi^k = (\pi^2)^{k/2} = (6\zeta(2))^{k/2}$. Therefore, for any $k \geq 1$, we see that $\zeta^*_{\mathcal{S}}(k) \in \zeta(2)\mathcal{Z}$ and hence $\zeta_{\mathcal{S}}(k) = 0$.

2. For indices of depth two, it is known that

$$\zeta_{\mathcal{S}}(k_1, k_2) = (-1)^{k_1} \binom{k_1 + k_2}{k_1} \zeta(k_1 + k_2) \bmod \zeta(2)\mathcal{Z}.$$

   See, e.g., [12].

Remark. In [29] Yasuda proves that the set of the values $\zeta^*_{\mathcal{S}}(\boldsymbol{k})$ spans the whole $\mathbb{Q}$-vector space $\mathcal{Z}$. Hence $\mathcal{Z}/\zeta(2)\mathcal{Z}$ is generated by SMZVs.

§ 3.3. Kaneko-Zagier conjecture

Now we formulate the conjecture due to Kaneko and Zagier precisely.

Kaneko-Zagier conjecture. (Kaneko-Zagier [14]) *There exists a $\mathbb{Q}$-algebra isomorphism*

$$\varphi_{KZ} : \mathcal{Z}_{\mathcal{A}} \longrightarrow \mathcal{Z}/\zeta(2)\mathcal{Z}$$

such that $\varphi_{KZ}(\zeta_{\mathcal{A}}(\boldsymbol{k})) = \zeta_{\mathcal{S}}(\boldsymbol{k})$ *for any index* $\boldsymbol{k}$.

According to the conjecture, $\mathbb{Q}$-linear relations among FMZVs should be satisfied by SMZVs. Here we give two examples of such relations, called duality and Ohno-type relation, which are proved to be correct. To state them, we define the *Hoffman dual* $\boldsymbol{k}^{\vee}$ as follows. Any non-empty index is written in the form $(1\square 1\square\cdots\square 1)$ in which $\square$ is either '+' (plus) or ',' (comma). For example, the index $\boldsymbol{k}=(3,1,2)$ is written as $\boldsymbol{k}=(1+1+1,1,1+1)$. Then the Hoffman dual $\boldsymbol{k}^{\vee}$ is defined to be the index obtained by replacing '+' by ',' and vice versa. Hence, if $\boldsymbol{k}=(3,1,2)$, then $\boldsymbol{k}^{\vee}=(1,1,1+1+1,1)=(1,1,3,1)$.

The duality is an identity of a variant of FMZV and SMZV called a "star-version" of them. For an index $\boldsymbol{k}=(k_1,\ldots,k_r)$ we set

$$H_m^{\star}(\boldsymbol{k}) = \sum_{m\ge m_1\ge\cdots\ge m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}.$$

It is written as a linear combination of the harmonic sums H_m. For example,

$$\begin{aligned}
H_m^{\star}(2,3,1) &= \sum_{m\ge m_1\ge m_2\ge m_3>0} \frac{1}{m_1^2 m_2^3 m_3} \\
&= \left(\sum_{m\ge m_1>m_2>m_3>0} + \sum_{m\ge m_1=m_2>m_3>0} + \sum_{m\ge m_1>m_2=m_3>0} + \sum_{m\ge m_1=m_2=m_3>0} \right) \frac{1}{m_1^2 m_2^3 m_3} \\
&= H_m(2,3,1) + H_m(5,1) + H_m(2,4) + H_m(6) \\
&= H_m(2,3,1) + H_m(2+3,1) + H_m(2,3+1) + H_m(2+3+1).
\end{aligned}$$

In general, we see that

$$H_m^{\star}(k_1,\ldots,k_r) = \sum_{\square=\text{'+' or ','}} H_m(k_1\square\cdots\square k_r). \tag{3.3}$$

Now we define the star-version of FMZV by

$$\zeta_{\mathcal{A}}^{\star}(\boldsymbol{k}) = (H_{p-1}^{\star}(\boldsymbol{k}) \bmod p)_p.$$

From (3.3) we see that

$$\zeta_{\mathcal{A}}^{\star}(k_1,\dots,k_r)=\sum_{\square='+'\text{ or }','}\zeta_{\mathcal{A}}(k_1\square\cdots\square k_r).$$

Motivated by this identity, we also define the star-version of SMZV by

$$\zeta_{\mathcal{S}}^{\star}(k_1,\dots,k_r)=\sum_{\square='+'\text{ or }','}\zeta_{\mathcal{S}}(k_1\square\cdots\square k_r).$$

Then we have the following relation:

Theorem 3.8 (Hoffman [8] for FMZV, Jarrosay [10] for SMZV)**.** *For any non-empty index* $\boldsymbol{k}$*, it holds that*

$$\zeta_{\mathcal{F}}^{\star}(\boldsymbol{k})=-\zeta_{\mathcal{F}}^{\star}(\boldsymbol{k}^{\vee})$$

for $\mathcal{F}=\mathcal{A}$ *or* $\mathcal{S}$*.*

Next we give the Ohno-type relation.

Theorem 3.9 (Oyama [21])**.** *Let* $\boldsymbol{k}$ *be a non-empty index, and set* $r=\mathrm{dep}(\boldsymbol{k})$ *and* $r'=\mathrm{dep}(\boldsymbol{k}^{\vee})$*. Then, for* $m\geq 0$*, it holds that*

$$\sum_{\substack{\boldsymbol{e}\in(\mathbb{Z}_{\geq 0})^r\\ \mathrm{wt}(\boldsymbol{e})=m}}\zeta_{\mathcal{F}}(\boldsymbol{k}+\boldsymbol{e})=\sum_{\substack{\boldsymbol{e}'\in(\mathbb{Z}_{\geq 0})^{r'}\\ \mathrm{wt}(\boldsymbol{e}')=m}}\zeta_{\mathcal{F}}((\boldsymbol{k}^{\vee}+\boldsymbol{e}')^{\vee})$$

for $\mathcal{F}=\mathcal{A}$ *or* $\mathcal{S}$*.*

Remark. FMZVs and SMZVs satisfy the following relations:

$$\zeta_{\mathcal{F}}(\boldsymbol{k}*\boldsymbol{l})=\zeta_{\mathcal{F}}(\boldsymbol{k})\zeta_{\mathcal{F}}(\boldsymbol{l}),\tag{3.4}$$

$$\zeta_{\mathcal{F}}(\boldsymbol{k}\,\text{ш}\,\boldsymbol{l})=(-1)^{\mathrm{wt}(\boldsymbol{k})}\zeta_{\mathcal{F}}(\overleftarrow{\boldsymbol{k}},\boldsymbol{l}),\tag{3.5}$$

where $*$ and ш are the harmonic and the shuffle product, respectively, and $\overleftarrow{\boldsymbol{k}}$ is the reversal of $\boldsymbol{k}$. For the details, see, e.g., [12]. In [21] Oyama proved that Theorem 3.9 follows from (3.4), (3.5) and the fact that $\zeta_{\mathcal{F}}(k_1,k_2,\dots,k_r)=0$ if $k_1=k_2=\cdots=k_r$, which can be derived from (3.4) and $\zeta_{\mathcal{F}}(k)=0$ for $k\geq 1$ (see [8, Theorem 2.3]).

§ 3.4. Finite multiple harmonic q-sum and Finite/Symmetric MZV

Here we see that FMZVs and SMZVs are simultaneously obtained from the finite multiple harmonic q-sum, which is a truncated version of the qMZV (2.1) defined by

$$H_m(\boldsymbol{k};q)=\sum_{m\geq m_1>\cdots>m_r>0}\frac{q^{(k_1-1)m_1+\cdots+(k_r-1)m_r}}{[m_1]^{k_1}\cdots[m_r]^{k_r}}\tag{3.6}$$

for an index $\boldsymbol{k} = (k_1, \ldots, k_r)$.

For $n \geq 1$, we set $\zeta_n = e^{2\pi i/n}$. If p is prime, the q-sum $H_{p-1}(\boldsymbol{k}; \zeta_p)$ at a root of unity belongs to the ring $\mathbb{Z}[\zeta_p]$ because $[m]|_{q=\zeta_p}$ is a unit of $\mathbb{Z}[\zeta_p]$ for $1 \leq m < p$. Then FMZVs are reconstructed from $H_{p-1}(\boldsymbol{k}; \zeta_p)$ as follows:

Theorem 3.10 (Bachmann-Takeyama-Tasaka [1]). *Under the identification $\mathbb{Z}[\zeta_p]/(1-\zeta_p)\mathbb{Z}[\zeta_p] \simeq \mathbb{F}_p$ for prime p, we have*

$$\left(H_{p-1}(\boldsymbol{k}; \zeta_p) \ \mathit{mod}\ (1-\zeta_p)\mathbb{Z}[\zeta_p]\right)_p = \zeta_{\mathcal{A}}(\boldsymbol{k})$$

in $\mathcal{A}$ for any index $\boldsymbol{k}$.

The SMZVs are obtained by taking the limit of the q-sum at a root of unity.

Theorem 3.11 (Bachmann-Takeyama-Tasaka [1]). *For any index $\boldsymbol{k}$, the limit*

$$\xi(\boldsymbol{k}) = \lim_{n\to\infty} H_{n-1}(\boldsymbol{k}; \zeta_n)$$

exists in $\mathcal{Z}[\pi i]$ and it is given by

$$\xi(k_1, \ldots, k_r) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta^*(k_a, k_{a-1}, \ldots, k_1; \frac{\pi i}{2})\, \zeta^*(k_{a+1}, k_{a+2}, \ldots, k_r; -\frac{\pi i}{2}),$$

where $\zeta^*(\boldsymbol{k}; T)$ is the polynomial determined by (3.2). *Hence it holds that*

$$\xi(\boldsymbol{k}) \ \mathit{mod}\ \pi i \mathcal{Z}[\pi i] = \zeta_{\mathcal{S}}(\boldsymbol{k})$$

under the identification $\mathcal{Z}[\pi i]/\pi i\mathcal{Z}[\pi i] \simeq \mathcal{Z}/\zeta(2)\mathcal{Z}$.

An important point is that, because of Theorem 3.10 and Theorem 3.11, we can obtain relations among FMZVs and SMZVs simultaneously from those among the finite multiple harmonic q-sums at a root of unity. For example, the duality (Theorem 3.8) and the Ohno-type relation (Theorem 3.9) are reproduced from the following identities, respectively.

Theorem 3.12 (Bachmann-Takeyama-Tasaka [1]). *We define the star-version $H^{\star}_m(\boldsymbol{k}; q)$ by* (3.6) *with the range $m \geq m_1 > \cdots > m_r > 0$ of the summation replaced by $m \geq m_1 \geq \cdots \geq m_r > 0$. Then, for any index $\boldsymbol{k}$ and $n > \mathrm{dep}(\boldsymbol{k})$, it holds that*

$$H^{\star}_{n-1}(\boldsymbol{k}; \zeta_n) = -\zeta_n^{\mathrm{wt}(\boldsymbol{k})} \overline{H^{\star}_{n-1}(\boldsymbol{k}^{\vee}; \zeta_n)},$$

where the bar on the right-hand side denotes complex conjugation.

Theorem 3.13 (Takeyama [25]). *Let* $\boldsymbol{k}$ *be a non-empty index, and set* $r = \mathrm{dep}(\boldsymbol{k})$ *and* $s = \mathrm{dep}(\boldsymbol{k}^\vee)$. *For* $m \geq 0$ *and* $n \geq r+m+1$, *it holds that*

$$\sum_{\substack{\boldsymbol{e}' \in (\mathbb{Z}_{\geq 0})^s \\ \mathrm{wt}(\boldsymbol{e}')=m}} H_{n-1}((\boldsymbol{k}^\vee + \boldsymbol{e}')^\vee; \zeta_n)$$
$$= \sum_{l=0}^{m} \frac{1}{n} \binom{n}{m-l+1} (1-\zeta_n)^{m-l} \sum_{\substack{\boldsymbol{e} \in (\mathbb{Z}_{\geq 0})^r \\ \mathrm{wt}(\boldsymbol{e})=l}} H_{n-1}(\boldsymbol{k}+\boldsymbol{e}; \zeta_n).$$

As seen above, the finite multiple harmonic q-sum $H_m(\boldsymbol{k}; q)$ at a root of unity plays a role of a bridge between FMZVs and SMZVs. We expect that this framework sheds some light on studying the Kaneko-Zagier conjecture.

§ 3.5. Finite and symmetric Mordell-Tornheim multiple zeta values

Lastly, we discuss a variant of MZVs called the Mordell-Tornheim multiple zeta values [16, 17, 27]:

Definition 3.14. For an index $\boldsymbol{k} = (k_1, \ldots, k_r)$ and a positive integer l, we define the *Mordell-Tornheim multiple zeta value* $\zeta^{MT}(\boldsymbol{k}; l)$ by

$$\zeta^{MT}(\boldsymbol{k}; l) = \sum_{m_1, \ldots, m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^l}.$$

It is known that $\zeta^{MT}(\boldsymbol{k}; l)$ belongs to $\mathcal{Z}_{\mathrm{wt}(\boldsymbol{k})+l}$, which is the weight $\mathrm{wt}(\boldsymbol{k}) + l$ part of the $\mathbb{Q}$-algebra of MZVs [4, 28].

In [11], Kamano introduces the finite Mordell-Tornheim multiple zeta value by

$$\zeta_{\mathcal{A}}^{MT}(\boldsymbol{k}; l) = \left(\sum_{\substack{m_1 + \cdots + m_r < p \\ m_1, \ldots, m_r > 0}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^l} \mod p \right)_p \in \mathcal{A}$$

and proves that it belongs to the $\mathbb{Q}$-algebra $\mathcal{Z}_{\mathcal{A}}$ of FMZVs. By setting $m_{r+1} = p - (m_1 + \cdots + m_r)$, we see that the above multiple sum is written in more symmetric form as

$$(-1)^l \sum_{\substack{m_1 + \cdots + m_{r+1} = p \\ m_1, \ldots, m_r, m_{r+1} > 0}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} m_{r+1}^l}$$

modulo p. Motivated by this expression, we introduce the following q-sum for an index $\boldsymbol{k} = (k_1, \ldots, k_r)$ with $r \geq 2$:

$$\omega_n(\boldsymbol{k}; q) = \sum_{\substack{m_1 + \cdots + m_r = n \\ m_1, \ldots, m_r > 0}} \frac{q^{(k_1-1)m_1 + \cdots + (k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}.$$

Then the algebraic/analytic limiting procedure as $q \to 1$ given in Theorem 3.10 and Theorem 3.11 also works for $\omega_n(\boldsymbol{k}; q)$, and the result is consistent with Kaneko-Zagier conjecture as follows.

Theorem 3.15 (Bachmann-Takeyama-Tasaka [2]).

1. *For a prime p, we identify $\mathbb{Z}[\zeta_p]/(1-\zeta_p)\mathbb{Z}[\zeta_p]$ with $\mathbb{F}_p$. For an index $\boldsymbol{k} = (k_1, \ldots, k_r)$ with $r \geq 2$, we set*

$$\omega_{\mathcal{A}}(\boldsymbol{k}) = (\omega_p(\boldsymbol{k}; \zeta_p) \bmod (1 - \zeta_p)\mathbb{Z}[\zeta_p]) \in \mathcal{A}.$$

Then it holds that $\omega_{\mathcal{A}}(\boldsymbol{k}) = (-1)^{k_r} \zeta_{\mathcal{A}}^{MT}(k_1, \ldots, k_{r-1}; k_r)$ and hence $\omega_{\mathcal{A}}(\boldsymbol{k}) \in \mathcal{Z}_{\mathcal{A}}$.

2. *For an index $\boldsymbol{k} = (k_1, \ldots, k_r)$ with $r \geq 2$, the limit*

$$\Omega(\boldsymbol{k}) = \lim_{n \to \infty} \omega_n(\boldsymbol{k}; \zeta_n)$$

exists and it holds that

$$\Omega(\boldsymbol{k}) = \sum_{a=1}^{r} (-1)^{k_a} \zeta^{MT}(k_1, \ldots, k_{a-1}, k_{a+1}, \ldots, k_r; k_a).$$

Hence $\Omega(\boldsymbol{k}) \in \mathcal{Z}_{\mathrm{wt}(\boldsymbol{k})}$.

3. *For an index $\boldsymbol{k} = (k_1, \ldots, k_r)$ with $r \geq 2$, we define*

$$\omega_{\mathcal{S}}(\boldsymbol{k}) = \Omega(\boldsymbol{k}) \bmod \zeta(2)\mathcal{Z}.$$

Then, if Kaneko-Zagier conjecture is true, we have

$$\varphi_{KZ}(\omega_{\mathcal{A}}(\boldsymbol{k})) = \omega_{\mathcal{S}}(\boldsymbol{k})$$

for any index $\boldsymbol{k}$ with $\mathrm{dep}(\boldsymbol{k}) \geq 2$.

Remark. In [2] we conjecture that the Mordell-Tornheim MZVs span the $\mathbb{Q}$-algebra $\mathcal{Z}$ of MZVs and, more strongly, the set $\{\Omega(\boldsymbol{k}) \,|\, \boldsymbol{k}\text{: index with } \mathrm{dep}(\boldsymbol{k}) \geq 2\}$ spans $\mathcal{Z}$. However, it is false if the conjectures (1.3) and (1.4) are true because of the following reason.

We set

$$\mathcal{W}_k = \sum_{\substack{\boldsymbol{k}:\text{index},\, l \geq 1 \\ \mathrm{wt}(\boldsymbol{k}) + l = k}} \mathbb{Q}\, \zeta^{MT}(\boldsymbol{k}; l).$$

and $w_k = \dim_{\mathbb{Q}} \mathcal{W}_k$. It is known that $\mathcal{W}_k \subset \mathcal{Z}_k$ for all $k \geq 2$. Note that $\zeta^{MT}(k_1, \ldots, k_r; l)$ is symmetric with respect to $k_1, \ldots, k_r$. Hence it holds that $w_k \leq \sum_{l=1}^{k-1} p(l)$, where $p(n)$ is the partition function. From the asymptotic formula due to Hardy and Ramanujan

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{2n/3}\right) \qquad (n \to \infty),$$

we see that

$$\log w_k = O(\sqrt{k}) \qquad (k \to \infty).$$

On the other hand, from the definition of d_k, (1.2), we have

$$\log d_k = O(k) \qquad (k \to \infty).$$

Hence, if (1.3) and (1.4) are true, $\mathcal{Z}_k$ is larger than $\mathcal{W}_k$ for large k.

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