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On symmetry of q-Painlevé equations and associated linear equations

By

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Abstract

We investigate the symmetry of the linear q-difference equations which are associated with some q-Painlevé equations. We apply it for adjustment of the expression of the time evolution on the q-Painlevé equations in terms of the Weyl group symmetry.

§ 1. Introduction

The Painlevé-type equations are non-linear generalizations of the hypergeometric-type equations. Several members of the discrete analogue of the Painlevé equation had been discovered individually in the 1990's, and a comprehensive list of the second order discrete Painlevé equations was provided by Sakai [3]. Each member of the q-difference Painlevé equation was essentially labelled by the affine root system from the symmetry. Sakai [3] realized the time evolution of the discrete Painlevé equations by the symmetry of the Weyl group, which originates from the Cremona action on a family of surfaces which are related with the space of initial conditions.

The q-Painlevé equation of type $D_5^{(1)}$, which is also denoted by q-$P(D_5^{(1)})$, is given by

$$f\overline{f} = \nu_3\nu_4 \frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{(g-1/\nu_1)(g-1/\nu_2)}, \quad g\underline{g} = \frac{1}{\nu_1\nu_2}\frac{(f-\kappa_1/\nu_7)(f-\kappa_1/\nu_8)}{(f-\nu_3)(f-\nu_4)}. \tag{1.1}$$

Here, $\overline{f}$ (resp. $\underline{f}$) is the consequence of the time evolution $t \mapsto qt$ (resp. the inverse time evolution $t \mapsto t/q$) to f. The time evolution for the parameters is given by $\overline{\kappa}_1 = \kappa_1/q$,

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$\overline{\kappa}_2 = q\kappa_2$ and $\overline{\nu}_i = \nu_i$ $(i = 1, 2, \ldots, 8)$. Note that the equation q-$P(D_5^{(1)})$ was originally introduced by Jimbo and Sakai [1] as a q-analogue of the sixth Painlevé equation, and it was obtained by considering a q-analogue of monodromy preserving deformation. Yamada studied the Lax pairs of the discrete Painlevé equations extensively [7, 8], and a list of the Lax pairs was provided in the survey [2] by Kajiwara, Noumi and Yamada. A Lax pair of the q-Painlevé equation q-$P(D_5^{(1)})$ was given as

$$
\begin{aligned}
L_1 &= \Big\{ \frac{z(g\nu_1 - 1)(g\nu_2 - 1)}{qg} - \frac{\nu_1\nu_2\nu_3\nu_4(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{fg} \Big\} \\
&\quad + \frac{\nu_1\nu_2(z - q\nu_3)(z - q\nu_4)}{q(qf - z)}(g - T_z^{-1}) + \frac{(z - \kappa_1/\nu_7)(z - \kappa_1/\nu_8)}{q(f - z)}\Big(\frac{1}{g} - T_z\Big), \\
L_2 &= \Big(1 - \frac{f}{z}\Big)T + T_z - \frac{1}{g}.
\end{aligned} \tag{1.2}
$$

Here T_z represents the transformation $z \mapsto qz$ and T represents the time evolution such as $T(f) = \overline{f}$. The equation q-$P(D_5^{(1)})$ is obtained as the compatibility condition for the Lax operators L_1 and L_2. In this paper we investigate the symmetry of the linear q-difference equation $L_1 y(z) = 0$, where L_1 (see Eq. (1.2)) is the Lax operator of the q-Painlevé equation of type $D_5^{(1)}$. Typical symmetries of the equation $L_1 y(z) = 0$ are given by the gauge transformations $y(z) \mapsto p(z)y(z)$ for some specific functions $p(z)$. We also investigate the symmetry of the associated linear q-difference equation with other q-Painlevé equations denoted by q-$P(E_6^{(1)})$ and q-$P(E_7^{(1)})$. The symmetry of the linear q-difference equation may induce the transformation of the parameters and some of them are described in terms of the affine Weyl group of the q-Painlevé equation. The gauge transformation $y(z) \mapsto z^\lambda y(z)$ for $\lambda \in \mathbb{C}$ and the dilation $z \mapsto az$ for $a \in \mathbb{C} \setminus \{0\}$ also induce the transformation of the parameters.

Based on Sakai's description of the symmetry of the discrete Painlevé equations, a systematic description of the symmetry of each member of the discrete Painlevé equations was provided in [2] by using the affine Weyl group. It seems that direct relationship between the time evolution of the q-Painlevé equations q-$P(D_5^{(1)})$, q-$P(E_6^{(1)})$ and q-$P(E_7^{(1)})$ derived from the Lax pairs and the symmetry in terms of the affine Weyl group was not described in [2]. On q-$P(D_5^{(1)})$, a candidate of the time evolution by the symmetry in terms of the affine Weyl group was essentially written in [3]. However, we found that some adjustment would be necessary to fit the time evolution from the Lax pair with the Weyl group symmetry in [2]. See Eqs. (2.17), (2.18) and the discussion around them. In this paper we describe the expression of the time evolution of q-Painlevé equations q-$P(D_5^{(1)})$, q-$P(E_6^{(1)})$ and q-$P(E_7^{(1)})$ in terms of the Weyl group symmetry and the symmetry coming from the gauge transformation and the dilation.

This paper is organized as follows. In section 2, we review the Weyl group action for the q-Painlevé equations q-$P(D_5^{(1)})$ by following [2], and investigate the symmetry for the linear q-difference equation $L_1 y(z) = 0$. We realize the time evolution of q-$P(D_5^{(1)})$ in terms of the actions of the generators of the Weyl group and the symmetry coming from the linear q-difference equation. In section 3, we introduce the q-Painlevé equations q-$P(E_6^{(1)})$. We also review the Lax pair and its Weyl group action of q-$P(E_6^{(1)})$ by following [2], and investigate the symmetry for the corresponding linear q-difference equation. We also investigate the time evolution of q-$P(E_6^{(1)})$, which was inspired by the study of the difference Painlevé equation by Tsuda [6]. In section 4, we give similar results for the q-Painlevé equations q-$P(E_7^{(1)})$.

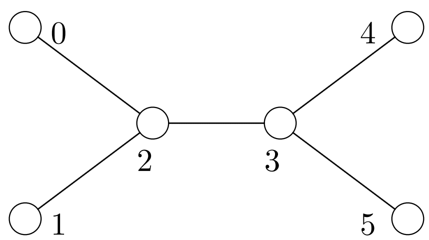
§ 2. The q-Painlevé equation of type $D_5^{(1)}$

We review the Weyl group symmetry of the q-Painlevé equation q-$P(D_5^{(1)})$. For this purpose, we recall the action of the operators $s_0, \ldots, s_5$, π_1 and π_2 for the parameters $(\kappa_1, \kappa_2, \nu_1, \ldots, \nu_8)$ and (f, g) introduced in [2].

(2.1)

$$\begin{aligned}
&s_0 : \nu_7 \leftrightarrow \nu_8, \quad s_1 : \nu_3 \leftrightarrow \nu_4, \quad s_4 : \nu_1 \leftrightarrow \nu_2, \quad s_5 : \nu_5 \leftrightarrow \nu_6,\\
&s_2 : \nu_3 \to \frac{\kappa_1}{\nu_7},\ \nu_7 \to \frac{\kappa_1}{\nu_3},\ \kappa_2 \to \frac{\kappa_1\kappa_2}{\nu_3\nu_7},\ g \to g\,\frac{f-\nu_3}{f-\kappa_1/\nu_7},\\
&s_3 : \nu_1 \to \frac{\kappa_2}{\nu_5},\ \nu_5 \to \frac{\kappa_2}{\nu_1},\ \kappa_1 \to \frac{\kappa_1\kappa_2}{\nu_1\nu_5},\ f \to f\,\frac{g-1/\nu_1}{g-\nu_5/\kappa_2},\\
&\pi_1 : q \to 1/q,\ \nu_1 \to 1/\nu_1,\ \nu_2 \to 1/\nu_2,\ \nu_3 \to 1/\nu_7,\ \nu_4 \to 1/\nu_8,\ \nu_5 \to 1/\nu_5,\ \nu_6 \to 1/\nu_6,\\
&\qquad \nu_7 \to 1/\nu_3,\ \nu_8 \to 1/\nu_4,\ \kappa_1 \to 1/\kappa_1,\ \kappa_2 \to 1/\kappa_2,\ f \to f/\kappa_1,\ g \to 1/g,\\
&\pi_2 : q \to 1/q,\ \nu_1 \to 1/\nu_7,\ \nu_2 \to 1/\nu_8,\ \nu_3 \to 1/\nu_5,\ \nu_4 \to 1/\nu_6,\ \nu_5 \to 1/\nu_3,\ \nu_6 \to 1/\nu_4,\\
&\qquad \nu_7 \to 1/\nu_1,\ \nu_8 \to 1/\nu_2,\ \kappa_1 \to 1/\kappa_2,\ \kappa_2 \to 1/\kappa_1,\ f \to 1/(\kappa_2 g),\ g \to \kappa_1/f.
\end{aligned}$$

The omitted variables are invariant by the action, e.g. $s_2(f) = f$. Then we can confirm that these operations satisfy the relations of the extended Weyl group $\widetilde{W}(D_5^{(1)})$ whose Dynkin diagram is as follows.



Namely we have $s_i^2 = \mathrm{id}$, $(i = 0, \ldots, 5)$, $\pi_1^2 = \pi_2^2 = \mathrm{id}$, $(\pi_1\pi_2)^4 = \mathrm{id}$, $\pi_1 s_0 = s_1 \pi_1$,

$\pi_1 s_j = s_j \pi_1$, $(j = 2, 3, 4, 5)$, $\pi_2 s_0 = s_4 \pi_2$, $\pi_2 s_1 = s_5 \pi_2$, $\pi_2 s_2 = s_3 \pi_2$ and

$$\begin{aligned} &s_i s_j s_i = s_j s_i s_j, \quad \{i,j\} = \{0,2\}, \{1,2\}, \{2,3\}, \{3,4\}, \{3,5\}, \\ &s_i s_j = s_j s_i, \quad \text{otherwise.} \end{aligned} \tag{2.2}$$

Recall that the operator L_1 is defined in Eq. (1.2) as one of the Lax pair, and the linear q-differential equation $L_1 y(z) = 0$ is written as

$$\begin{aligned} &\Big\{ \frac{z(g\nu_1 - 1)(g\nu_2 - 1)}{qg} - \frac{\nu_1\nu_2\nu_3\nu_4(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{fg} \Big\} y(z) \\ &+ \frac{\nu_1\nu_2(z - q\nu_3)(z - q\nu_4)}{q(qf - z)} (gy(z) - y(z/q)) \\ &+ \frac{(z - \kappa_1/\nu_7)(z - \kappa_1/\nu_8)}{q(f - z)} \Big(\frac{1}{g} y(z) - y(qz) \Big) = 0. \end{aligned} \tag{2.3}$$

Note that Eq. (2.3) was studied in [4] from the aspect of the initial-value space of q-$P(D_5^{(1)})$, and it was observed that the q-Heun equation appears.

In this paper, we investigate the action of the symmetry in Eq. (2.1) on Eq. (2.3). It follows immediately that Eq. (2.3) is invariant under the actions of s_0, s_1, s_4 and s_5. It was discussed in [5] that the action of s_3 is related to the q-middle convolution.

Next, we investigate the action of s_2 on Eq. (2.3). It follows from Eq. (2.1) that ν_3 and κ_1/ν_7 are exchanged by the action of s_2. Thus, we set

$$y(z) = \tilde{y}(z) \frac{(q\nu_3/z; q)_\infty}{(q\kappa_1/(\nu_7 z); q)_\infty} = \tilde{y}(z) \prod_{j=0}^{\infty} \frac{1 - q^{j+1}\nu_3/z}{1 - q^{j+1}\kappa_1/(\nu_7 z)}. \tag{2.4}$$

Then we have

$$\begin{aligned} y(qz) &= \tilde{y}(qz) \frac{(1 - \nu_3/z)}{(1 - \kappa_1/(\nu_7 z))} \frac{(q\nu_3/z; q)_\infty}{(q\kappa_1/(\nu_7 z); q)_\infty}, \\ y(z/q) &= \tilde{y}(z/q) \frac{(1 - q\kappa_1/(\nu_7 z))}{(1 - q\nu_3/z)} \frac{(q\nu_3/z; q)_\infty}{(q\kappa_1/(\nu_7 z); q)_\infty}. \end{aligned} \tag{2.5}$$

Therefore, if $y(z)$ is a solution of Eq. (2.3), then $\tilde{y}(z)$ satisfies

$$\begin{aligned} &\Big\{ \frac{z(g\nu_1 - 1)(g\nu_2 - 1)}{qg} - \frac{\nu_1\nu_2\nu_3\nu_4(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{fg} \Big\} \tilde{y}(z) \\ &+ \frac{\nu_1\nu_2(z - q\nu_3)(z - q\nu_4)}{q(qf - z)} g\tilde{y}(z) - \frac{\nu_1\nu_2(z - q\kappa_1/\nu_7)(z - q\nu_4)}{q(qf - z)} \tilde{y}(z/q) \\ &+ \frac{(z - \kappa_1/\nu_7)(z - \kappa_1/\nu_8)}{q(f - z)} \frac{1}{g} \tilde{y}(z) - \frac{(z - \nu_3)(z - \kappa_1/\nu_8)}{q(f - z)} \tilde{y}(qz) = 0. \end{aligned} \tag{2.6}$$

It follows from a straightforward calculation that Eq. (2.6) is written as

$$\begin{aligned}(2.7)\qquad &\Big\{\frac{z(\tilde{g}\nu_1-1)(\tilde{g}\nu_2-1)}{q\tilde{g}}-\frac{\nu_1\nu_2\tilde{\nu}_3\nu_4(\tilde{g}-\nu_5/\tilde{\kappa}_2)(\tilde{g}-\nu_6/\tilde{\kappa}_2)}{f\tilde{g}}\Big\}\tilde{y}(z)\\ &+\frac{\nu_1\nu_2(z-q\tilde{\nu}_3)(z-q\nu_4)}{q(qf-z)}(\tilde{g}\tilde{y}(z)-\tilde{y}(z/q))\\ &+\frac{(z-\kappa_1/\tilde{\nu}_7)(z-\kappa_1/\nu_8)}{q(f-z)}\Big(\frac{1}{\tilde{g}}\tilde{y}(z)-\tilde{y}(qz)\Big)=0\end{aligned}$$

by setting $\tilde{\nu}_3=\kappa_1/\nu_7$, $\tilde{\nu}_7=\kappa_1/\nu_3$, $\tilde{\kappa}_2=\kappa_1\kappa_2/(\nu_3\nu_7)$ and $\tilde{g}=g(f-\nu_3)/(f-\kappa_1/\nu_7)$. Therefore, the gauge transformation defined by Eq. (2.4) induces the symmetry s_2 of the q-Painlevé equation of type $D_5^{(1)}$. The gauge transformation defined by

$$(2.8)\qquad y(z)=\tilde{y}(z)\frac{(q\nu_3/z;q)_\infty(q\nu_4/z;q)_\infty}{(q\kappa_1/(\nu_7z);q)_\infty(q\kappa_1/(\nu_8z);q)_\infty}$$

induces the symmetry $s_2s_1s_0s_2$ of the q-Painlevé equation of type $D_5^{(1)}$.

We examine another gauge-transformation. Let $y(z)$ be a solution of Eq. (2.3) and set $y(z)=z^d\tilde{y}(z)$. Then the function $\tilde{y}(z)$ satisfies

$$\begin{aligned}(2.9)\qquad &\Big\{\frac{z(q^dgq^{-d}\nu_1-1)(q^dgq^{-d}\nu_2-1)}{qq^dg}\\ &\qquad-\frac{q^{-d}\nu_1q^{-d}\nu_2\nu_3\nu_4(q^dg-q^d\nu_5/\kappa_2)(q^dg-q^d\nu_6/\kappa_2)}{fq^dg}\Big\}\tilde{y}(z)\\ &+\frac{q^{-d}\nu_1q^{-d}\nu_2(z-q\nu_3)(z-q\nu_4)}{q(qf-z)}(q^dg\tilde{y}(z)-\tilde{y}(z/q))\\ &+\frac{(z-\kappa_1/\nu_7)(z-\kappa_1/\nu_8)}{q(f-z)}\Big(\frac{1}{q^dg}\tilde{y}(z)-\tilde{y}(qz)\Big)=0.\end{aligned}$$

Hence, the correspondence of the parameters is written as $(\nu_1,\nu_2,\nu_5/\kappa_2,\nu_6/\kappa_2,g)\mapsto(q^{-d}\nu_1,q^{-d}\nu_2,q^d\nu_5/\kappa_2,q^d\nu_5/\kappa_2,q^dg)$, while the parameters $\nu_3,\nu_4,\nu_7/\kappa_1,\nu_8/\kappa_1,f$ are unchanged. We denote by $G[s]$ the transformation of the parameters

$$\begin{aligned}(2.10)\qquad &(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5/\kappa_2,\nu_6/\kappa_2,\nu_7/\kappa_1,\nu_8/\kappa_1,f,g)\\ &\mapsto(\nu_1/s,\nu_2/s,\nu_3,\nu_4,s\nu_5/\kappa_2,s\nu_6/\kappa_2,\nu_7/\kappa_1,\nu_8/\kappa_1,f,sg).\end{aligned}$$

We investigate the dilation $z\mapsto cz$ for the non-zero constant c. Set $z=u/c$ and $y(z)=\tilde{y}(u)$. Let $y(z)$ be a solution of Eq. (2.3). Then the function $\tilde{y}(u)$ satisfies

$$\begin{aligned}(2.11)\qquad &\Big\{u\frac{(\nu_1g-1)(\nu_2g-1)}{qg}-\nu_1\nu_2c\nu_3c\nu_4\frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{cfg}\Big\}\tilde{y}(u)\\ &+\frac{\nu_1\nu_2(u-qc\nu_3)(u-qc\nu_4)}{q(qcf-u)}\Big(g\tilde{y}(u)-\tilde{y}(u/q)\Big)\\ &+\frac{(u-c\kappa_1/\nu_7)(u-c\kappa_1/\nu_8)}{q(cf-u)}\Big(\frac{1}{g}\tilde{y}(u)-\tilde{y}(qu)\Big)=0.\end{aligned}$$

Hence, the correspondence for the parameters is written as

$$(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5/\kappa_2,\nu_6/\kappa_2,\nu_7/\kappa_1,\nu_8/\kappa_1,f,g) \tag{2.12}$$
$$\mapsto (\nu_1,\nu_2,c\nu_3,c\nu_4,\nu_5/\kappa_2,\nu_6/\kappa_2,\nu_7/\kappa_1/c,\nu_8/\kappa_1/c,cf,g),$$

which we denote by $D[c]$.

We investigate the symmetry related to $z \leftrightarrow 1/z$. More precisely we take into account a more general situation $z = c/u$ and $y(z) = z^d\tilde{y}(u) = z^d\tilde{y}(c/z)$, where $y(z)$ is a solution of Eq. (2.3). Then we have

$$\begin{aligned}&\Big\{\frac{c(g\nu_1-1)(g\nu_2-1)}{uqg} - \frac{\nu_1\nu_2\nu_3\nu_4(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{fg}\Big\}\tilde{y}(u)\\ &+\frac{\nu_1\nu_2(c/u-q\nu_3)(c/u-q\nu_4)}{q(qf-c/u)}(g\tilde{y}(u)-q^{-d}\tilde{y}(qu))\\ &+\frac{(c/u-\kappa_1/\nu_7)(c/u-\kappa_1/\nu_8)}{q(f-c/u)}\Big(\frac{1}{g}\tilde{y}(u)-q^d\tilde{y}(u/q)\Big)=0.\end{aligned} \tag{2.13}$$

Therefore,

$$\begin{aligned}&\Big\{\frac{u((1/g)\nu_5/\kappa_2-1)((1/g)\nu_6/\kappa_2-1)}{q/(gq^d)} - \frac{c\nu_5\nu_6\nu_7\nu_8(1/g-\nu_1)(1/g-\nu_2)}{q\kappa_1^2\kappa_2^2/(fgq^d)}\Big\}\tilde{y}(u)\\ &+\frac{q^{2d}\nu_5\nu_6(u-c\nu_7/\kappa_1)(u-c\nu_8/\kappa_1)}{\kappa_2^2q(c/f-u)}\Big(\frac{1}{gq^d}\tilde{y}(u)-\tilde{y}(u/q)\Big)\\ &+\frac{(u-c/(q\nu_3))(u-c/(q\nu_4))}{q(c/(qf)-u)}(q^dg\tilde{y}(u)-\tilde{y}(qu))=0.\end{aligned} \tag{2.14}$$

In the case $q^d = \kappa_2$ and $c = q\kappa_1$, the equation is written as

$$\begin{aligned}&\Big\{\frac{u(\nu_5/(g\kappa_2)-1)(\nu_6/(g\kappa_2)-1)}{q/(g\kappa_2)}\\ &\quad-\frac{\nu_5\nu_6\nu_7\nu_8(1/(g\kappa_2)-\nu_1/\kappa_2)(1/(g\kappa_2)-\nu_2/\kappa_2)}{\kappa_1/(fg\kappa_2)}\Big\}\tilde{y}(u)\\ &+\frac{\nu_5\nu_6(u-q\nu_7)(u-q\nu_8)}{q(q\kappa_1/f-u)}\Big(\frac{1}{g\kappa_2}\tilde{y}(u)-\tilde{y}(u/q)\Big)\\ &+\frac{(u-\kappa_1/\nu_3)(u-\kappa_1/\nu_4)}{q(\kappa_1/f-u)}(g\kappa_2\tilde{y}(u)-\tilde{y}(qu))=0,\end{aligned} \tag{2.15}$$

and the correspondence of the parameters is described by the action of $\pi_2\pi_1\pi_2\pi_1(= \pi_1\pi_2\pi_1\pi_2)$,

$$(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7,\nu_8;\kappa_1,\kappa_2;f,g) \tag{2.16}$$
$$\mapsto (\nu_5,\nu_6,\nu_7,\nu_8,\nu_1,\nu_2,\nu_3,\nu_4;\kappa_1,\kappa_2;\kappa_1/f,1/(\kappa_2 g)).$$

On q-$P(D_5^{(1)})$, a candidate of the time evolution by the symmetry of $\widetilde{W}(D_5^{(1)})$ is $(\pi_2\pi_1 s_2 s_1 s_0 s_2)^2$, which was essentially written in Sakai [3]. By a direct calculation, we have

$$(2.17)\quad (\pi_2\pi_1 s_2 s_1 s_0 s_2)^2(f) = \pi_2\pi_1 s_2 s_1 s_0 s_2(1/g) = \frac{q\nu_3\nu_4\nu_7\nu_8}{f\kappa_1}\frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{(g-1/\nu_1)(g-1/\nu_2)},$$

which does not coincide with

$$(2.18)\qquad \overline{f} = \frac{\nu_3\nu_4}{f}\frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{(g-1/\nu_1)(g-1/\nu_2)}.$$

Moreover we have $(\pi_2\pi_1 s_2 s_1 s_0 s_2)^2(\nu_3) = q\nu_3\nu_7\nu_8/\kappa_1 (\neq \nu_3)$. We can adjust the expressions as the following theorem.

Theorem 2.1. *(i) Let Ξ be the transformation of the parameters defined by*

$$(2.19)\qquad \begin{aligned}&(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7,\nu_8;\kappa_1,\kappa_2;f,g)\\ &\mapsto \Big(\frac{\nu_1\nu_5\nu_6}{\kappa_2},\frac{\nu_2\nu_5\nu_6}{\kappa_2},\frac{\kappa_1}{q\nu_4},\frac{\kappa_1}{q\nu_3},\frac{\nu_5\nu_1\nu_2}{\kappa_2},\frac{\nu_6\nu_1\nu_2}{\kappa_2},\frac{\kappa_1}{q\nu_8},\frac{\kappa_1}{q\nu_7};\\ &\qquad\qquad \frac{\kappa_1^3}{q^2\nu_3\nu_4\nu_7\nu_8},\frac{\nu_1\nu_2\nu_5\nu_6}{\kappa_2};\frac{f\kappa_1}{q\nu_3\nu_4},\frac{g\kappa_2}{\nu_5\nu_6}\Big).\end{aligned}$$

Let T be the transformation of the parameters defined by

$$(2.20)\qquad T = \Xi\cdot(\pi_2\pi_1 s_2 s_1 s_0 s_2)^2.$$

Then

$$(2.21)\qquad \begin{aligned}T(f) &= \frac{\nu_3\nu_4}{f}\frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{(g-1/\nu_1)(g-1/\nu_2)},\\ T(g) &= \frac{1}{g\nu_1\nu_2}\frac{(T(f)-T(\kappa_1/\nu_7))(T(f)-T(\kappa_1/\nu_8))}{(T(f)-\nu_3)(T(f)-\nu_4)}.\end{aligned}$$

Namely the operator T represents the time evolution of q-$P(D_5^{(1)})$ (see Eq. (1.1)). On the other parameters, we have

$$(2.22)\qquad T(\nu_i)=\nu_i\ (i=1,2,\dots,8),\ T(\kappa_1)=\kappa_1/q,\ T(\kappa_2)=q\kappa_2.$$

(ii) Let $G[s]$ and $D[c]$ be the operators in Eqs. (2.10) and (2.12). For any elements ζ generated by $\nu_1,\nu_2,\nu_3,\nu_4,\nu_5/\kappa_2,\nu_6/\kappa_2,\nu_7/\kappa_1,\ \nu_8/\kappa_1,\ f$ and g, we have

$$(2.23)\qquad \Xi(\zeta) = G\Big[\frac{\kappa_2}{\nu_5\nu_6}\Big]D\Big[\frac{\kappa_1}{q\nu_7\nu_8}\Big](\zeta).$$

Proof. Set $s=\pi_2\pi_1 s_2 s_1 s_0 s_2$. It follows from Eq. (2.1) that

$$(2.24)\qquad \begin{aligned}&s(\nu_1)=\nu_7,\ s(\nu_2)=\nu_8,\ s(\nu_3)=\kappa_2/\nu_6,\ s(\nu_4)=\kappa_2/\nu_5,\\ &s(\nu_5)=\nu_3,\ s(\nu_6)=\nu_4,\ s(\nu_7)=\kappa_2/\nu_2,\ s(\nu_8)=\kappa_2/\nu_1,\\ &s(\kappa_1)=\kappa_2,\ s(\kappa_2)=\kappa_1\kappa_2^2/(\nu_1\nu_2\nu_5\nu_6)=q\nu_3\nu_4\nu_7\nu_8/\kappa_1.\end{aligned}$$

Then

$$s^2(\nu_1)=\kappa_2/\nu_2,\ s^2(\nu_2)=\kappa_2/\nu_1,\ s^2(\nu_3)=q\nu_3\nu_7\nu_8/\kappa_1,\ s^2(\nu_4)=q\nu_4\nu_7\nu_8/\kappa_1,$$
$$s^2(\nu_5)=\kappa_2/\nu_6,\ s^2(\nu_6)=\kappa_2/\nu_5,\ s^2(\nu_7)=q\nu_3\nu_4\nu_7/\kappa_1,\ s^2(\nu_8)=q\nu_3\nu_4\nu_8/\kappa_1,$$
$$s^2(\kappa_1)=q\nu_3\nu_4\nu_7\nu_8/\kappa_1,\ s^2(\kappa_2)=q\kappa_2^3/(\nu_1\nu_2\nu_5\nu_6). \tag{2.25}$$

Let Ξ be the operator defined in Eq. (2.19). We have

$$\Xi s^2(\nu_i)=\nu_i,\ (i=1,2,\dots,8),\ \Xi s^2(\kappa_1)=\kappa_1/q,\ \Xi s^2(\kappa_2)=q\kappa_2. \tag{2.26}$$

Therefore, we obtain Eq. (2.22).

It follows from Eq. (2.1) that $s(f)=1/g$ and

$$s(g)=\frac{\kappa_1 f}{q\nu_3\nu_4\nu_7\nu_8}\frac{(g-1/\nu_1)(g-1/\nu_2)}{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}. \tag{2.27}$$

Then

$$s^2(f)=\frac{1}{s(g)}=\frac{q\nu_3\nu_4\nu_7\nu_8}{\kappa_1 f}\frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{(g-1/\nu_1)(g-1/\nu_2)}. \tag{2.28}$$

Since $\Xi(f)=f\kappa_1/(q\nu_3\nu_4)$ and $\Xi(g)=g\kappa_2/(\nu_5\nu_6)$, we have

$$\Xi s^2(f)=\Xi\Big(\frac{q\nu_3\nu_4\nu_7\nu_8}{f\kappa_1}\Big)\Xi\Big(\frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{(g-1/\nu_1)(g-1/\nu_2)}\Big)=\frac{\nu_3\nu_4}{f}\frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}{(g-1/\nu_1)(g-1/\nu_2)}. \tag{2.29}$$

Therefore we obtain the first equation of (2.21).

It follows from Eq. (2.27) that

$$\begin{aligned} s^2(g)&=s(f)s\Big(\frac{\kappa_1}{q\nu_3\nu_4\nu_7\nu_8}\Big)s\Big(\frac{(g-1/\nu_1)(g-1/\nu_2)}{(g-/\kappa_2)(g-\nu_6/\kappa_2)}\Big)\\ &=\frac{1}{g}\frac{1}{s^2(\kappa_2)}\frac{(1/s^2(f)-1/\nu_7)(1/s^2(f)-1/\nu_8)}{(1/s^2(f)-s^2(1/\nu_4))(1/s^2(f)-s^2(1/\nu_3))}. \end{aligned} \tag{2.30}$$

We apply the operation Ξ. It follows from $T=\Xi s^2$ and Eq. (2.26) that

$$\begin{aligned} T(g)&=\frac{1}{\Xi(g)}\frac{1}{q\kappa_2}\frac{(1/T(f)-1/\Xi(\nu_7))(1/T(f)-1/\Xi(\nu_8))}{(1/T(f)-1/\nu_4)(1/T(f)-1/\nu_3)}\\ &=\frac{\nu_5\nu_6}{gq\kappa_2^2}\frac{\nu_3\nu_4}{\Xi(\nu_7)\Xi(\nu_8)}\frac{(T(f)-\Xi(\nu_8))(T(f)-\Xi(\nu_7))}{(T(f)-\nu_3)(T(f)-\nu_4)}. \end{aligned} \tag{2.31}$$

Then the second equation of (2.21) follows from $\Xi(\nu_7)=\kappa_1/(q\nu_8)$, $\Xi(\nu_8)=\kappa_1/(q\nu_7)$ and the relation $\kappa_1^2\kappa_2^2=q\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8$. Therefore we obtain (i).

(ii) follows from confirming the relation $\Xi(\nu_5/\kappa_2)=1/\nu_6=G[\kappa_2/(\nu_5\nu_6)](\nu_5/\kappa_2)$ and so on. □

Theorem 2.1 asserts that the time evolution is essentially expressed as a composition of Weyl group symmetry and the symmetry originated from a gauge transformation and a dilation.

§ 3. The q-Painlevé equation of type $E_6^{(1)}$

The q-Painlevé equation of type $E_6^{(1)}$ (q-$P(E_6^{(1)})$) was given as

$$\begin{aligned}\frac{(fg-1)(\overline{f}g-1)}{f\overline{f}} &= \frac{(g-1/\nu_1)(g-1/\nu_2)(g-1/\nu_3)(g-1/\nu_4)}{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)},\\ \frac{(fg-1)(f\underline{g}-1)}{g\underline{g}} &= \frac{(f-\nu_1)(f-\nu_2)(f-\nu_3)(f-\nu_4)}{(f-\kappa_1/\nu_7)(f-\kappa_1/\nu_8)}.\end{aligned} \tag{3.1}$$

The equation q-$P(E_6^{(1)})$ is realized by the compatibility condition for the Lax pair L_1 and L_2, where

$$\begin{aligned}L_1 =&\frac{z(g\nu_1-1)(g\nu_2-1)(g\nu_3-1)(g\nu_4-1)}{g(fg-1)(gz-q)} - \frac{(g\kappa_2/\nu_5-1)(g\kappa_2/\nu_6-1){\kappa_1}^2}{qfg\nu_7\nu_8}\\ &+ \frac{(\nu_1-z/q)(\nu_2-z/q)(\nu_3-z/q)(\nu_4-z/q)}{f-z/q}\Big\{\frac{g}{1-gz/q} - T_z^{-1}\Big\}\\ &+ \frac{(\kappa_1/\nu_7-z)(\kappa_1/\nu_8-z)}{q(f-z)}\Big\{\Big(\frac{1}{g}-z\Big)-T_z\Big\},\\ L_2 =&\Big(1-\frac{f}{z}\Big)T+T_z-\Big(\frac{1}{g}-z\Big).\end{aligned} \tag{3.2}$$

Then the linear q-differential equation $L_1y(z)=0$ is written as

$$\begin{aligned}&\Big\{\frac{z(g\nu_1-1)(g\nu_2-1)(g\nu_3-1)(g\nu_4-1)}{g(fg-1)(gz-q)} - \frac{(g\kappa_2/\nu_5-1)(g\kappa_2/\nu_6-1){\kappa_1}^2}{qfg\nu_7\nu_8}\Big\}y(z)\\ &+ \frac{(\nu_1-z/q)(\nu_2-z/q)(\nu_3-z/q)(\nu_4-z/q)}{f-z/q}\Big(\frac{g}{1-gz/q}y(z)-y(z/q)\Big)\\ &+ \frac{(\kappa_1/\nu_7-z)(\kappa_1/\nu_8-z)}{q(f-z)}\Big(\Big(\frac{1}{g}-z\Big)y(z)-y(qz)\Big)=0.\end{aligned} \tag{3.3}$$

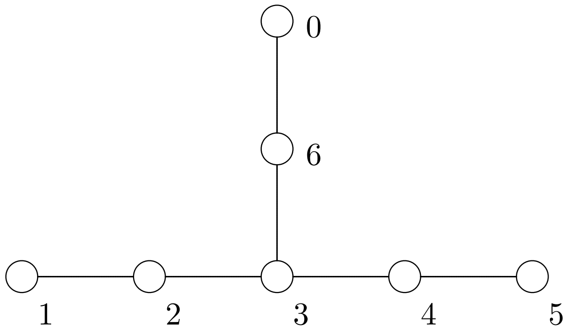
Note that Eq. (3.3) was studied in [4] from the aspect of the initial-value space of q-$P(E_6^{(1)})$, and it was observed that a variant of the q-Heun equation appears.

The Weyl group symmetry of the q-Painlevé equation q-$P(E_6^{(1)})$ was given by the following action of the operators $s_0,\dots,s_6$, π_1 and π_2 for the parameters $(\kappa_1,\kappa_2,\nu_1,\dots,\nu_8)$

and (f, g) in [2].

$$(3.4)\quad \begin{aligned}&s_0 : \nu_7 \leftrightarrow \nu_8, \quad s_1 : \nu_5 \leftrightarrow \nu_6, \quad s_3 : \nu_1 \leftrightarrow \nu_2, \quad s_4 : \nu_2 \leftrightarrow \nu_3, \quad s_5 : \nu_3 \leftrightarrow \nu_4,\\
&s_2 : \nu_1 \to \frac{\kappa_2}{\nu_6},\ \nu_6 \to \frac{\kappa_2}{\nu_1},\ \kappa_1 \to \frac{\kappa_1\kappa_2}{\nu_1\nu_6},\ f \to f\,\frac{\kappa_2(\nu_1 g - 1)}{-(\kappa_2 - \nu_1\nu_6)fg + \nu_1\kappa_2 g - \nu_1\nu_6},\\
&s_6 : \nu_1 \to \frac{\kappa_1}{\nu_7},\ \nu_7 \to \frac{\kappa_1}{\nu_1},\ \kappa_2 \to \frac{\kappa_1\kappa_2}{\nu_1\nu_7},\ g \to g\,\frac{\nu_7(\nu_1 - f)}{\kappa_1 - \nu_7 f + (\nu_1\nu_7 - \kappa_1)fg},\\
&\pi_1 : q \to 1/q,\ \nu_1 \to \nu_2/\kappa_2,\ \nu_2 \to \nu_1/\kappa_2,\ \nu_3 \to 1/\nu_6,\ \nu_4 \to 1/\nu_5,\ \nu_5 \to 1/\nu_4,\\
&\qquad \nu_6 \to 1/\nu_3,\ \nu_7 \to 1/\nu_7,\ \nu_8 \to 1/\nu_8,\ \kappa_1 \to \nu_1\nu_2/(\kappa_1\kappa_2),\ \kappa_2 \to 1/\kappa_2,\\
&\qquad f \to \frac{\nu_1\nu_2(1 - fg)}{\kappa_2\{\nu_1\nu_2 g + f - (\nu_1 + \nu_2)fg\}},\ g \to \kappa_2 g,\\
&\pi_2 : q \to 1/q,\ \nu_1 \to 1/\nu_1,\ \nu_2 \to 1/\nu_2,\ \nu_3 \to 1/\nu_3,\ \nu_4 \to 1/\nu_4,\ \nu_5 \to 1/\nu_8,\\
&\qquad \nu_6 \to 1/\nu_7,\ \nu_7 \to 1/\nu_6,\ \nu_8 \to 1/\nu_5,\ \kappa_1 \to 1/\kappa_2,\ \kappa_2 \to 1/\kappa_1,\ f \leftrightarrow g.\end{aligned}$$

Then we can confirm that these operations satisfy the relations of the extended Weyl group $\widetilde{W}(E_6^{(1)})$ whose Dynkin diagram is as follows.



We investigate the action of the symmetry in Eq. (3.4) on the equation $L_1 y(z) = 0$. It follows immediately that Eq. (3.3) is invariant under the actions of s_0, s_1, s_3, s_4 and s_5. We investigate the action of s_6 on Eq. (3.3). It follows from Eq. (3.4) that ν_1 and κ_1/ν_7 are exchanged by the action of s_6. Thus, we set

$$(3.5)\qquad y(z) = \tilde{y}(z)\frac{(q\nu_1/z; q)_\infty}{(q\kappa_1/(\nu_7 z); q)_\infty}.$$

Then it is shown as the case of $D_5^{(1)}$ that, if $y(z)$ is a solution of Eq. (3.3), then $\tilde{y}(z)$ satisfies

$$(3.6)\quad \begin{aligned}&\Big\{\frac{z(\tilde{g}\tilde{\nu}_1 - 1)(\tilde{g}\nu_2 - 1)(\tilde{g}\nu_3 - 1)(\tilde{g}\nu_4 - 1)}{\tilde{g}(f\tilde{g} - 1)(\tilde{g}z - q)} - \frac{(\tilde{g}\tilde{\kappa}_2/\nu_5 - 1)(\tilde{g}\tilde{\kappa}_2/\nu_6 - 1){\kappa_1}^2}{qf\tilde{g}\tilde{\nu}_7\nu_8}\Big\}\tilde{y}(z)\\
&+ \frac{(\tilde{\nu}_1 - z/q)(\nu_2 - z/q)(\nu_3 - z/q)(\nu_4 - z/q)}{f - z/q}\Big(\frac{\tilde{g}}{1 - \tilde{g}z/q}\tilde{y}(z) - \tilde{y}(z/q)\Big)\\
&+ \frac{(\kappa_1/\tilde{\nu}_7 - z)(\kappa_1/\nu_8 - z)}{q(f - z)}\Big(\Big(\frac{1}{\tilde{g}} - z\Big)\tilde{y}(z) - \tilde{y}(qz)\Big) = 0,\end{aligned}$$

where

$$\tilde{g} = g\,\frac{\nu_7(\nu_1 - f)}{\kappa_1 - \nu_7 f + (\nu_1\nu_7 - \kappa_1)fg},\ \tilde{\nu}_1 = \frac{\kappa_1}{\nu_7},\ \tilde{\nu}_7 = \frac{\kappa_1}{\nu_1}, \tilde{\kappa}_2 = \frac{\kappa_1\kappa_2}{\nu_1\nu_7}. \tag{3.7}$$

Therefore, the gauge transformation defined in Eq. (3.5) induces the symmetry s_6 of the q-Painlevé equation of type $E_6^{(1)}$.

We investigate the actions of the gauge transformation and dilation in Eq. (3.4) on the equation $L_1 y(z) = 0$. Set $z = u/c$, $y(z) = z^d \tilde{y}(u) = z^d \tilde{y}(cz)$. Then the function $\tilde{y}(u)$ satisfies

$$\begin{aligned}
&\left\{\frac{u(g\nu_1 - 1)(g\nu_2 - 1)(g\nu_3 - 1)(g\nu_4 - 1)}{q(g/c)(fg - 1)(ug/(cq) - 1)} - c^2\frac{(g\kappa_2/\nu_5 - 1)(g\kappa_2/\nu_6 - 1){\kappa_1}^2}{qfg\nu_7\nu_8}\right\}\tilde{y}(u)\\
&+ \frac{(c\nu_1 - u/q)(c\nu_2 - u/q)(c\nu_3 - u/q)(c\nu_4 - u/q)}{cf - u/q}\left\{\frac{g/c}{1 - ug/(cq)}\tilde{y}(u) - \frac{\tilde{y}(u/q)}{cq^d}\right\}\\
&+ \frac{(c\kappa_1/\nu_7 - u)(c\kappa_1/\nu_8 - u)}{q(cf - u)}\left\{\left(\frac{c}{g} - u\right)\tilde{y}(u) - cq^d\tilde{y}(qu)\right\} = 0.
\end{aligned} \tag{3.8}$$

If $cq^d = 1$, then it is written in the form of Eq.(3.3), where the parameters are changed as

$$\begin{aligned}
&(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5/\kappa_2, \nu_6/\kappa_2, \nu_7/\kappa_1, \nu_8/\kappa_1, f, g)\\
&\mapsto (c\nu_1, c\nu_2, c\nu_3, c\nu_4, \nu_5/\kappa_2/c, \nu_6/\kappa_2/c, \nu_7/\kappa_1/c, \nu_8/\kappa_1/c, cf, g/c).
\end{aligned} \tag{3.9}$$

We denote the transformation of the parameters in Eq. (3.9) by $S_{E_6}[c]$.

The time evolution of the q-Painlevé equation of type $E_6^{(1)}$ is expressed as the following theorem.

Theorem 3.1. *(i) Let Ξ be the transformation of the parameters defined by*

$$\begin{aligned}
&(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8; \kappa_1, \kappa_2; f, g)\\
&\mapsto \Big(\frac{\nu_1\kappa_2}{\nu_5\nu_6\kappa_1^2}, \frac{\nu_2\kappa_2}{\nu_5\nu_6\kappa_1^2}, \frac{\nu_3\kappa_2}{\nu_5\nu_6\kappa_1^2}, \frac{\nu_4\kappa_2}{\nu_5\nu_6\kappa_1^2}, \frac{q\nu_5\kappa_1}{\kappa_2}, \frac{q\nu_6\kappa_1}{\kappa_2}, \frac{\kappa_1}{q\nu_8}, \frac{\kappa_1}{q\nu_7};\\
&\qquad \frac{\kappa_2}{q\nu_5\nu_6\nu_7\nu_8}, \frac{\kappa_2}{q\nu_5\nu_6\kappa_1}; \frac{f\kappa_2}{\nu_5\nu_6\kappa_1^2}, \frac{g\nu_5\nu_6\kappa_1^2}{\kappa_2}\Big).
\end{aligned} \tag{3.10}$$

Let T be the transformation of the parameters defined by

$$T = \Xi \cdot (\pi_1\pi_2 s_4 s_5 s_3 s_6 s_4 s_3 s_0 s_6)^2. \tag{3.11}$$

Then

$$\begin{aligned}
\frac{(T(f)g - 1)(fg - 1)}{T(f)f} &= \frac{(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)}{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)},\\
\frac{(T(f)g - 1)(T(f)T(g) - 1)}{gT(g)} &= \frac{(T(f) - \nu_1)(T(f) - \nu_2)(T(f) - \nu_3)(T(f) - \nu_4)}{(T(f) - T(\kappa_1/\nu_7))(T(f) - T(\kappa_1/\nu_8))}.
\end{aligned} \tag{3.12}$$

Namely the operator T represents the time evolution of q-$P(E_6^{(1)})$ (see Eq. (3.1)). On the other parameters, we have

$$T(\nu_i) = \nu_i \ (i = 1, 2, \dots, 8), \ T(\kappa_1) = \kappa_1/q, \ T(\kappa_2) = q\kappa_2. \tag{3.13}$$

(ii) For any element ζ generated by $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5/\kappa_2, \nu_6/\kappa_2, \nu_7/\kappa_1, \ \nu_8/\kappa_1, \ f$ and g, we have

$$\Xi(\zeta) = S_{E_6}\Big[\frac{\kappa_2}{\nu_5\nu_6\kappa_1^2}\Big](\zeta). \tag{3.14}$$

Proof. Set $s = \pi_1\pi_2 s_4 s_5 s_3 s_6 s_4 s_3 s_0 s_6$. It follows from Eq. (3.4) that

$$\begin{aligned} &s(\nu_1) = \kappa_2/\nu_4, \ s(\nu_2) = \kappa_2/\nu_3, \ s(\nu_3) = \kappa_2/\nu_2, \ s(\nu_4) = \kappa_2/\nu_1, \\ &s(\nu_5) = \nu_8, \ s(\nu_6) = \nu_7, \ s(\nu_7) = \kappa_2/\nu_5, \ s(\nu_8) = \kappa_2/\nu_6, \\ &s(\kappa_1) = \kappa_2, \ s(\kappa_2) = \kappa_1\kappa_2^3/(\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6) = q\nu_7\nu_8\kappa_2/\kappa_1. \end{aligned} \tag{3.15}$$

Then

(3.16)
$$\begin{aligned} &s^2(\nu_1) = q\nu_1\nu_7\nu_8/\kappa_1, \ s^2(\nu_2) = q\nu_2\nu_7\nu_8/\kappa_1, \ s^2(\nu_3) = q\nu_3\nu_7\nu_8/\kappa_1, \ s^2(\nu_4) = q\nu_4\nu_7\nu_8/\kappa_1, \\ &s^2(\nu_5) = \kappa_2/\nu_6, \ s^2(\nu_6) = \kappa_2/\nu_5, \ s^2(\nu_7) = q\nu_7\kappa_2/\kappa_1, \ s^2(\nu_8) = q\nu_8\kappa_2/\kappa_1, \\ &s^2(\kappa_1) = q\nu_7\nu_8\kappa_2/\kappa_1, \ s^2(\kappa_2) = q^2\nu_7\nu_8\kappa_2^2/(\nu_5\nu_6\kappa_1). \end{aligned}$$

Let Ξ be the operator defined in Eq. (3.10). Then we have

$$\Xi s^2(\nu_i) = \nu_i, \ (i = 1, 2, \dots, 8), \ \Xi s^2(\kappa_1) = \kappa_1/q, \ \Xi s^2(\kappa_2) = q\kappa_2. \tag{3.17}$$

Hence we obtain Eq. (3.13).

It follows from Eq. (3.4) that $s(f) = \kappa_2 g$ and

$$\frac{1}{\kappa_2 s(g)} = g + \frac{f(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)}{(1 - fg)(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}. \tag{3.18}$$

Then

$$\frac{1}{s^2(f)} = \frac{1}{s(\kappa_2)s(g)} = \frac{\kappa_1}{q\nu_7\nu_8}\Big\{g + \frac{f(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)}{(1 - fg)(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}\Big\}. \tag{3.19}$$

It follows from Eq. (3.10) that

(3.20)
$$\begin{aligned} \frac{1}{T(f)} &= \frac{1}{\Xi s^2(f)} = \Xi\Big(\frac{\kappa_1}{q\nu_7\nu_8}\Big)\Xi\Big\{g + \frac{f(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)}{(1 - fg)(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}\Big\} \\ &= \frac{\kappa_2}{\nu_5\nu_6\kappa_1^2}\frac{\nu_5\nu_6\kappa_1^2}{\kappa_2}\Big\{g + \frac{f(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)}{(1 - fg)(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}\Big\}. \end{aligned}$$

It is equivalent to

$$\frac{(T(f)g-1)(fg-1)}{T(f)f} = \frac{(g-1/\nu_1)(g-1/\nu_2)(g-1/\nu_3)(g-1/\nu_4)}{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}, \tag{3.21}$$

and we obtain the first equation of (3.12).

It follows from Eq. (3.18) that

$$\begin{aligned} \frac{1}{s^2(g)} &= s(\kappa_2)s\Big(g + \frac{f(g-1/\nu_1)(g-1/\nu_2)(g-1/\nu_3)(g-1/\nu_4)}{(1-fg)(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)}\Big) \\ &= s(\kappa_2 g) + \frac{s(f)\prod_{j=1}^4 (s(\kappa_2 g)-s(\kappa_2/\nu_j))}{s(\kappa_2)(1-s(f)s(g))(s(\kappa_2 g)-s(\nu_5))(s(\kappa_2 g)-s(\nu_6))} \\ &= s^2(f) + \frac{\kappa_2 g(s^2(f)-s^2(\nu_4))(s^2(f)-s^2(\nu_3))(s^2(f)-s^2(\nu_2))(s^2(f)-s^2(\nu_1))}{(s(\kappa_2)-\kappa_2 g s^2(f))(s^2(f)-\nu_8)(s^2(f)-\nu_7)}. \end{aligned} \tag{3.22}$$

We apply the operation Ξ. Then we have

$$\begin{aligned} \frac{1}{T(g)} &= T(f) + \frac{(T(f)-T(\nu_4))(T(f)-T(\nu_3))(T(f)-T(\nu_2))(T(f)-T(\nu_1))}{(\Xi(q\nu_7\nu_8\kappa_2/\kappa_1)/\Xi(\kappa_2 g)-T(f))(T(f)-\Xi(\nu_8))(T(f)-\Xi s(\nu_7))} \\ &= T(f) + \frac{g(T(f)-\nu_1)(T(f)-\nu_2)(T(f)-\nu_3)(T(f)-\nu_4)}{(1-T(f)g)(T(f)-\kappa_1/(q\nu_7))(T(f)-\kappa_1/(q\nu_8))}. \end{aligned} \tag{3.23}$$

Therefore we obtain the second equation of (3.12).

(ii) follows similarly to the case $D_5^{(1)}$. □

Note that the expression $(\pi_1\pi_2 s_4 s_5 s_3 s_6 s_4 s_3 s_0 s_6)^2$ in Eq. (3.11) can be essentially found in Tsuda's paper [6].

§ 4. The q-Painlevé equation of type $E_7^{(1)}$

The q-Painlevé equation of type $E_7^{(1)}$ (q-$P(E_7^{(1)})$) was given as

$$\begin{aligned} \frac{(fg-\kappa_1/\kappa_2)(\overline{f}g-\kappa_1/(q\kappa_2))}{(fg-1)(\overline{f}g-1)} &= \frac{(g-\nu_5/\kappa_2)(g-\nu_6/\kappa_2)(g-\nu_7/\kappa_2)(g-\nu_8/\kappa_2)}{(g-1/\nu_1)(g-1/\nu_2)(g-1/\nu_3)(g-1/\nu_4)}, \\ \frac{(fg-\kappa_1/\kappa_2)(f\underline{g}-q\kappa_1/\kappa_2)}{(fg-1)(f\underline{g}-1)} &= \frac{(f-\kappa_1/\nu_5)(f-\kappa_1/\nu_6)(f-\kappa_1/\nu_7)(f-\kappa_1/\nu_8)}{(f-\nu_1)(f-\nu_2)(f-\nu_3)(f-\nu_4)}. \end{aligned} \tag{4.1}$$

The equation q-$P(E_7^{(1)})$ is realized by the compatibility condition for the Lax pair L_1 and L_2, where

$$(4.2)\quad \begin{aligned} L_1 = & \frac{q(\kappa_1-\kappa_2)(g\kappa_2-\nu_5)(g\kappa_2-\nu_6)(g\kappa_2-\nu_7)(g\kappa_2-\nu_8)}{g\kappa_1{\kappa_2}^2(fg\kappa_2-\kappa_1)(g\kappa_2 z-\kappa_1)} \\ & - \frac{q(\kappa_1-\kappa_2)(g\nu_1-1)(g\nu_2-1)(g\nu_3-1)(g\nu_4-1)}{g(fg-1)\kappa_1\nu_1\nu_2\nu_3\nu_4(gz-q)} \\ & + \frac{(q\nu_1-z)(q\nu_2-z)(q\nu_3-z)(q\nu_4-z)\{(g\kappa_2 z-\kappa_1 q)-\kappa_1(gz-q)T_z^{-1}\}}{\kappa_1 q\nu_1\nu_2\nu_3\nu_4(fq-z)z^2(q-gz)} \\ & - \frac{q(\kappa_1-\nu_5 z)(\kappa_1-\nu_6 z)(\kappa_1-\nu_7 z)(\kappa_1-\nu_8 z)\{\kappa_1(gz-1)-(g\kappa_2 z-\kappa_1)T_z\}}{{\kappa_1}^4(f-z)z^2(g\kappa_2 z-\kappa_1)}, \\ L_2 = & (1-zg\kappa_2/\kappa_1)T_z-(1-zg)+z(z-f)gT\,. \end{aligned}$$

Then the linear q-differential equation $L_1 y(z)=0$ is written as

$$(4.3)\quad \begin{aligned} & \Big\{\frac{q(\kappa_1-\kappa_2)(g\kappa_2-\nu_5)(g\kappa_2-\nu_6)(g\kappa_2-\nu_7)(g\kappa_2-\nu_8)}{g\kappa_1{\kappa_2}^2(fg\kappa_2-\kappa_1)(g\kappa_2 z-\kappa_1)} \\ & - \frac{q(\kappa_1-\kappa_2)(g\nu_1-1)(g\nu_2-1)(g\nu_3-1)(g\nu_4-1)}{g(fg-1)\kappa_1\nu_1\nu_2\nu_3\nu_4(gz-q)}\Big\}y(z) \\ & + \frac{(q\nu_1-z)(q\nu_2-z)(q\nu_3-z)(q\nu_4-z)}{q\nu_1\nu_2\nu_3\nu_4(fq-z)z^2}\Big\{\frac{g\kappa_2 z-\kappa_1 q}{\kappa_1(q-gz)}y(z)+y(z/q)\Big\} \\ & + \frac{q(\kappa_1-\nu_5 z)(\kappa_1-\nu_6 z)(\kappa_1-\nu_7 z)(\kappa_1-\nu_8 z)}{{\kappa_1}^4(f-z)z^2}\Big\{\frac{\kappa_1(1-gz)}{g\kappa_2 z-\kappa_1}y(z)+y(qz)\Big\}=0. \end{aligned}$$

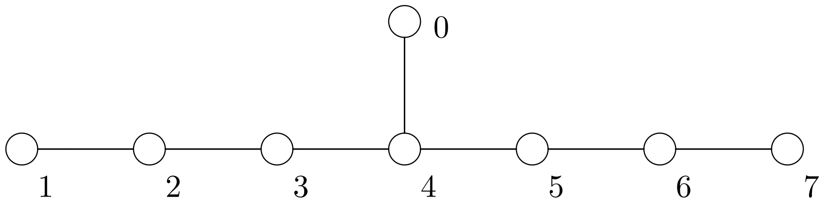
Note that Eq. (4.3) was studied in [4] from the aspect of the initial-value space of q-$P(E_7^{(1)})$, and it was observed that a variant of the q-Heun equation appears.

The Weyl group symmetry of the q-Painlevé equation q-$P(E_7^{(1)})$ was given by the following action of the operators $s_0,\dots,s_6$, π_1 and π_2 for the parameters $(\kappa_1,\kappa_2,\nu_1,\dots,\nu_8)$ and (f,g) in [2].

(4.4)

$$\begin{aligned} & s_0:\kappa_1\leftrightarrow\kappa_2,\ f\to 1/g,\ g\to 1/f, \\ & s_1:\nu_3\leftrightarrow\nu_4,\quad s_2:\ \nu_2\leftrightarrow\nu_3,\quad s_3:\nu_1\leftrightarrow\nu_2, \\ & s_5:\nu_5\leftrightarrow\nu_6,\quad s_6:\ \nu_6\leftrightarrow\nu_7,\quad s_7:\ \nu_7\leftrightarrow\nu_8 \\ & s_4:\nu_1\to\frac{\kappa_2}{\nu_5},\ \nu_5\to\frac{\kappa_2}{\nu_1},\ \kappa_1\to\frac{\kappa_1\kappa_2}{\nu_1\nu_5}, \\ & \qquad f\to\frac{-\kappa_2(\nu_1\nu_5-\kappa_1)fg-\nu_5(\kappa_1-\kappa_2)f+\kappa_1(\nu_1\nu_5-\kappa_2)}{\nu_5\{-(\nu_1\nu_5-\kappa_2)fg+\nu_1(\kappa_1-\kappa_2)g+(\nu_1\nu_5-\kappa_1)\}}, \\ & \pi:q\to 1/q,\ \nu_1\to 1/\nu_5,\ \nu_2\to 1/\nu_6,\ \nu_3\to 1/\nu_7,\ \nu_4\to 1/\nu_8,\ \nu_5\to 1/\nu_1,\ \nu_6\to 1/\nu_2, \\ & \qquad \nu_7\to 1/\nu_3,\ \nu_8\to 1/\nu_4,\ \kappa_1\to 1/\kappa_1,\ \kappa_2\to 1/\kappa_2,\ f\to f/\kappa_1,\ g\to\kappa_2 g, \end{aligned}$$

Then we can confirm that these operations satisfy the relations of the extended Weyl group $\widetilde{W}(E_7^{(1)})$ whose Dynkin diagram is as follows.



We investigate the action of the symmetry in Eq. (4.4) on the equation $L_1 y(z) = 0$. It follows immediately that Eq. (4.3) is invariant under the actions of s_1, s_2, s_3, s_5, s_6 and s_7. The action of s_4 does not exchange ν_1 and κ_1/ν_5, which are related with zeros of the coefficient of $y(z/q)$ and $y(qz)$, and the action of $s_0 s_4 s_0$ exchanges them. More precisely, the action is written as

$$(4.5)\qquad \begin{aligned} s_0 s_4 s_0 : \nu_1 \to \frac{\kappa_1}{\nu_5},\ \nu_5 \to \frac{\kappa_1}{\nu_1},\ \kappa_2 \to \frac{\kappa_1\kappa_2}{\nu_1\nu_5},& \\ g \to \frac{\nu_5\{-(\nu_1\nu_5 - \kappa_1) + \nu_1(\kappa_2 - \kappa_1)g + (\nu_1\nu_5 - \kappa_2)fg\}}{-\kappa_1(\nu_1\nu_5 - \kappa_2) - \nu_5(\kappa_2 - \kappa_1)f + \kappa_2(\nu_1\nu_5 - \kappa_1)fg}&. \end{aligned}$$

Set

$$(4.6)\qquad y(z) = \tilde{y}(z)\frac{(q\nu_1/z;q)_\infty}{(q\kappa_1/(\nu_5 z);q)_\infty}.$$

It follows that, if $y(z)$ is a solution of Eq. (4.3), then $\tilde{y}(z)$ satisfies the equation as Eq. (4.3) whose parameters are changed by the action of $s_0 s_4 s_0$.

We investigate the action of another gauge transformation in Eq. (4.4) on the equation $L_1 y(z) = 0$. Set $z = u/c$ and $y(z) = \tilde{y}(u)$. If $y(z)$ is a solution of Eq. (4.3), then the function $\tilde{y}(u)$ satisfies Eq. (4.3) for the variable u, where the parameters are changed as

$$(4.7)\qquad \begin{aligned} &(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5/\kappa_1, \nu_6/\kappa_1, \nu_7/\kappa_1, \nu_8/\kappa_1, \kappa_2/\kappa_1, f, g) \\ &\mapsto (c\nu_1, c\nu_2, c\nu_3, c\nu_4, (\nu_5/\kappa_1)/c, (\nu_6/\kappa_1)/c, (\nu_7/\kappa_1)/c, (\nu_8/\kappa_1)/c, \kappa_2/\kappa_1, cf, g/c). \end{aligned}$$

We denote the transformation of the parameters in Eq. (4.7) by $S_{E_7}[c]$.

The time evolution of the q-Painlevé equation of type $E_7^{(1)}$ is expressed as the following theorem.

Theorem 4.1. *(i) Let Ξ be the transformation of the parameters defined by*

$$(4.8)\qquad \begin{aligned} &(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7, \nu_8; \kappa_1, \kappa_2; f, g) \\ &\mapsto \Big(\frac{\nu_1\kappa_1}{q\kappa_2}, \frac{\nu_2\kappa_1}{q\kappa_2}, \frac{\nu_3\kappa_1}{q\kappa_2}, \frac{\nu_4\kappa_1}{q\kappa_2}, \frac{\nu_5\kappa_1}{q\kappa_2}, \frac{\nu_6\kappa_1}{q\kappa_2}, \frac{\nu_7\kappa_1}{q\kappa_2}, \frac{\nu_8\kappa_1}{q\kappa_2}; \frac{\kappa_1^3}{q^2\kappa_2^2}, \frac{\kappa_1^2}{q^2\kappa_2}; \frac{f\kappa_1}{q\kappa_2}, \frac{gq\kappa_2}{\kappa_1}\Big). \end{aligned}$$

Let T be the transformation of the parameters defined by

$$T = \Xi \cdot (s_4 s_5 s_1 s_4 s_6 s_5 s_1 s_2 s_4 s_7 s_6 s_5 s_1 s_2 s_3 s_4 s_0)^2. \tag{4.9}$$

Then

(4.10)
$$\frac{(T(f)g - \kappa_1/(q\kappa_2))(fg - \kappa_1/\kappa_2)}{(T(f)g - 1)(fg - 1)} = \frac{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)(g - \nu_7/\kappa_2)(g - \nu_8/\kappa_2)}{(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)},$$
$$\frac{(T(g)T(f) - \kappa_1/(q^2\kappa_2))(T(f)g - \kappa_1/(q\kappa_2))}{(T(g)T(f) - 1)(T(f)g - 1)}$$
$$= \frac{(T(f) - T(\kappa_1/\nu_5))(T(f) - T(\kappa_1/\nu_6))(T(f) - T(\kappa_1/\nu_7))(T(f) - T(\kappa_1/\nu_8))}{(T(f) - \nu_1)(T(f) - \nu_2)(T(f) - \nu_3)(T(f) - \nu_4)}.$$

Namely the operator T represents the time evolution of q-$P(E_7^{(1)})$ (see Eq. (4.1)). On the other parameters, we have

$$T(\nu_i) = \nu_i \ (i = 1, 2, \dots, 8), \ T(\kappa_1) = \kappa_1/q, \ T(\kappa_2) = q\kappa_2. \tag{4.11}$$

(ii) For any elements ζ generated by $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5/\kappa_1, \nu_6/\kappa_1, \nu_7/\kappa_1, \nu_8/\kappa_1, \kappa_2/\kappa_1, f$ and g, we have

$$\Xi(\zeta) = S_{E_7}\left[\frac{\kappa_1}{q\kappa_2}\right](\zeta). \tag{4.12}$$

Proof. Set $s = s_4 s_5 s_1 s_4 s_6 s_5 s_1 s_2 s_4 s_7 s_6 s_5 s_1 s_2 s_3 s_4 s_0$. Then it follows from Eq. (4.4) that $s(\nu_i) = \kappa_2/\nu_{9-i}$ $(i = 1, \dots, 8)$, $s(\kappa_1) = \kappa_2$, $s(\kappa_2) = q\kappa_2^2/\kappa_1$, $s(f) = 1/g$ and

(4.13)
$$\frac{(s(g)/g - \kappa_1/(q\kappa_2))(fg - 1)}{(s(g)/g - 1)(fg - \kappa_1/\kappa_2)} = \frac{\kappa_1/(q\kappa_2)(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)}{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)(g - \nu_7/\kappa_2)(g - \nu_8/\kappa_2)}.$$

Therefore, $s^2(\nu_j) = q\nu_j\kappa_2/\kappa_1$ $(j = 1, \dots 8)$, $s^2(\kappa_1) = q\kappa_2^2/\kappa_1$ and $s^2(\kappa_2) = q^3\kappa_2^3/\kappa_1^2$.

Let Ξ be the operator defined in Eq. (4.8). We have

$$\Xi s^2(\nu_i) = \nu_i, \ (i = 1, 2, \dots, 8), \ \Xi s^2(\kappa_1) = \kappa_1/q, \ \Xi s^2(\kappa_2) = q\kappa_2. \tag{4.14}$$

Then we obtain Eq. (4.11). Note that $\Xi(\kappa_2/\kappa_1) = \kappa_2/\kappa_1$ and $\Xi(\nu_i/\kappa_2) = (q\kappa_2/\kappa_1)(\nu_i/\kappa_2)$ $(i = 1, \dots, 8)$.

It follows from Eq. (4.13) and $s^2(f) = 1/s(g)$ that

$$\frac{(q\kappa_2/\kappa_1 - gs^2(f))(fg - 1)}{(1 - gs^2(f))(fg - \kappa_1/\kappa_2)} = \frac{(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)}{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)(g - \nu_7/\kappa_2)(g - \nu_8/\kappa_2)}. \tag{4.15}$$

We apply the operation Ξ. It follows from Eq. (4.8) that

(4.16)
$$\frac{(1 - gT(f))(fg - 1)}{(\kappa_1/(q\kappa_2) - gT(f))(fg - \kappa_1/\kappa_2)} = \frac{(g - 1/\nu_1)(g - 1/\nu_2)(g - 1/\nu_3)(g - 1/\nu_4)}{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)(g - \nu_7/\kappa_2)(g - \nu_8/\kappa_2)}.$$

Therefore we obtain the first equation of (4.10).

We apply the operation s to Eq. (4.13). It follows from $s(f) = 1/g$ and $s(g) = 1/s^2(f)$ that

$$\begin{aligned}(4.17)\qquad &\frac{(s^2(g)s^2(f) - \kappa_1/\kappa_2/q^2)(gs^2(f) - 1)}{(s^2(g)s^2(f) - 1)(gs^2(f) - q\kappa_2/\kappa_1)} \\ &= \frac{(s^2(f) - \kappa_2/\nu_5)(s^2(f) - \kappa_2/\nu_6)(s^2(f) - \kappa_2/\nu_7)(s^2(f) - \kappa_2/\nu_8)}{(s^2(f) - s^2(\nu_1))(s^2(f) - s^2(\nu_2))(s^2(f) - s^2(\nu_3))(s^2(f) - s^2(\nu_4))}.\end{aligned}$$

By applying the operation Ξ, we obtain the second equation of (4.10). □

Note that the expression $(s_4 s_5 s_1 s_4 s_6 s_5 s_1 s_2 s_4 s_7 s_6 s_5 s_1 s_2 s_3 s_4 s_0)^2$ in Eq. (4.9) can be essentially found in [6].

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