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HKLL bulk reconstruction for small Δ

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Abstract: We discuss the extension of the HKLL (Hamilton, Kabat, Lifschytz, and Lowe) bulk reconstruction for non-interacting scalar fields corresponding to conformal weights Δ smaller than the original condition $\Delta > d-1$. We give explicit formulas for the cases $d-2 < \Delta \leq d-1$ and $\Delta = d-s$ with integer s. In the latter case we show that smearing CFT fields over a region of the boundary consisting of points light-like separated from the bulk point is sufficient for bulk reconstruction, whereas in general smearing over all light-like and space-like separated points is required.

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Contents

1 Introduction and motivation 1

2 Review of HKLL bulk reconstruction 2
- 2.1 BDHM relation 2
- 2.2 Bulk-boundary mapping 4

3 Bulk reconstruction for the range $d-2<\Delta\leq d-1$ 6

4 Bulk reconstruction for $\Delta = d-s$ with an integer s 7
- 4.1 Results for $\Delta = d-s$ with odd d 8
- 4.2 Results for $\Delta = d-s$ with even d 8

5 Bulk reconstruction at generic points for small integer Δ 9
- 5.1 Bulk reconstruction for $\Delta = d-1$ with odd d at generic bulk points 9
- 5.2 Bulk reconstruction for $\Delta = d-2$ with odd d at generic bulk points 10

6 Conclusions and discussion 11

A Canonical quantization of the free scalar field 11
- A.1 Radial quantization 13
- A.2 Boundary conditions 14
- A.3 Spectrum and eigenfunctions 14
  - A.3.1 Δ_+ case 15
  - A.3.2 Δ_- case 15
  - A.3.3 Final form of the solution 15

B Derivation of the bulk reconstruction for $\Delta > d-1$ with odd d 16

C Geometry of the AdS space 17
- C.1 Geodesics 17
- C.2 Infinitesimal transformations 17

D Bulk reconstruction for $\Delta > d-1$ at generic bulk points 18
- D.1 The explicit form of $I(Y,x)$ 18
- D.2 The explicit form of $T(Y,x)$ 18

E Bulk reconstruction for $\Delta > d-2$ 19

F Details of the derivation of bulk reconstruction for small integer Δ 20
- F.1 Bulk reconstruction for $\Delta = d-1$ at generic bulk points 20
- F.2 Bulk reconstruction for $\Delta = d-2$ at generic bulk points 21
  - F.2.1 Delta function identities 21
  - F.2.2 Expansions 22

G BDHM relation for $\Delta = d-1$ 23
- G.1 Calculation of A 25
- G.2 The result 26

1 Introduction and motivation

The AdS/CFT correspondence [1, 2] plays a central role to investigate the holographic nature of gravity, which may give a hint for quantum gravity. Even though much evidence has appeared after the first proposal, the fundamental mechanism why the AdS/CFT correspondence holds has not been completely understood yet. While the correspondence may be explained by the close string/open string duality, an alternative but more universal mechanism might exist because of the holographic nature of gravity.

One of the key questions one may naturally ask is how the additional dimension of the AdS emerges from CFT, which lives on the boundary of the AdS spacetime. An approach to this problem, called the HKLL (Hamilton, Kabat, Lifschytz, and Lowe) bulk reconstruction, is to relate a bulk local field operator in the AdS to CFT operators at its boundary [3, 4]. For example, let us consider a massive free scalar field operator $\Phi(X)$ with mass squared $m^2 = \Delta(\Delta - d)/R^2$ in the AdS with a radius R. Then one may define the CFT field operator $O(t, \Omega)$ with a conformal weight Δ from $\Phi(X)$ through the BDHM relation [5] as

$$O(x) = \lim_{\rho\to\infty} (\sinh \rho)^{\Delta} \Phi(t, \rho, \Omega), \quad X := (t, \rho, \Omega), \ x := (t, \Omega), \tag{1.1}$$

where ρ is the radial coordinate of the $d+1$ dimensional AdS with its boundary at $\rho \to \infty$, t is a time coordinate, and Ω is a $d-1$ dimensional angular variable (see section 2). The HKLL bulk reconstruction is the inverse mapping: using this $O(t, \Omega)$, the bulk field can be reconstructed as

$$\Phi(X) = \int_{\Sigma_X} dy\, K(X, y) O(y), \tag{1.2}$$

where $K(X, y)$ is a smearing function, and the integration at the boundary should be performed in a region Σ_X space-like separated from the bulk point X. We refer to [6, 7] for recent reviews.

The result of this explicit construction can be elegantly reproduced in a somewhat abstract way [8]. The starting point of the abstract construction is the space-like Green function in the bulk (which vanishes if its arguments are *not* space-like separated). With the help of the space-like Green function not only the free case is easily reproduced but can also be used to introduce interactions. In the original HKLL paper (and also in this paper) the case of a free massive scalar is considered. See also [9] for an alternative derivation based on Gel'fand-Graev-Radon transforms. Later the reconstruction has been extended to higher spins as well [10–14]. Recently an interesting connection between the bulk reconstruction and the theory of quantum error correcting codes was pointed out [15].

The HKLL bulk reconstruction provides the operator to operator relation in the AdS/CFT correspondence. Recently Terashima argued under reasonable assumptions in the large N limit that the relation (1.2) follows from CFT considerations *without* assuming the BDHM relation [16]. In other words, the BDHM relation (1.1) is shown explicitly. Moreover, he claimed that the integration in the space-like region Σ_X in (1.2) can be effectively replaced by an integration over a much smaller region $\Sigma_X^{(0)}$, which is the boundary of Σ_X and consists of boundary points light-like separated from X [17]. (See also [18].)

Although it was not explicitly mentioned in the original papers, (1.2) holds only for $\Delta > d-1$, due to the convergence for the integral. For applications of the AdS/CFT correspondence in the case of supersymmetric gauge theories and in particular in the prime example of the $\mathcal{N}=4$ SUSY U(N) gauge theory in $d=4$ dimensions, this restriction is not essential since the conformal dimensions of physically relevant operators are typically (much) larger than this lower bound. However, there is an other family of models often used in the AdS/CFT context, namely, the O(N) vector models and their holographic duals: higher spin theories in the bulk [19, 20]. In the most interesting $d=3$ case, for example, the simplest singlet operator has $\Delta=1$ $(d-2)$ and its square, the only relevant operator which can be used to introduce interactions, is of $\Delta=2$ $(d-1)$. These singlet scalar operators in the free $O(N)$ vector model cannot be related to the bulk operator by blindly applying (1.2).

The case $\Delta = d-1$ was studied in [10] in Poincare coordinates. It was found that in this case the support of the smearing function is the intersection of the light-cone of the bulk point and the boundary. In [21] the range of allowed Δ was extended to $d/2 \leq \Delta \leq d-1$ by analytic continuation. Our purpose here is to find a direct derivation of the generalized HKLL formula for Δ values below the original lower bound $d-1$.

In this paper we present two results for conformal weights smaller than the lower bound mentioned above. We derive an extension of the HKLL bulk reconstruction to the range $d-2 < \Delta \leq d-1$, which is the first main result and is given in (3.4). Our result agrees with that of [21] (if their limit is explicitly evaluated) in the range where they overlap. We cannot confirm Terashima's claim in general, but show that the bulk operator $\Phi(X)$ is expressed in terms of CFT operators living on $\Sigma_X^{(0)}$ (points light-like separated from X at the boundary) for the special cases $\Delta = d-s$, where s is a positive integer. (s is limited by the requirement that the conformal weight satisfies the unitarity bound $\Delta > (d-2)/2$.) This is the second main result of this paper.

2 Review of HKLL bulk reconstruction

In this section we review the HKLL bulk reconstruction [3, 4] for a massive free scalar boson field with conformal weight $\Delta > d-1$ in $d+1$ dimensional AdS spacetime. This construction is very well-known, and our pupose here is to introduce our notation and conventions and also some tools which will be needed later in the paper when we extend the validity of the construction to smaller values of Δ.

2.1 BDHM relation

In the Lorenttzian AdS$_{d+1}$ space we will use the usual global coordinates (t, ρ, n^i) $(n \cdot n = 1)$ with the metric

$$\mathrm{d}s^2 = R^2(\mathrm{d}\rho)^2 - R^2(\cosh\rho)^2(\mathrm{d}t)^2 + R^2(\sinh\rho)^2\mathrm{d}n^i\mathrm{d}n^i, \tag{2.1}$$

where R is the AdS radius. We will denote a bulk point in AdS$_{d+1}$ by Y with global coordinates $Y^\mu = (t, \rho, n^i)$ (with corresponding derivatives $\partial_\mu = \partial/\partial Y^\mu$). Similarly a boundary point will be denoted by x with coordinates $x^A : (\tilde{t}, \tilde{n}^i)$ and derivatives $\partial_A =$

JHEP02(2022)015

$\partial/\partial x^A$. We will also use the "flat" coordinates ($T = Rt, y^i = R\sinh\rho\, n^i$) and the notation $y = \sqrt{y^i y^i} = R\sinh\rho$ for the radial coordinate. The metric in these coordinates is given by

$$\mathrm{d}s^2 = -\frac{y^2+R^2}{R^2}(\mathrm{d}T)^2 + \left(\delta_{ij} - \frac{y^i y^j}{y^2+R^2}\right)\mathrm{d}y^i\mathrm{d}y^j. \tag{2.2}$$

In appendix A, we review the complete canonical quantization of a free bulk scalar field Φ in terms of canonical creation and annihilation operators $\mathcal{A}^{+}_{n\ell\underline{m}}$ and $\mathcal{A}_{n\ell\underline{m}}$, which is given by

$$\Phi(t,y,\Omega) = \sum_{n\ell\underline{m}} \sqrt{\frac{\mathcal{N}R}{2\nu_{n\ell}}}\Big\{u_{n\ell}(y)Y_{\ell\underline{m}}(\Omega)\mathcal{A}_{n\ell\underline{m}}\,\mathrm{e}^{-i\nu_{n\ell}t} + u_{n\ell}(y)Y_{\ell\underline{m}}(\Omega)\mathcal{A}^{\dagger}_{n\ell\underline{m}}\,\mathrm{e}^{i\nu_{n\ell}t}\Big\}, \tag{2.3}$$

where $\mathcal{N}$ is a normalization constant related to the free Lagrangian, $\nu_{n\ell} = \Delta + \ell + 2n$ is the eigenfrequency, $u_{n\ell}(y)$ is the radial wave function, and $Y_{\ell\underline{m}}(\Omega)$ are hyper-spherical harmonics[1] for the $d-1$ dimensional sphere parametrized alternatively by the angular variables Ω or by the d dimensional unit vector n^i.

The value of Φ at the middle of (the global coordinate system of) the AdS space becomes

$$\begin{aligned}\mathcal{A}(t) = \Phi(t,0,\Omega) = \sum_n \sqrt{\frac{\mathcal{N}R}{2\nu_{n0}}}\Bigg\{&\mathrm{e}^{-i\nu_{n0}t}\,(-1)^n\frac{P_n(d/2)}{n!}\mathcal{N}_{n0}\frac{1}{\sqrt{\Omega_d}}\mathcal{A}_{n0\underline{0}}\\ &+\mathrm{e}^{i\nu_{n0}t}\,(-1)^n\frac{P_n(d/2)}{n!}\mathcal{N}_{n0}\frac{1}{\sqrt{\Omega_d}}\mathcal{A}^{\dagger}_{n0\underline{0}}\Bigg\},\end{aligned} \tag{2.4}$$

where $P_n(z)$ is the Pochhammer symbol, defined by

$$P_n(z) := \frac{\Gamma(n+z)}{\Gamma(z)} = z(z+1)\cdots(z+n-1), \qquad P_0(z) = 1, \tag{2.5}$$

$\Omega_d = \dfrac{2\pi^{d/2}}{\Gamma(d/2)}$ is a volume factor, and the normalization constant $\mathcal{N}_{n\ell}$ is given by (A.37), but it is not needed explicitly in our analysis.

With the rescaled Fock space operator,

$$d_n = \sqrt{\frac{\mathcal{N}R}{2\nu_{n0}}}\,(-1)^n\frac{P_n(d/2)}{n!}\mathcal{N}_{n0}\frac{1}{\sqrt{\Omega_d}}\mathcal{A}_{n0\underline{0}}, \tag{2.6}$$

the middle-point field is expressed simply as

$$\mathcal{A}(t) = \mathrm{e}^{-i\Delta t}D\left(\mathrm{e}^{-2it}\right) + \mathrm{e}^{i\Delta t}D_1\left(\mathrm{e}^{2it}\right), \tag{2.7}$$

where formally holomorphic operators are defined by

$$D(z) = \sum_n d_n z^n, \qquad D_1(z) = \sum_n d_n^{\dagger} z^n. \tag{2.8}$$

[1] We use real hyper-spherical harmonics for simplicity. This will not be important in our analysis since we only use the hyper-spherical harmonics $Y_{\ell\underline{0}}$, which are real anyway.

JHEP02(2022)015

The BDHM relation [5] gives the boundary field $\mathcal{O}(t,\Omega)$ of conformal weight Δ as

$$\mathcal{O}(t,\Omega) := \lim_{y\to\infty}\left(\frac{y}{R}\right)^{\Delta}\Phi(t,y,\Omega) = \sum_{n\ell\underline{m}}\sqrt{\frac{\mathcal{N}R}{2\nu_{n\ell}}}\left\{\mathrm{e}^{-i\nu_{n\ell}t}\,\frac{P_n(1+\alpha)}{n!}\mathcal{N}_{n\ell}Y_{\ell\underline{m}}(\Omega)\mathcal{A}_{n\ell\underline{m}}\right.$$
$$\left.+\mathrm{e}^{i\nu_{n\ell}t}\,\frac{P_n(1+\alpha)}{n!}\mathcal{N}_{n\ell}Y_{\ell\underline{m}}(\Omega)\mathcal{A}^{\dagger}_{n\ell\underline{m}}\right\}, \tag{2.9}$$

where $\alpha := \Delta - d/2$ (see appendix A). It is clear that $O(t,\Omega)$ in the above expression is not a canonical field operator, since it does not satisfy the canonical commutation relation $[O(t,\Omega),\partial_t O(t,\Omega')] = i\delta(\Omega-\Omega')$.

An integration over the angular variables simplifies the above formula as

$$\mathcal{C}(t) := \int \mathrm{d}\Omega\,\mathcal{O}(t,\Omega) = e^{-i\Delta t}B(-e^{-2it}) + e^{i\Delta t}B_1(-e^{2it}), \tag{2.10}$$

where an other pair of formally holomorphic operators is given by

$$B(z) = \sum_n b_n z^n, \qquad\qquad B_1(z) = \sum_n b_n^{\dagger} z^n \tag{2.11}$$

in terms of Fock space operators rescaled differently from d_n as

$$b_n = \sqrt{\frac{\mathcal{N}R}{2\nu_{n0}}}(-1)^n\frac{P_n(1+\alpha)}{n!}\mathcal{N}_{n0}\sqrt{\Omega_d}\mathcal{A}_{n0\underline{0}} = \Omega_d\,\frac{P_n(1+\alpha)}{P_n(d/2)}d_n. \tag{2.12}$$

2.2 Bulk-boundary mapping

Following HKLL [3, 4], we relate the holomorphic functions D and B as

$$D(w) = \sum_n \frac{1}{\Omega_d}\frac{P_n(d/2)}{P_n(1+\alpha)}w^n\,\frac{1}{2\pi i}\oint\frac{\mathrm{d}z}{z^{n+1}}B(z), \tag{2.13}$$

which, by reversing the order of summation and integration, is rewritten as

$$\begin{aligned} D(w) &= \frac{1}{2\pi i\Omega_d}\oint\frac{\mathrm{d}z}{z}B(z)\sum_n\frac{P_n(d/2)}{P_n(1+\alpha)}\left(\frac{w}{z}\right)^n = \frac{1}{2\pi i\Omega_d}\oint\frac{\mathrm{d}z}{z}B(z)\,{}_2F_1(1,d/2;1+\alpha;w/z)\\ &= \frac{1}{2\pi i\Omega_d}\oint\frac{\mathrm{d}z}{z}B(wz)\,{}_2F_1(1,d/2;1+\alpha;1/z). \end{aligned} \tag{2.14}$$

The integration contour in the last formula must lie outside the unit circle for the sum defining the hypergeometric function to be convergent.

In this paper, for simplicity,[2] we mainly (except in subsection 4.2) consider the case d odd. The derivation of the explicit form of the linear relation between the bulk field "at the middle" and the integrated boundary field found by HKLL is reproduced in appendix B. Although it was not emphasized in the original HKLL paper [4], this derivation is valid for the range

$$\Delta > d-1 \tag{2.15}$$

[2] Similar results hold also for even d, but some of the formulas receive logarithmic corrections [3, 4].

only. The result is given by

$$\mathcal{A}(t) = \xi \int_{t-\pi/2}^{t+\pi/2} \mathrm{d}u [2\cos(t-u)]^{\Delta-d} \mathcal{C}(u), \tag{2.16}$$

where the overall constant is

$$\xi = \frac{1}{\pi\Omega_d} \frac{\Gamma(1-d/2)\Gamma(1+\alpha)}{\Gamma(\Delta-d+1)}. \tag{2.17}$$

We can see that the HKLL result (2.16) is valid for the range (2.15) only, because for $\Delta \leq d-1$ this integral is divergent. In the next section and appendix E, we extend the calculation for $\Delta > d-2$ and consider the most interesting special case $\Delta = d-1$ in detail.

We finish the review of the HKLL construction by transforming the result, calculated above for the "middle" of the AdS space, to an arbitrary point in AdS space. The result (2.16) for the "middle" point $Y_o = (t=0, \rho=0, \Omega)$ is rewritten as

$$\Phi(Y_o) = \int \mathcal{D}x \, \mathcal{K}(x) \mathcal{O}(x), \tag{2.18}$$

where

$$x = (\tilde{t}, \tilde{\Omega}), \quad \mathcal{D}x = \mathrm{d}\tilde{t}\mathrm{d}\tilde{\Omega}, \qquad \mathcal{K}(x) = \xi(2\cos\tilde{t})^{\Delta-d}\Theta\left(\frac{\pi}{2}-\tilde{t}\right)\Theta\left(\tilde{t}+\frac{\pi}{2}\right) \tag{2.19}$$

with the step function Θ.

In what follows we will make use the symmetry properties of the solution and use the notations introduced in appendix C. Applying the Hilbert space isometry action to both sides of the equation, Φ for a generic bulk point $Y = g^{-1}Y_o$ is represented as

$$\Phi(Y=g^{-1}Y_o) = \int \mathcal{D}x \, \mathcal{K}(x)[J(g^{-1},x)]^{\Delta}\mathcal{O}(g^{-1}x) = \int \mathcal{D}y \, \mathcal{K}(gy)[J(g,y)]^{d-\Delta}\mathcal{O}(y), \tag{2.20}$$

where (C.3) is used for the second equality. The solution to the above equation is given by

$$\Phi(Y=g^{-1}Y_o) = \int \mathcal{D}x \, I^{\Delta-d}(Y,x)T(Y,x)\mathcal{O}(x), \tag{2.21}$$

where I and T have to satisfy

$$I(gY,gx) = J(g,x)I(Y,x), \qquad I(Y_o,x) = 2\cos\tilde{t}, \tag{2.22}$$

$$T(gY,gx) = T(Y,x), \qquad T(Y_o,x) = \xi\Theta\left(\tilde{t}+\frac{\pi}{2}\right)\Theta\left(\frac{\pi}{2}-\tilde{t}\right). \tag{2.23}$$

Now it is easy to see that (2.21) satisfies (2.20) since

$$\begin{aligned}\Phi(Y=g^{-1}Y_o) &= \int \mathcal{D}y \, J^{d-\Delta}(g,y)I^{\Delta-d}(Y_o,gy)T(Y_o,gy)\mathcal{O}(y) \\ &= \int \mathcal{D}y \, J^{d-\Delta}(g,y)\mathcal{K}(gy)O(y).\end{aligned} \tag{2.24}$$

I and T for $Y = (t, \rho, n^i)$ and $x = (\tilde{t}, \tilde{n}^i)$ are explicitly constructed in appendix D:

$$I(Y,x) = 2[\cosh\rho\cos(t-\tilde{t}) - \sinh\rho \, n\cdot\tilde{n}], \quad T(Y,x) = \xi\Theta(X_1)\Theta(X_2), \tag{2.25}$$

JHEP02(2022)015

where $X_1 = \tilde{t} - T_1$, $X_2 = T_2 - \tilde{t}$, and $T_{1,2}$ are defined in (C.1) and (C.2). Geometrically, if $X_1(Y, x) = 0$ or $X_2(Y, x) = 0$, Y and x can be connected by a past or future oriented light-like geodesic, respectively. Thus $\Theta(X_1)\Theta(X_2)$ is only non-vanishing if $T_1 < \tilde{t} < T_2$, which means that Y and x can be connected by a space-like geodesic. This last observation leads to the introduction of the space-like Green function, which is useful to introduce interactions in the bulk. (See [6] for a review.)

3 Bulk reconstruction for the range $d - 2 < \Delta \leq d - 1$

We have seen that the derivation of the HKLL formula is only valid for the range (2.15). (The a priori lower limit for a scalar field is $\Delta > (d-2)/2$, which is smaller.) Here we extend the possible range to

$$\Delta > d - 2. \tag{3.1}$$

Our starting point is the last line of (2.14) and the identity (B.1). We note that this hypergeometric identity is valid for odd d and $\Delta \neq$ integer. This last requirement is only temporary and later we extend the results (by taking limits) to integer Δ, too.

To circumvent the restriction (2.15), we rewrite (2.14) by adding and subtracting $B(w)$ under the integral as

$$\begin{aligned} D(w) = {} & \frac{B(w)}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z} \, {}_2F_1(1, d/2; 1+\alpha; 1/z) \\ & + \frac{1}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z} [B(wz) - B(w)] \, {}_2F_1(1, d/2; 1+\alpha; 1/z). \end{aligned} \tag{3.2}$$

Using this form, the manipulations in appendix B remain valid for the extended range $\Delta > d - 2$ and we obtain

$$D(w) = \frac{B(w)}{\Omega_d} + \xi \int_{-\pi/2}^{\pi/2} \mathrm{d}u \, \mathrm{e}^{-iu\Delta} [2\cos(u)]^{\Delta - d} \{B(-w\mathrm{e}^{-2iu}) - B(w)\}, \tag{3.3}$$

where singularities near $u = \pm\frac{\pi}{2}$ of the integrand become integrable for $\Delta > d - 2$ thanks to the subtraction of $B(w)$.

Employing the above expression for $D(w)$ and a similar one for $D_1(w)$ (see appendix E for the details of the derivation), we obtain one of our main results in this paper:

$$\begin{aligned} \mathcal{A}(t) = {} & \frac{\eta}{2\Omega_d}[\mathcal{C}(t - \pi/2) + \mathcal{C}(t + \pi/2)] + \xi \int_{t-\pi/2}^{t} \mathrm{d}u [2\cos(u - t)]^{\Delta - d} \{\mathcal{C}(u) - \mathcal{C}(t - \pi/2)\} \\ & + \xi \int_{t}^{t+\pi/2} \mathrm{d}u [2\cos(u - t)]^{\Delta - d} \{\mathcal{C}(u) - \mathcal{C}(t + \pi/2)\}, \end{aligned} \tag{3.4}$$

which is valid for the extended range (3.1). Here

$$\eta = \frac{\Gamma(1 - d/2)\Gamma(1+\alpha)}{\Gamma^2(1 + \frac{\Delta - d}{2})}. \tag{3.5}$$

Our explicit derivation confirms the result found in [21] (if the limit is explicitly evaluated) at least in their overlapping range of validity. For the original range, $\Delta > d - 1$, (3.4) gives back the original HKLL result (2.16), since the subtracted terms, which now can be integrated separately by using the identity

$$\int_0^{\pi/2} \mathrm{d}u (2\cos u)^A = \frac{\pi}{2} \frac{\Gamma(1+A)}{\Gamma^2(1+A/2)}, \qquad A > -1, \tag{3.6}$$

exactly cancel the first term.

An interesting special case is obtained if we take the limit $\Delta \to d-1$. In this limit, the integrals do not contribute as $\xi = 0$, and η simplifies to $\eta = (-1)^{\Delta/2}$. We thus obtain

$$\mathcal{A}(t) = \xi_o[\mathcal{C}(t-\pi/2) + \mathcal{C}(t+\pi/2)], \quad \xi_o := \frac{(-1)^{\Delta/2}}{2\Omega_d}, \tag{3.7}$$

which means that the bulk field at the middle point in the global AdS is expressed in terms of the CFT field values only at boundary points connected to the middle point by light-like geodesics. This is the other main result in this paper, which is in agreement with the result in [10] and confirms the claim in [18] for the special case $\Delta = d-1$. We will consider this interesting case and its generalization to $\Delta = d-s$ with an integer s in the next section.

It is also straightforward to extend the range to $\Delta > d-3$, by rewriting (2.14) as

$$\begin{aligned} D(w) = &\frac{1}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z} \left[B(w) + B'(w)w(z-1)\right] {}_2F_1(1, d/2; 1+\alpha; 1/z) \\ &+ \frac{1}{2\pi i \Omega_d} \oint \frac{\mathrm{d}z}{z} [B(wz) - B(w) - B'(w)w(z-1)]\, {}_2F_1(1, d/2; 1+\alpha; 1/z), \end{aligned} \tag{3.8}$$

but we do not pursue this direction further in this paper.

4 Bulk reconstruction for $\Delta = d - s$ with an integer s

In this section we consider the special cases $\Delta = d-s$ with integer $s < (d+2)/2$ satisfying the lower bound, $\Delta > (d-2)/2$. For these special cases we have found a simpler derivation of the bulk reconstruction formulas, in particular for (3.7) with odd d, without using the limiting procedure starting from integrals like (3.4). Interestingly the bulk field operator at the middle point can be expressed in terms of CFT field operators and their t derivatives only at boundary points light-like separated from the middle point. This is shown by (4.7) and (4.8), which are also one of our main results in this paper. For even d, we can derive similar results, which however also contain a derivative with respect to Δ.

From (2.12) we see that the bulk field can be written in terms of boundary operators b_n and $b_n^\dagger$ as

$$\mathcal{A}(t) = \frac{1}{\Omega_n} \sum_n X_n^\Delta \left\{ e^{-i(\Delta+2n)t} b_n + e^{i(\Delta+2n)t} b_n^\dagger \right\}, \quad X_n^\Delta := \frac{P_n(\frac{d}{2})}{P_n(\alpha+1)}. \tag{4.1}$$

On the other hand we have

$$\begin{aligned} C_\pm(t) &:= \mathcal{C}\left(t + \frac{\pi}{2}\right) \pm \mathcal{C}\left(t - \frac{\pi}{2}\right) \\ &= \left(e^{-i\Delta\frac{\pi}{2}} \pm e^{i\Delta\frac{\pi}{2}}\right) \sum_n \left\{ e^{-i(\Delta+2n)t} b_n \pm e^{i(\Delta+2n)t} b_n^\dagger \right\}. \end{aligned} \tag{4.2}$$

Thus $C_\pm(t) = 0$ if Δ is an odd/even integer.

4.1 Results for $\Delta = d - s$ with odd d

As a warmup, we first give a much simpler derivation of (3.7) for $\Delta = d-1$. Since $X_n^\Delta = 1$ in this case, we have

$$\mathcal{A}(t) = \frac{1}{\Omega_d}\sum_n \left\{e^{-i(\Delta+2n)t}b_n + e^{i(\Delta+2n)t}b_n^\dagger\right\} = \frac{(-1)^{\Delta/2}}{2\Omega_d}C_+(t), \quad \Delta = d-1, \tag{4.3}$$

which reproduces (3.7), because $C_+(t) = 2\mathcal{C}(t \pm \frac{\pi}{2})$.

For $\Delta = d-2$, since $X_n^\Delta = (\Delta+2n)/(d-2)$, we obtain

$$\mathcal{A}(t) = \frac{1}{\Omega_d}\sum_n X_n^\Delta \left\{e^{-i(\Delta+2n)t}b_n + e^{i(\Delta+2n)t}b_n^\dagger\right\} = -\frac{(-1)^{\frac{\Delta-1}{2}}}{2(d-2)\Omega_d}\frac{\partial}{\partial t}C_-(t), \tag{4.4}$$

where in this case $C_-(t) = \pm 2\mathcal{C}(t \pm \frac{\pi}{2})$.

For general $\Delta = d-s$ with $s < (d+2)/2$, we have

$$X_n^{d-(2\ell+1)} = X_n^{d-1}\frac{\prod_{k=1}^{\ell}(\Delta_n+2k-1)(\Delta_n-2k+1)}{\prod_{k=1}^{2\ell}(d-2k)}, \quad X_n^{d-1} = 1, \tag{4.5}$$

$$X_n^{d-(2\ell+2)} = X_n^{d-2}\frac{\prod_{k=1}^{\ell}(\Delta_n+2k)(\Delta_n-2k)}{\prod_{k=2}^{2\ell+1}(d-2k)}, \quad X_n^{d-2} = \frac{\Delta_n}{d-2} \tag{4.6}$$

for $\ell = 1, 2, \cdots$, where $\Delta_n := \Delta + 2n$. We thus obtain

$$\mathcal{A}(t) = \frac{(-1)^{\frac{d-1}{2}}}{2\Omega_d}\frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2}+(2k-1)^2\right\}}{\prod_{k=1}^{2\ell}(d-2k)}C_+(t), \tag{4.7}$$

for $\Delta = d-(2\ell+1)$, where $C_+(t) = 2\mathcal{C}(t \pm \frac{\pi}{2})$, while

$$\mathcal{A}(t) = \frac{(-1)^{\frac{d-1}{2}}}{2\Omega_d}\frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2}+4k^2\right\}}{\prod_{k=1}^{2\ell+1}(d-2k)}\frac{\partial}{\partial t}C_-(t), \tag{4.8}$$

for $\Delta = d - 2(\ell+1)$, where $C_-(t) = \pm 2\mathcal{C}(t \pm \frac{\pi}{2})$. (4.7) and (4.8) cover all cases $\Delta = d-s$ for odd d.

4.2 Results for $\Delta = d - s$ with even d

For an even dimension d, the bulk field operator becomes

$$\mathcal{A}(t) = \frac{(-1)^\ell}{\Omega_d}\frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2}+(2k-1)^2\right\}}{\prod_{k=1}^{2\ell}(d-2k)}\sum_n\left\{e^{-i(\Delta+2n)t}b_n + e^{i(\Delta+2n)t}b_n^\dagger\right\} \tag{4.9}$$

for $\Delta = d-(2\ell+1)$, while

$$\mathcal{A}(t) = \frac{(-1)^\ell}{\Omega_d}\frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2}+4k^2\right\}}{\prod_{k=1}^{2\ell+1}(d-2k)}\frac{i\partial}{\partial t}\sum_n\left\{e^{-i(\Delta+2n)t}b_n - e^{i(\Delta+2n)t}b_n^\dagger\right\} \tag{4.10}$$

JHEP02(2022)015

for $\Delta = d - 2(\ell+1)$. On the other hand, the boundary field operators satisfy

$$\frac{\partial}{\partial\Delta}C_+(t)\Big|_{\Delta=d-(2\ell+1)} = \pi(-1)^{d/2-\ell}\sum_n \left\{e^{-i(\Delta+2n)t}b_n + e^{i(\Delta+2n)t}b_n^\dagger\right\}, \tag{4.11}$$

$$\frac{\partial}{\partial\Delta}C_-(t)\Big|_{\Delta=d-2(\ell+1)} = (-i)\pi(-1)^{d/2-\ell-1}\sum_n \left\{e^{-i(\Delta+2n)t}b_n - e^{i(\Delta+2n)t}b_n^\dagger\right\}. \tag{4.12}$$

Combining these, we obtain

$$\mathcal{A}(t) = \frac{(-1)^{d/2}}{\pi\Omega_d}\frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2}+(2k-1)^2\right\}}{\prod_{k=1}^{2\ell}(d-2k)}\frac{\partial}{\partial\Delta}C_+(t)\Big|_{\Delta=d-(2\ell+1)}, \tag{4.13}$$

$$\mathcal{A}(t) = \frac{(-1)^{d/2}}{\pi\Omega_d}\frac{\prod_{k=1}^{\ell}\left\{\frac{\partial^2}{\partial t^2}+4k^2\right\}}{\prod_{k=1}^{2\ell+1}(d-2k)}\frac{\partial}{\partial t}\frac{\partial}{\partial\Delta}C_-(t)\Big|_{\Delta=d-2(\ell+1)}. \tag{4.14}$$

For $\Delta = d-1, d-2$, for example, we have

$$\mathcal{A}(t) = \frac{(-1)^{d/2}}{\pi\Omega_d}\frac{\partial}{\partial\Delta}C_+(t)\Big|_{\Delta=d-1}, \quad \mathcal{A}(t) = \frac{(-1)^{d/2}}{(d-2)\pi\Omega_d}\frac{\partial}{\partial t}\frac{\partial}{\partial\Delta}C_-(t)\Big|_{\Delta=d-2}. \tag{4.15}$$

5 Bulk reconstruction at generic points for small integer Δ

In this section we derive the bulk field operator at generic points for $\Delta = d-1$ and $\Delta = d-2$ with odd d, along the same logic we used in section 2 for the (2.15) case.

5.1 Bulk reconstruction for $\Delta = d - 1$ with odd d at generic bulk points

For the middle point Y_o, we write

$$\Phi(Y_o) = \int \mathcal{D}x\, k(x)\mathcal{O}(x), \qquad k(x) := \xi_o[\delta_o(\tilde{t}+\pi/2)+\delta_o(\tilde{t}-\pi/2)], \tag{5.1}$$

where δ_o is the standard delta function of one argument. Making the isometry transformation in the Hilbert space as before, we obtain

$$\Phi(g^{-1}Y_o) = \int \mathcal{D}y J^d(g,y)J^\Delta(g^{-1},gy)k(gy)\mathcal{O}(y) = \int \mathcal{D}y\, k(gy)J(g,y)\mathcal{O}(y), \tag{5.2}$$

which is solved by

$$\Phi(Y = g^{-1}Y_o) = \int \mathcal{D}x\, D(Y,x)\mathcal{O}(x), \tag{5.3}$$

where $D(Y,x)$ has to satisfy

$$D(gY,gx) = \frac{D(Y,x)}{J(g,x)}, \quad D(Y_o,x) = k(x), \tag{5.4}$$

since, with this definition,

$$\Phi(g^{-1}Y_o) = \int \mathcal{D}x\, D(g^{-1}Y_o,x)\mathcal{O}(x) = \int \mathcal{D}x\, J(g,x)\, k(gx)\mathcal{O}(x). \tag{5.5}$$

JHEP02(2022)015

The kernel function $D(Y,x)$ is constructed explicitly in appendix F and is given by

$$D(Y,x) = \frac{\xi_o}{\mathcal{R}(Y,x)}[\delta_o(X_1) + \delta_o(X_2)], \tag{5.6}$$

where

$$\mathcal{R}(Y,x) = \cosh\rho\cos\Psi = \sqrt{\cosh^2\rho - \sinh^2\rho(n\cdot\tilde{n})^2}, \qquad \mathcal{R}(Y_o,x) = 1. \tag{5.7}$$

The final result for the bulk reconstruction for $\Delta = d-1$ with odd d is

$$\Phi(Y) = \xi_o \int \mathrm{d}\tilde{\Omega} \frac{1}{\mathcal{R}(Y,x)}[\mathcal{O}(T_1,\tilde{\Omega}) + \mathcal{O}(T_2,\tilde{\Omega})], \tag{5.8}$$

which again shows that the field operator at a generic bulk point is reconstructed from operators having support only on boundary points light-like separated from the bulk point.

Although the BDHM relation [5] was our starting point in the construction, it is by far not obvious that the representation (5.8) reproduces this relation. It is a nice check on our results that, as explicitly shown in appendix G, $\Phi(Y)$ in (5.8) for $Y = (t,\rho,\Omega)$ does satisfy the BDHM relation

$$\lim_{\rho\to\infty}(\sinh\rho)^\Delta\Phi(t,\rho,\Omega) = \mathcal{O}(t,\Omega). \tag{5.9}$$

5.2 Bulk reconstruction for $\Delta = d-2$ with odd d at generic bulk points

For this special case the bulk reconstruction at the origin can be written in a symmetric way as

$$\mathcal{A}(t) = -\tilde{\xi}_o\frac{\partial}{\partial t}[\mathcal{C}(t+\pi/2) - \mathcal{C}(t-\pi/2)], \qquad \tilde{\xi}_o = \frac{(-1)^{\frac{\Delta-1}{2}}}{2(d-2)\Omega_d}. \tag{5.10}$$

To extend the result to an arbitrary bulk point we can proceed analogously to the $\Delta = d-1$ case.

In a more compact notation (5.10) for the middle point Y_o can be written as

$$\Phi(Y_o) = \int \mathcal{D}x k_2(x)\mathcal{O}(x), \qquad k_2(x) = -\tilde{\xi}_o[\delta_o'(\tilde{t}+\pi/2) - \delta_o'(\tilde{t}-\pi/2)]. \tag{5.11}$$

Here δ_o' is the derivative of the delta function. Making the isometry transformation in the Hilbert space, we find

$$\Phi(g^{-1}Y_o) = \int \mathcal{D}x\, k_2(x)J^\Delta(g^{-1},x)\mathcal{O}(g^{-1}x) = \int \mathcal{D}y\, k_2(gy)J^2(g,y)\mathcal{O}(y). \tag{5.12}$$

Motivated by the $\Delta = d-1$ result we now take the ansatz $\Phi(Y) = \int \mathcal{D}x\, D_2(Y,x)\mathcal{O}(x)$, where we require that $D_2(Y,x)$ satisfies

$$D_2(gY,gx) = \frac{D(Y,x)}{J^2(g,x)}, \quad D_2(Y_o,x) = k_2(x). \tag{5.13}$$

We can now verify that (5.12) holds:

$$\Phi(g^{-1}Y_o) = \int \mathcal{D}x\, D_2(g^{-1}Y_o,x)\mathcal{O}(x) = \int \mathcal{D}x\, J^2(g,x)\, k_2(gx)\mathcal{O}(x). \tag{5.14}$$

The derivation of the solution of (5.13) is given in appendix F:

$$D_2(Y,x) = -\frac{\tilde{\xi}_o}{\mathcal{R}^2(Y,x)}\left\{[\delta_o'(X_1) + \delta_o'(X_2)] - \tan\Psi[\delta_o(X_1) + \delta_o(X_2)]\right\}. \tag{5.15}$$

6 Conclusions and discussion

In this paper we extended the applicability of the HKLL bulk reconstruction for non-interacting scalar theories, which was restricted to $\Delta > d-1$, to smaller conformal weights Δ of the boundary CFT in the range $d-2 < \Delta \leq d-1$. The explicit formula is given in (3.4). In addition, we have derived a simple formula for $\Delta = d-s$ with positive integer s, which (for these special cases) confirms Terashima's claim that a field at a point X in the AdS bulk can be reconstructed from CFT fields smeared only over boundary points connected to X by light-like geodesic curves [17, 18].

Results in this paper enable us to apply the HKLL bulk reconstruction to $O(N)$ vector models, which are expected to be dual to higher spin theories [19, 20]. Moreover, explicit demonstration of Terashima's claim, even though only for the above special cases, may bring new insights [18] to the sub-region duality and its relation to quantum error corrections [15].

It would be interesting to generalize the Green function method so that it covers the extended range and to reproduce (3.4) with this technique. We hope that this will enable us to introduce interactions systematically.

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A Canonical quantization of the free scalar field

Quantization in a general curved background is difficult, but it is straightforward if there exists a global time t and the metric has a form,

$$\mathrm{d}s^2 = -\mathcal{H}(\mathrm{d}t)^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{A.1}$$

where $\{x^1, \ldots, x^d\}$ are the space coordinates, and both $\mathcal{H}$ and g_{ij} are time-independent. In such a background the Lagrangian of a (dimensionless) free scalar Φ is defined as

$$L = \frac{1}{2\mathcal{N}} \int \mathrm{d}^d x \frac{\sqrt{-g}}{\mathcal{H}} [\dot{\Phi}^2 - \Phi K \Phi], \tag{A.2}$$

where $\mathcal{N} = B^{d-1}$, B is a parameter of length dimension,

$$K = -\frac{\mathcal{H}}{\sqrt{-g}} \partial_i \left(\sqrt{-g} g^{ij} \partial_j\right) + \mathcal{H} m^2, \tag{A.3}$$

and m^2 is a parameter of dimension mass squared. The consistency of the quantization procedure requires that the operator K is self-adjoint such that $\langle f_1|Kf_2\rangle = \langle Kf_1|f_2\rangle$ for any two functions f_1, f_2 in the domain of definition of K, where the scalar product of two functions is defined with the measure $\sqrt{-g}/\mathcal{H}$ as

$$\langle f_1|f_2\rangle = \int \mathrm{d}^d x \frac{\sqrt{-g}}{\mathcal{H}} f_1\, f_2. \tag{A.4}$$

The Euler-Lagrange equations following from the Lagrangian (A.2) can be written in a covariant form as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = m^2\Phi. \tag{A.5}$$

We will expand the free field in terms of eigenfunctions of K satisfying

$$K\psi_a = \omega_a^2\psi_a, \qquad\qquad \langle\psi_a|\psi_b\rangle = \delta_{ab}, \tag{A.6}$$

where the frequencies ω_a are all real because K (for large enough m^2) is positive self-adjoint. Writing the field as

$$\Phi(x) = \sum_a Q_a\psi_a(x) \tag{A.7}$$

the Lagrangian becomes

$$L = \frac{1}{2\mathcal{N}}\sum_a[\dot{Q}_a^2 - \omega_a^2 {Q_a}^2]. \tag{A.8}$$

A complete set of solutions to the equations of motion (A.5) is $\{f_a(t,x)\}$, $\{f_a^*(t,x)\}$, where $f_a(t,x) = \mathrm{e}^{-i\omega_a t}\psi_a(x)$, so that the general solution is expanded in terms of constant amplitudes $\{\beta_a\}$ as

$$\Phi(t,x) = \sum_a[f_a(t,x)\beta_a + f_a^*(t,x)\beta_a^*]. \tag{A.9}$$

We introduce canonical momentum variables and the Hamiltonian of the system as

$$H = \frac{1}{2}\sum_a\left(\mathcal{N}p_a^2 + \frac{\omega_a^2}{\mathcal{N}}Q_a^2\right) = \frac{2}{\mathcal{N}}\sum_a\omega_a^2\beta_a^*\beta_a, \quad p_a := \frac{1}{\mathcal{N}}\dot{Q}_a, \tag{A.10}$$

then we promote the canonical variables p_a, Q_a to operators satisfying $[p_a, Q_b] = -i\delta_{ab}$. The quantized amplitudes become

$$\beta_a = \sqrt{\frac{\mathcal{N}}{2\omega_a}}\mathcal{A}_a, \qquad \beta_a^+ = \sqrt{\frac{\mathcal{N}}{2\omega_a}}\mathcal{A}_a^\dagger, \tag{A.11}$$

where $\mathcal{A}_a$, $\mathcal{A}_a^\dagger$ are operators in a Fock space such that $[\mathcal{A}_a, \mathcal{A}_b^\dagger] = \delta_{ab}$ and $\mathcal{A}_a|0\rangle = 0$, and the corresponding quantum Hamiltonian becomes

$$H = E_0 + \sum_a\omega_a\mathcal{A}_a^\dagger\mathcal{A}_a, \tag{A.12}$$

where E_0 is the vacuum energy and $\{\omega_a\}$ is the spectrum of 1-particle states in the Fock space. Finally the canonical quantum field operator is expanded as

$$\Phi(t,x) = \sum_a\sqrt{\frac{\mathcal{N}}{2\omega_a}}\left[f_a(t,x)\mathcal{A}_a + f_a^*(t,x)\mathcal{A}_a^\dagger\right]. \tag{A.13}$$

A.1 Radial quantization

We will apply the above quantization scheme to the global AdS space. For the metric (2.2), we have

$$\mathcal{H} = \frac{y^2+R^2}{R^2}, \qquad \sqrt{-g}=1, \qquad g^{ij} = \delta_{ij} + \frac{y^i y^j}{R^2}, \tag{A.14}$$

and the operator K becomes

$$K = \mathcal{H}\left\{\frac{L^2}{y^2} - \left(\frac{1}{y^2}+\frac{1}{R^2}\right)(\mathcal{D}^2 + d\mathcal{D}) + \frac{2}{y^2}\mathcal{D} + m^2\right\}, \tag{A.15}$$

where

$$L^2 := -\frac{1}{2}L_{ij}L_{ij}, \qquad L_{ij} := y^i\frac{\partial}{\partial y^j} - y^j\frac{\partial}{\partial y^i}, \qquad \mathcal{D} := y^i\frac{\partial}{\partial y^i}. \tag{A.16}$$

Eigenfunctions of the Casimir operator L^2 are hyper-spherical harmonics $Y_{\ell\underline{m}}(\Omega)$, where Ω are the angular variables (n^i). The spectrum is given by

$$L^2\, Y_{\ell\underline{m}}(\Omega) = \ell(\ell+d-2)\, Y_{\ell\underline{m}}(\Omega), \tag{A.17}$$

where $\ell = 0, 1, \ldots$ and $\underline{m}$ is a multi-index. The hyper-spherical harmonics in a real basis are normalized to

$$\int \mathrm{d}\Omega\, Y_{\ell'\underline{m}'}(\Omega) Y_{\ell\underline{m}}(\Omega) = \delta_{\ell'\ell}\delta_{\underline{m}'\underline{m}}, \tag{A.18}$$

where $\mathrm{d}\Omega$ is the measure of the angular integration, $\int \mathrm{d}^d y = \int_0^\infty \mathrm{d}y\, y^{d-1}\mathrm{d}\Omega$. Using the ansatz $\psi_{\ell\underline{m}}(y,\Omega) = u_\ell(y) Y_{\ell\underline{m}}(\Omega)$ for eigenfunctions, the radial functions $u_\ell(y)$ must satisfy the differential equation

$$\mathcal{H}\left\{\frac{\ell(\ell+d-2)}{y^2} - \left(\frac{1}{y^2}+\frac{1}{R^2}\right)(\mathcal{D}^2+d\mathcal{D}) + \frac{2}{y^2}\mathcal{D} + m^2\right\} u_\ell(y) = \omega_\ell^2\, u_\ell(y), \tag{A.19}$$

where now $\mathcal{D} = y\frac{\mathrm{d}}{\mathrm{d}y}$, and the radial scalar product is defined by

$$\langle u_\ell^{(1)} | u_\ell^{(2)}\rangle = \int_0^\infty \mathrm{d}y\, y^{d-1}\frac{R^2}{y^2+R^2} u_\ell^{(1)}(y)\, u_\ell^{(2)}(y). \tag{A.20}$$

Introducing dimensionless quantities

$$\mu = mR, \qquad \nu_\ell = \omega_\ell R, \qquad \xi = \frac{y^2}{y^2+R^2}, \qquad 1-\xi = \frac{R^2}{y^2+R^2}, \tag{A.21}$$

(A.19) becomes

$$\begin{aligned} K_\ell^{\mathrm{rad}} u_\ell(\xi) &= \frac{1}{1-\xi}\left\{\frac{\ell(\ell+d-2)(1-\xi)}{\xi} - \frac{1}{\xi}(\mathcal{D}^2+d\mathcal{D}) + \frac{2(1-\xi)}{\xi}\mathcal{D} + \mu^2\right\} u_\ell(\xi) \\ &= \nu_\ell^2\, u_\ell(\xi). \end{aligned} \tag{A.22}$$

Using the ansatz

$$u_\ell(\xi) = \xi^{\frac{\ell}{2}}(1-\xi)^{\frac{\Delta_+}{2}}\mathcal{F}(\xi), \qquad \Delta_\pm := \frac{d}{2} \pm \bar{\alpha}, \quad \bar{\alpha} := \sqrt{\frac{d^2}{4}+\mu^2} \geq 0, \tag{A.23}$$

we can verify that $\mathcal{F}(\xi)$ must satisfiy the hypergeometric equation with parameters

$$a = \frac{\Delta_+ + \ell - \nu_\ell}{2}, \qquad b = \frac{\Delta_+ + \ell + \nu_\ell}{2}, \qquad c = \ell + \frac{d}{2}. \tag{A.24}$$

A.2 Boundary conditions

At this point it is necessary to discuss boundary conditions. First of all, we notice that the point $y^1 = y^2 = \cdots = y^d = 0$ is just as any other point in AdS (and can be transformed to any other point) therefore $\psi_{\ell\underline{m}}$ must be analytic at $y^i = 0$. Since $y^\ell Y_{\ell\underline{m}}(\Omega)$ is a polynomial in y^i (of order ℓ), we have to require $u_\ell(y) = y^\ell f(y^2)$ near $y = 0$ with an analytic $f(y^2)$. Since the hypergeometric equation with $c = \ell + \frac{d}{2}$ has two linearly independent solutions, one is constant at $\xi = 0$ (this is given by the hypergeometric function), the other is singular like $\mathcal{F}(\xi) \sim (1/\xi)^{\ell-1+d/2}$, we conclude that the radial solution must be of the form

$$u_\ell(y) = \mathcal{M}_\ell \left(\frac{R^2}{y^2+R^2}\right)^{\frac{\Delta_+}{2}} \left(\frac{y^2}{y^2+R^2}\right)^{\frac{\ell}{2}} {}_2F_1\left(a, b;, c; \frac{y^2}{y^2+R^2}\right), \tag{A.25}$$

where $\mathcal{M}_\ell$ is a normalization constant to be determined later.

Next we discuss the $y \to \infty$ behaviour of the solutions. We assume that it is of the form $u_\ell(y) \sim y^{-L}[1 + \mathrm{O}(y^{-2})]$. In principle the domain of definition may consist of several such classes of functions with different asymptotic behaviour $L = L_1,\, L_2,\, \cdots$. Since the solutions are normalizable with respect to the scalar product (A.20), we require $2L + 2 > d$. Another condition is that the radial operator defined by (A.22) is self-adjoint such that

$$\int_0^\infty \mathrm{d}y\, y^{d-1} \frac{R^2}{y^2+R^2} u_\ell^{(1)}(y) K_\ell^{\mathrm{rad}} u_\ell^{(2)}(y) = \int_0^\infty \mathrm{d}y\, y^{d-1} \frac{R^2}{y^2+R^2} u_\ell^{(2)}(y) K_\ell^{\mathrm{rad}} u_\ell^{(1)}(y), \tag{A.26}$$

where we have to ensure that the boundary terms (emerging from an integration by part) do not contribute. For this condition, we find that

- If $u_\ell^{(1)}$ and $u_\ell^{(2)}$ belong to the same class then the self-adjointness conditions require $2L_1 + 2 > d$ and $2L_2 + 2 > d$, which is the same as coming from normalizability.
- If $u_\ell^{(1)}$ and $u_\ell^{(2)}$ belong to different classes then the self-adjointness condition becomes $L_1 + L_2 > d$.

A.3 Spectrum and eigenfunctions

Let us assume (temporarily) that $\bar{\alpha}$ is not integer. Then using identities satisfied by the hypergeometric function we can write our solution (A.25) in an alternative form

$$\begin{aligned} u_\ell(y) = {} & \mathcal{M}_\ell \Gamma(\ell + d/2) \left(\frac{y^2}{y^2+R^2}\right)^{\frac{\ell}{2}} \times \\ & \left\{ \left(\frac{R^2}{y^2+R^2}\right)^{\frac{\Delta_+}{2}} \frac{\Gamma(-\bar{\alpha})}{\Gamma\left(\frac{\ell+\Delta_- -\nu_\ell}{2}\right)\Gamma\left(\frac{\ell+\Delta_- +\nu_\ell}{2}\right)} {}_2F_1\left(\frac{\ell+\Delta_+ -\nu_\ell}{2}, \frac{\ell+\Delta_+ +\nu_\ell}{2}; 1+\bar{\alpha}; \frac{R^2}{y^2+R^2}\right)\right. \\ & \left. + \left(\frac{R^2}{y^2+R^2}\right)^{\frac{\Delta_-}{2}} \frac{\Gamma(\bar{\alpha})}{\Gamma\left(\frac{\ell+\Delta_+ -\nu_\ell}{2}\right)\Gamma\left(\frac{\ell+\Delta_+ +\nu_\ell}{2}\right)} {}_2F_1\left(\frac{\ell+\Delta_- -\nu_\ell}{2}, \frac{\ell+\Delta_- +\nu_\ell}{2}; 1-\bar{\alpha}; \frac{R^2}{y^2+R^2}\right) \right\}. \end{aligned} \tag{A.27}$$

JHEP02(2022)015

The first term has asymptotic exponent $L_1 = \Delta_+$ and the second terms has $L_2 = \Delta_-$. Since $\Delta_+ + \Delta_- = d$, the second condition $L_1 + L_2 > d$ can never be satisfied. This means that both terms cannot simultaneously be present in (A.27). Since $2\Delta_+ + 2 = d + 2 + 2\bar{\alpha} > d$, the first term is always normalizable. On the other hand, $2\Delta_- + 2 = d + 2 - 2\bar{\alpha} > d$ is satisfied only if $\bar{\alpha} < 1$.

A.3.1 Δ_+ case

The second term in (A.27) is absent if we choose

$$\frac{\Delta_+ + \ell - \nu_\ell}{2} = -n \quad n = 0, 1, \cdots, \qquad \nu_\ell = \nu_{n\ell} = \Delta_+ + \ell + 2n, \tag{A.28}$$

since the inverse Gamma function in front of the second term then vanishes. In this case the first term simplifies to

$$\begin{aligned} u_{n\ell}(y) = \mathcal{M}_{n\ell}(-1)^n \frac{P_n(\bar{\alpha}+1)}{P_n(\ell + d/2)} \left(\frac{y^2}{y^2+R^2}\right)^{\frac{\ell}{2}} \left(\frac{R^2}{y^2+R^2}\right)^{\frac{\Delta_+}{2}} \\ \times {}_2F_1\left(-n, \Delta_+ + \ell + n; 1 + \bar{\alpha}; \frac{R^2}{y^2+R^2}\right). \end{aligned} \tag{A.29}$$

We see that the limit $\bar{\alpha} \to$ integer is smooth.

A.3.2 Δ_- case

If we choose

$$\frac{\Delta_- + \ell - \nu_\ell}{2} = -n \quad n = 0, 1, \cdots, \qquad \nu_\ell = \nu_{n\ell} = \Delta_- + \ell + 2n, \tag{A.30}$$

the first term in (A.27) vanishes and the second term becomes

$$\begin{aligned} u_{n\ell}(y) = \mathcal{M}_{n\ell}(-1)^n \frac{P_n(1-\bar{\alpha})}{P_n(\ell + d/2)} \left(\frac{y^2}{y^2+R^2}\right)^{\frac{\ell}{2}} \left(\frac{R^2}{y^2+R^2}\right)^{\frac{\Delta_-}{2}} \\ \times {}_2F_1\left(-n, \Delta_- + \ell + n; 1 - \bar{\alpha}; \frac{R^2}{y^2+R^2}\right). \end{aligned} \tag{A.31}$$

A.3.3 Final form of the solution

The possible range of asymptotic exponents (which later become conformal weights) is $\frac{d-2}{2} < \Delta$. If we introduce the parameters

$$\alpha = \Delta - \frac{d}{2} \quad (\alpha > -1; \quad \bar{\alpha} = |\alpha|), \qquad \beta = \ell + \frac{d}{2} - 1, \tag{A.32}$$

the solutions (A.29) and (A.31) can be uniformly written as

$$u_{n\ell}(y) = \mathcal{N}_{n\ell}\, \xi^{\ell/2}(1-\xi)^{\Delta/2} P_n^{(\alpha,\beta)}(x), \qquad \nu_{n\ell} = \Delta + \ell + 2n, \tag{A.33}$$

where

$$\mathcal{N}_{n\ell} = (-1)^n \frac{n!}{P_n(\ell + d/2)} \mathcal{M}_{n\ell}, \qquad x = 2\xi - 1, \tag{A.34}$$

and $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial given by

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} {}_2F_1\left(-n, \alpha+\beta+1+n; \alpha+1; \frac{1-x}{2}\right). \tag{A.35}$$

Using the known orthogonality properties of the Jacobi polynomials, we can make our set of solutions orthonormal:

$$\int_0^\infty \mathrm{d}y\, y^{d-1} \frac{R^2}{y^2+R^2} u_{n\ell}(y)\, u_{m\ell}(y) = \delta_{nm}. \tag{A.36}$$

This requirement fixes the normalization constants as

$$\mathcal{N}_{n\ell}^2 = \frac{2\nu_{n\ell}}{R^d} \frac{n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}. \tag{A.37}$$

For later use we note that

$$y \to \infty: \qquad u_{n\ell}(y) \approx \frac{P_n(\alpha+1)}{n!} \mathcal{N}_{n\ell} \left(\frac{R}{y}\right)^\Delta, \tag{A.38}$$

$$y \to 0: \qquad u_{n\ell}(y) \approx (-1)^n \frac{P_n(\beta+1)}{n!} \mathcal{N}_{n\ell} \left(\frac{y}{R}\right)^\ell. \tag{A.39}$$

To summarize, we have found the expansion of the free scalar on the AdS background in terms of mode functions

$$f_{n\ell\underline{m}}(t,y,\Omega) = \mathrm{e}^{-i\nu_{n\ell}t} u_{n\ell}(y) Y_{\ell\underline{m}}(\Omega) \tag{A.40}$$

for all possible boundary conditions/conformal weights.

B Derivation of the bulk reconstruction for $\Delta > d-1$ with odd d

We evaluate the integral (2.14), using the hypergeometric function identity (valid for odd d)

$$\begin{aligned} {}_2F_1(1, d/2; 1+\alpha; 1/z) = {} & \frac{2\alpha z}{2-d} {}_2F_1(1, 1-\alpha; 2-d/2; z) \\ & + \frac{\Gamma(1-d/2)\Gamma(1+\alpha)}{\Gamma(\Delta-d+1)} \left(-\frac{1}{z}\right)^{-d/2} (1-z)^{\Delta-d}, \end{aligned} \tag{B.1}$$

where the first term is regular except for a cut starting at $z=1$. Around the branch point $z=1$, its behaviour is

$$\text{regular} + \text{const.}\,(1-z)^{\Delta-d}. \tag{B.2}$$

When calculating the integral of this first term in (2.14), we can shrink our contour so that it becomes a very small circle around the branch point $z=1$, and then, its contribution vanishes in our case (2.15), because a value of the integral gets smaller and smaller as our integral contour gets smaller and smaller.

The second term has a cut starting already at $z=0$. The contour can be shrunken so that it becomes just the unit circle, since the singularity around the second branch point

JHEP02(2022)015

$z = 1$ is an integrable one for (2.15). After a change of integration variable $z = -\mathrm{e}^{-2iu}$, the integral along the unit circle becomes

$$D(w) = \frac{1}{\pi\Omega_d}\frac{\Gamma(1-d/2)\Gamma(1+\alpha)}{\Gamma(\Delta-d+1)}\int_{-\pi/2}^{\pi/2}\mathrm{d}uB\left(-w\mathrm{e}^{-2iu}\right)\mathrm{e}^{-i\Delta u}(2\cos u)^{\Delta-d}. \tag{B.3}$$

Thus we obtain

$$\mathrm{e}^{-i\Delta t}D\left(\mathrm{e}^{-2it}\right) = \xi\int_{t-\pi/2}^{t+\pi/2}\mathrm{d}u\,\mathrm{e}^{-i\Delta u}B\left(-\mathrm{e}^{-2iu}\right)[2\cos(t-u)]^{\Delta-d} \tag{B.4}$$

with overall constant ξ in (2.17). If we repeat the whole calculation for D_1, we have

$$\mathrm{e}^{i\Delta t}D_1\left(\mathrm{e}^{2it}\right) = \xi\int_{t-\pi/2}^{t+\pi/2}\mathrm{d}u\,\mathrm{e}^{i\Delta u}B_1\left(-\mathrm{e}^{2iu}\right)[2\cos(t-u)]^{\Delta-d}. \tag{B.5}$$

We can simply add the two contributions to arrive at (2.16).

C Geometry of the AdS space

C.1 Geodesics

An important feature of the geometry of AdS space is that Y and x can be connected with a past directed light-like geodesic if

$$\tilde{t} = T_1, \qquad T_1 = t - \frac{\pi}{2} + \Psi, \qquad \Psi = \arcsin[(\tanh\rho)\,n\cdot\tilde{n}]. \tag{C.1}$$

Similarly, Y and x can be connected with a future directed light-like geodesic if

$$\tilde{t} = T_2, \qquad T_2 = t + \frac{\pi}{2} - \Psi. \tag{C.2}$$

Finally, Y and x can be connected with a space-like geodesic if $T_1 < \tilde{t} < T_2$.

C.2 Infinitesimal transformations

There is a symmetry action by the isometry grop $\mathrm{SO}(d,2)$ on AdS space. The transformed point will be denoted by gY, where g is the group element and Y is transformed to gY. Since it is a group action, the relation $g_2(g_1Y) = (g_2g_1)Y$ is satisfied. The corresponding group action in the Hilbert space is given by the unitary operators $U(g)$ satisfying $U(g_2)U(g_1) = U(g_2g_1)$, under which a scalar field $\Phi(Y)$ transforms as $U^\dagger(g)\Phi(Y)U(g) = \Phi(g^{-1}Y)$. The symmetry group acts also on boundary points by conformal transformations as $x \to gx$ satisfying $g_2(g_1x) = (g_2g_1)x$. A primary scalar field $\mathcal{O}(x)$ transforms under the conformal transformation by the unitary operator as

$$U^\dagger(g)\mathcal{O}(x)U(g) = J^\Delta(g^{-1},x)\mathcal{O}(g^{-1}x), \quad J(g,x) := \left|\det\frac{\partial(gx)^A}{\partial x^B}\right|^{\frac{1}{d}} = \frac{1}{J(g^{-1},gx)}. \tag{C.3}$$

The infinitesimal version of the symmetry transformations are given by

$$\delta\Phi = -\delta Y^\mu\partial_\mu\Phi, \qquad \delta\mathcal{O} = -\delta x^A\partial_A\mathcal{O} - \frac{\omega\Delta}{d}\mathcal{O}, \quad \omega := \frac{\partial\delta x^A}{\partial x^A}. \tag{C.4}$$

The infinitesimal parameters of the $\mathrm{SO}(d,2)$ transformations are $E^{AB}=-E^{BA}$, $A,B=0,D,i$ $(i=1,\dots,d)$. The infinitesimal bulk transformations are given explicitly by

$$\begin{aligned}
\delta t &= -E^{0D}+\tanh\rho(n\cdot E^0\sin t+n\cdot E^D\cos t), \qquad n\cdot E^0:=n^iE^{i0}, \quad n\cdot E^D:=n^iE^{iD},\\
\delta\rho &= -n\cdot E^0\cos t+n\cdot E^D\sin t,\\
\delta n^i &= E^{ij}n^j+\coth\rho(n\cdot E^0\,n^i-E^{i0})\cos t-\coth\rho(n\cdot E^D\,n^i-E^{iD})\sin t.
\end{aligned} \tag{C.5}$$

The boundary (conformal) version of the above is

$$\begin{aligned}
\delta\tilde{t} &= -E^{0D}+\tilde{n}\cdot E^0\sin\tilde{t}+\tilde{n}\cdot E^D\cos\tilde{t},\\
\delta\tilde{n}^i &= E^{ij}\tilde{n}^j+(\tilde{n}\cdot E^0\,\tilde{n}^i-E^{i0})\cos\tilde{t}-(\tilde{n}\cdot E^D\,\tilde{n}^i-E^{iD})\sin\tilde{t}.
\end{aligned} \tag{C.6}$$

For the boundary points there is no $\tilde{\rho}$ coordinate, but for later use we keep the notation $\delta\tilde{\rho}=-\tilde{n}\cdot E^0\cos\tilde{t}+\tilde{n}\cdot E^D\sin\tilde{t}$. Also for later use we note that $\delta J=-\delta\tilde{\rho}=\omega/d$.

D Bulk reconstruction for $\Delta > d-1$ at generic bulk points

In this appendix we construct the building blocks $I(Y,x)$ and $T(Y,x)$, which are necessary to complete the bulk reconstruction for generic bulk points discussed in section 2.

D.1 The explicit form of $I(Y,x)$

The infinitesimal version of the first requirement in (2.22) is $\delta I=(\delta J)I=\frac{\omega}{d}I=-\delta\tilde{\rho}\,I$. We start from the invariant function depending on two bulk points in AdS given by

$$S=\cosh\rho\cosh\tilde{\rho}\cos(t-\tilde{t})-\sinh\rho\sinh\tilde{\rho}\,n\cdot\tilde{n}, \qquad \delta S=0. \tag{D.1}$$

For large $\tilde{\rho}$, the second point goes to the boundary and we have approximately

$$S\approx\frac{1}{4}\mathrm{e}^{\tilde{\rho}}I, \qquad I=2[\cosh\rho\cos(t-\tilde{t})-\sinh\rho\,n\cdot\tilde{n}]. \tag{D.2}$$

In this limit, the infinitesimal variation gives $0=\delta S=\frac{1}{4}\mathrm{e}^{\tilde{\rho}}(\delta\tilde{\rho}\,I+\delta I)$, which leads to $\delta I=-\delta\tilde{\rho}\,I$. The first requirement in its infinitesimal form is thus satisfied by this I, which also satisfies the second requirement since $I(Y_o,x)=2\cos\tilde{t}$ for $t=\rho=0$.

D.2 The explicit form of $T(Y,x)$

A natural guess is to take $T(Y,x)=\xi\Theta(X_1)\Theta(X_2)$, where $X_1=\tilde{t}-T_1$ and $X_2=T_2-\tilde{t}$. The second requirement in (2.23) is satisfied with this choice since $T(Y_o,x)=\xi\Theta\left(\tilde{t}+\frac{\pi}{2}\right)\Theta\left(\frac{\pi}{2}-\tilde{t}\right)$.

The infinitesimal variation of X_1 can be calculated using the formulas given in (C.5) and (C.6). After some calculation, we obtain

$$\begin{aligned}
\delta X_1=2\sin\frac{X_1}{2}\Bigg\{&n\cdot E^0\left(-\frac{\tanh\rho}{\cos\Psi}\sin\frac{\tilde{t}+T_1}{2}\right)+n\cdot E^D\left(-\frac{\tanh\rho}{\cos\Psi}\cos\frac{\tilde{t}+T_1}{2}\right)\\
&+\tilde{n}\cdot E^0\left(\tan\Psi\sin\frac{\tilde{t}+T_1}{2}+\cos\frac{\tilde{t}+T_1}{2}\right)+\tilde{n}\cdot E^D\left(\tan\Psi\cos\frac{\tilde{t}+T_1}{2}-\sin\frac{\tilde{t}+T_1}{2}\right)\Bigg\},
\end{aligned} \tag{D.3}$$

so that $X_1 = 0$ implies $\delta X_1 = 0$. Thus $\Theta(X_1)$ is invariant: $\Theta(X_1) = \Theta(X_1 + \delta X_1)$ for infinitesimal changes. We also observe that $\lim_{X_1 \to 0} \delta X_1 / X_1 = \delta\lambda$, where

$$\begin{aligned} \delta\lambda = n \cdot E^0 \left(-\frac{\tanh\rho}{\cos\Psi} \sin T_1 \right) + n \cdot E^D \left(-\frac{\tanh\rho}{\cos\Psi} \cos T_1 \right) \\ + \tilde{n} \cdot E^0 (\tan\Psi \sin\tilde{t} + \cos\tilde{t}) + \tilde{n} \cdot E^D (\tan\Psi \cos\tilde{t} - \sin\tilde{t}). \end{aligned} \tag{D.4}$$

We can make similar calculations and draw similar conclusions for X_2. For its infinitesimal variation, we obtain

$$\begin{aligned} \delta X_2 = -2\sin\frac{X_2}{2} \Bigg\{ n \cdot E^0 \left(-\frac{\tanh\rho}{\cos\Psi} \sin\frac{\tilde{t}+T_2}{2} \right) + n \cdot E^D \left(-\frac{\tanh\rho}{\cos\Psi} \cos\frac{\tilde{t}+T_2}{2} \right) \\ + \tilde{n} \cdot E^0 \left(\tan\Psi \sin\frac{\tilde{t}+T_2}{2} - \cos\frac{\tilde{t}+T_2}{2} \right) + \tilde{n} \cdot E^D \left(\tan\Psi \cos\frac{\tilde{t}+T_2}{2} + \sin\frac{\tilde{t}+T_2}{2} \right) \Bigg\}, \end{aligned} \tag{D.5}$$

so that $X_2 = 0$ implies $\delta X_2 = 0$. Finally $\lim_{X_2 \to 0} \delta X_2 / X_2 = \delta\bar{\lambda}$, where

$$\begin{aligned} \delta\bar{\lambda} = n \cdot E^0 \left(\frac{\tanh\rho}{\cos\Psi} \sin T_2 \right) + n \cdot E^D \left(\frac{\tanh\rho}{\cos\Psi} \cos T_2 \right) \\ - \tilde{n} \cdot E^0 (\tan\Psi \sin\tilde{t} - \cos\tilde{t}) - \tilde{n} \cdot E^D (\tan\Psi \cos\tilde{t} + \sin\tilde{t}). \end{aligned} \tag{D.6}$$

E Bulk reconstruction for $\Delta > d - 2$

We separate the bulk and boundary fields, $\mathcal{A}(t)$ and $\mathcal{C}(t)$, into positive/negative frequency parts, $\mathcal{A}_+(t)/\mathcal{A}_-(t)$ and $\mathcal{C}_+(t)/\mathcal{C}_-(t)$, which are given by the two terms of (2.7) and (2.10), respectively. Using these definitions, we have the identity

$$\mathrm{e}^{-i\Delta t} B(\mathrm{e}^{-2it}) = \mathrm{e}^{-i\frac{\Delta\pi}{2}} \mathcal{C}_+(t - \pi/2) = \mathrm{e}^{i\frac{\Delta\pi}{2}} \mathcal{C}_+(t + \pi/2). \tag{E.1}$$

Thus (3.3) leads to

$$\begin{aligned} \mathcal{A}_+(t) = \frac{1}{\Omega_d} \mathrm{e}^{-\frac{i\Delta\pi}{2}} \mathcal{C}_+(t - \pi/2) + \xi \int_{-\pi/2}^{0} \mathrm{d}u [2\cos(u)]^{\Delta - d} \{ \mathcal{C}_+(u+t) - \mathrm{e}^{-i(u+\pi/2)\Delta} \mathcal{C}_+(t - \pi/2) \} \\ + \xi \int_0^{\pi/2} \mathrm{d}u [2\cos(u)]^{\Delta - d} \{ \mathcal{C}_+(u+t) - \mathrm{e}^{-i(u-\pi/2)\Delta} \mathcal{C}_+(t + \pi/2) \}. \end{aligned} \tag{E.2}$$

Next by adding and subtracting an integral proportional to k_+ for the first integral and k_- for the second integral, where

$$k_\pm = \xi \int_0^{\pi/2} \mathrm{d}u (2\cos u)^{\Delta - d} [1 - \mathrm{e}^{\pm i\Delta(u - \pi/2)}], \tag{E.3}$$

which are convergent for $\Delta > d - 2$, we obtain

$$\begin{aligned} \mathcal{A}_+(t) = \left\{ \frac{1}{\Omega_d} + \mathrm{e}^{\frac{i\Delta\pi}{2}} k_+ + \mathrm{e}^{\frac{-i\Delta\pi}{2}} k_- \right\} \mathrm{e}^{-\frac{i\Delta\pi}{2}} \mathcal{C}_+(t - \pi/2) \\ + \xi \int_{t-\pi/2}^{t} \mathrm{d}u [2\cos(u - t)]^{\Delta - d} \{ \mathcal{C}_+(u) - \mathcal{C}_+(t - \pi/2) \} \\ + \xi \int_t^{t+\pi/2} \mathrm{d}u [2\cos(u - t)]^{\Delta - d} \{ \mathcal{C}_+(u) - \mathcal{C}_+(t + \pi/2) \}. \end{aligned} \tag{E.4}$$

JHEP02(2022)015

Analogously, repeating the calculation with $D_1(w)$, $B_1(w)$, $\mathcal{A}_-$ and $\mathcal{C}_-$, we have

$$e^{i\Delta t}B_1(e^{2it}) = e^{i\frac{\Delta\pi}{2}}\mathcal{C}_-(t-\pi/2) = e^{-i\frac{\Delta\pi}{2}}\mathcal{C}_-(t+\pi/2) \tag{E.5}$$

and

$$\begin{aligned}\mathcal{A}_-(t) = &\left\{\frac{1}{\Omega_d} + e^{\frac{i\Delta\pi}{2}}k_+ + e^{\frac{-i\Delta\pi}{2}}k_-\right\}e^{\frac{i\Delta\pi}{2}}\mathcal{C}_-(t-\pi/2)\\ &+\xi\int_{t-\pi/2}^{t}du[2\cos(u-t)]^{\Delta-d}\{\mathcal{C}_-(u)-\mathcal{C}_-(t-\pi/2)\}\\ &+\xi\int_{t}^{t+\pi/2}du[2\cos(u-t)]^{\Delta-d}\{\mathcal{C}_-(u)-\mathcal{C}_-(t+\pi/2)\}.\end{aligned} \tag{E.6}$$

These results can be further simplified by using the following two identities.

$$\begin{aligned}\mathcal{C}(t-\pi/2)+\mathcal{C}(t+\pi/2) &= \mathcal{C}_+(t-\pi/2)+\mathcal{C}_-(t-\pi/2)+\mathcal{C}_+(t+\pi/2)+\mathcal{C}_-(t+\pi/2)\\ &= 2\cos\frac{\Delta\pi}{2}\left\{e^{-\frac{i\Delta\pi}{2}}\mathcal{C}_+(t-\pi/2)+e^{\frac{i\Delta\pi}{2}}\mathcal{C}_-(t-\pi/2)\right\},\end{aligned} \tag{E.7}$$

$$\int_0^{\pi/2}du(2\cos u)^A\left\{\cos\frac{B\pi}{2}-\cos Bu\right\} = \frac{\pi}{2}\Gamma(1+A)\left\{\frac{\cos\frac{B\pi}{2}}{\Gamma^2(1+A/2)}-\frac{1}{\Gamma(1+\frac{A-B}{2})\Gamma(1+\frac{A+B}{2})}\right\}, \tag{E.8}$$

for $A>-2$, $B=A+d$, $d=3,5,7,\cdots$. Using the second identity we find

$$e^{\frac{i\Delta\pi}{2}}k_+ + e^{-\frac{i\Delta\pi}{2}}k_- = \xi\int_0^{\pi/2}du\,(2\cos u)^{\Delta-d}\left\{2\cos\frac{\Delta\pi}{2}-2\cos\Delta u\right\} = \frac{1}{\Omega_d}\left\{\eta\cos\frac{\Delta\pi}{2}-1\right\}. \tag{E.9}$$

Finally, adding $\mathcal{A}_+$ and $\mathcal{A}_-$ we obtain the final result (3.4), valid for the extended range $\Delta > d-2$.

F Details of the derivation of bulk reconstruction for small integer Δ

F.1 Bulk reconstruction for $\Delta = d-1$ at generic bulk points

We will look for a solution to the infinitesimal form of the first requirement, $\delta D = -\delta J\, D = \delta\tilde{\rho}\, D$. For the ansatz $D(Y,x) = \xi_o[f(Y,x)\delta_o(X_1) + g(Y,x)\delta_o(X_2)]$, the second requirement is satisfied if $f(Y_o,x) = g(Y_o,x) = 1$. Using the delta function relations

$$\begin{aligned}\delta_o(X_1+\delta X_1) &= \delta_o(X_1+\delta\lambda X_1) = (1-\delta\lambda)\delta_o(X_1),\\ \delta_o(X_2+\delta X_2) &= \delta_o(X_2+\delta\bar{\lambda}X_2) = (1-\delta\bar{\lambda})\delta_o(X_2)\end{aligned} \tag{F.1}$$

we see that the first requirement is equivalent to

$$\frac{\delta f}{f} = \delta\lambda + \delta\tilde{\rho} \qquad \text{(on the } X_1 = 0 \text{ hyperplane)}, \tag{F.2}$$

$$\frac{\delta g}{g} = \delta\bar{\lambda} + \delta\tilde{\rho} \qquad \text{(on the } X_2 = 0 \text{ hyperplane)}. \tag{F.3}$$

Considering

$$\sin T_1 = \sin\Psi \sin t - \cos\Psi \cos t, \qquad \cos T_1 = \cos\Psi \sin t + \sin\Psi \cos t, \tag{F.4}$$

we have

$$\begin{aligned}
\delta\lambda &= -\tanh\rho\, n\cdot E^0(\tan\Psi \sin t - \cos t) - \tanh\rho\, n\cdot E^D(\sin t + \tan\Psi\cos t)\\
&\quad + \tilde{n}\cdot E^0(\tan\Psi \sin\tilde{t} + \cos\tilde{t}) + \tilde{n}\cdot E^D(\tan\Psi\cos\tilde{t} - \sin\tilde{t})\\
&= -\tan\Psi(\delta t + E^{0D}) - \tanh\rho\,\delta\rho + \tan\Psi(\delta\tilde{t} + E^{0D}) - \delta\tilde{\rho}\\
&= \tan\Psi\delta(\tilde{t} - t) - \tanh\rho\,\delta\rho - \delta\tilde{\rho}.
\end{aligned} \tag{F.5}$$

Under the condition $X_1 = 0$

$$\delta\lambda + \delta\tilde{\rho} = -\tanh\rho\,\delta\rho + \tan\Psi\delta\Psi = -\frac{\delta\cosh\rho}{\cosh\rho} - \frac{\delta\cos\Psi}{\cos\Psi}. \tag{F.6}$$

Thus the solution to (F.2) is obtained by $f = \dfrac{1}{\mathcal{R}}$, where $\mathcal{R} = \cosh\rho\cos\Phi$ is given in (5.7).

Similarly we have

$$\delta\bar{\lambda} = \tan\Psi\delta(t - \tilde{t}) - \tanh\rho\,\delta\rho - \delta\tilde{\rho}, \tag{F.7}$$

which, under the condition $X_2 = 0$, implies

$$\delta\bar{\lambda} + \delta\tilde{\rho} = -\tanh\rho\,\delta\rho + \tan\Psi\delta\Psi = -\frac{\delta\mathcal{R}}{\mathcal{R}}. \tag{F.8}$$

We find that (F.3) has the same solution, $g = \dfrac{1}{\mathcal{R}}$. Thus the result for the complete kernel function is given by (5.6).

F.2 Bulk reconstruction for $\Delta = d - 2$ at generic bulk points

Let us first concentrate on the X_1 part of D_2. Motivated by the $\Delta = d - 1$ result we take the ansatz

$$D_2(Y,x) = -\tilde{\xi}_o[f_2(Y,x)\delta_o'(X_1) + p_2(Y,x)\delta_o(X_1)] + X_2 \text{ part}. \tag{F.9}$$

The first requirement is $\delta D_2 = -2\delta J D_2 = 2\delta\bar{\rho}D_2$. The second requirement will be satisfied if $f_2(Y_o,x) = 1$ and $p_2(Y_o,x) = 0$.

F.2.1 Delta function identities

We start from the well-known delta function relation

$$\delta_o(f(x)) = \frac{1}{|f'(0)|}\delta_o(x), \tag{F.10}$$

where we assume that the only zero of $f(x)$ is at $x = 0$. Then

$$\begin{aligned}
\int \mathrm{d}x\delta_o'(f(x))\mathcal{F}(x) &= \int \mathrm{d}x\delta_o'(f(x))f'(x)\frac{\mathcal{F}(x)}{f'(x)} = -\int \mathrm{d}x\delta_o(f(x))\Big[\frac{\mathcal{F}(x)}{f'(x)}\Big]'\\
&= -\frac{\mathcal{F}'(0)}{|f'(0)|f'(0)} + \frac{\mathcal{F}(0)f''(0)}{|f'(0)|f'^2(0)}.
\end{aligned} \tag{F.11}$$

This gives the delta function identity

$$\delta_o'(f(x)) = \frac{1}{|f'(0)|f'(0)}\delta_o'(x) + \frac{f''(0)}{|f'(0)|f'^2(0)}\delta_o(x). \tag{F.12}$$

We will apply the above identities to the infinitesimal variation of X_1:

$$\delta X_1 = \varepsilon_0 X_1 + \varepsilon_1 X_1^2 + \mathrm{O}(X_1^3). \tag{F.13}$$

In this case, $\delta_o'(X_1+\delta X_1) = (1-2\varepsilon_0)\delta_o'(X_1) + 2\varepsilon_1\delta_o(X_1)$. The infinitesimal change of the delta function and its derivative is thus

$$\delta[\delta_o(X_1)] = -\varepsilon_0\delta_0(X_1), \qquad \delta[\delta_o'(X_1)] = -2\varepsilon_0\delta_0'(X_1) + 2\varepsilon_1\delta_o(X_1). \tag{F.14}$$

Later we will also use the identity $X_1\delta_o'(X_1) = -\delta_o(X_1)$.

F.2.2 Expansions

For later use we now calculate and simplify the expansion coefficients in (F.13) and in the expansion of $\delta\tilde{\rho}$ (defined under (C.6)): $\delta\tilde{\rho} = r_0 + r_1 X_1 + \mathrm{O}(X_1^2)$. Using

$$\tilde{t} = T_1 + X_1, \qquad \frac{\tilde{t}+T_1}{2} = T_1 + \frac{X_1}{2} \tag{F.15}$$

we find from (D.3)

$$\begin{aligned}\varepsilon_0 &= n\cdot E^0\left(-\frac{\tanh\rho}{\cos\Psi}\sin T_1\right) + n\cdot E^D\left(-\frac{\tanh\rho}{\cos\Psi}\cos T_1\right)\\ &\quad + \tilde{n}\cdot E^0\left(\tan\Psi\sin T_1 + \cos T_1\right) + \tilde{n}\cdot E^D\left(\tan\Psi\cos T_1 - \sin T_1\right),\end{aligned} \tag{F.16}$$

$$\begin{aligned}2\varepsilon_1 &= n\cdot E^0\left(-\frac{\tanh\rho}{\cos\Psi}\cos T_1\right) + n\cdot E^D\left(\frac{\tanh\rho}{\cos\Psi}\sin T_1\right)\\ &\quad + \tilde{n}\cdot E^0\left(\tan\Psi\cos T_1 - \sin T_1\right) - \tilde{n}\cdot E^D\left(\tan\Psi\sin T_1 + \cos T_1\right).\end{aligned} \tag{F.17}$$

For the expansion of $\delta\tilde{\rho}$ we find

$$r_0 = -\tilde{n}\cdot E^0\cos T_1 + \tilde{n}\cdot E^D\sin T_1, \quad r_1 = \tilde{n}\cdot E^0\sin T_1 + \tilde{n}\cdot E^D\cos T_1. \tag{F.18}$$

Using the relations (F.4), (C.5), (C.6) we make the following calculations (for later use).

$$\begin{aligned}\varepsilon_0 + r_0 &= \tanh\rho\, n\cdot E^0(\cos t - \tan\Psi\sin t) - \tanh\rho\, n\cdot E^D(\sin t + \tan\Psi\cos t)\\ &\quad + \tan\Psi[\tilde{n}\cdot E^0\sin(\tilde{t}-X_1) + \tilde{n}\cdot E^D\cos(\tilde{t}-X_1)]\\ &= -\tan\Psi(\delta t + E^{0D}) - \tanh\rho\,\delta\rho + \tan\Psi[(\delta\tilde{t} + E^{0D}) + r_0X_1] + \mathrm{O}(X_1^2)\\ &= -\tanh\rho\,\delta\rho + \tan\Psi(\delta\Psi + \delta X_1 + r_0X_1) + \mathrm{O}(X_1^2)\\ &= \tan\Psi\,\delta\Psi - \tanh\rho\,\delta\rho + \tan\Psi(\varepsilon_0 + r_0)X_1 + \mathrm{O}(X_1^2).\end{aligned} \tag{F.19}$$

$$\begin{aligned}2\varepsilon_1 + 2r_1 &= -\tanh\rho\, n\cdot E^0(\sin t + \tan\Psi\cos t) - \tanh\rho\, n\cdot E^D(\cos t - \tan\Psi\sin t)\\ &\quad + \tan\Psi(\tilde{n}\cdot E^0\cos\tilde{t} - \tilde{n}\cdot E^D\sin\tilde{t}) + (\tilde{n}\cdot E^0\sin\tilde{t} + \tilde{n}\cdot E^D\cos\tilde{t}) + \mathrm{O}(X_1)\\ &= -(\delta t + E^{0D}) + \tan\Psi\tanh\rho\,\delta\rho - r_0\tan\Psi + (\delta\tilde{t} + E^{0D}) + \mathrm{O}(X_1)\\ &= \delta\Psi + \tan\Psi\tanh\rho\,\delta\rho - r_0\tan\Psi + \mathrm{O}(X_1).\end{aligned} \tag{F.20}$$

After temporarily dropping the $-\tilde{\xi}_o$ factor the first requirement (for the X_1 part) reads

$$\delta[f_2\delta_o'(X_1)+p_2\delta_o(X_1)]=2\delta\tilde{\rho}[f_2\delta_o'(X_1)+p_2\delta_o(X_1)], \tag{F.21}$$

which can be expanded as

$$\delta f_2\delta_o'+f_2[-2\varepsilon_0\delta_o'+2\varepsilon_1\delta_o]+\delta p_2\delta_o-\varepsilon_0 p_2\delta_o=2r_0f_2\delta_o'-2r_1f_2\delta_o+2r_0p_2\delta_o. \tag{F.22}$$

Reducing to zero and dividing by f_2 we get

$$\left[\frac{\delta f_2}{f_2}-2(\varepsilon_0+r_0)\right]\delta_o'+\delta_o\left[2(\varepsilon_1+r_1)+\frac{\delta p_2}{f_2}-(2r_0+\varepsilon_0)\frac{p_2}{f_2}\right]=0. \tag{F.23}$$

Thus the condition that the leading term multiplying δ_o' vanishes leads to $f_2=\mathcal{R}^{-2}$. Introducing the parametrization $p_2=f_2\omega_2$, the requirement that the coefficient of δ_o vanishes becomes

$$2\tan\Psi(\varepsilon_0+r_0)+2(\varepsilon_1+r_1)+2(\varepsilon_0+r_0)\omega_2+\delta\omega_2-(2r_0+\varepsilon_0)\omega_2=0. \tag{F.24}$$

This can be simplified to

$$\begin{aligned}&2\tan\Psi(\varepsilon_0+r_0)+\delta\Psi+\tan\Psi\tanh\rho\delta\rho-r_0\tan\Psi+\delta\omega_2+\varepsilon_0\omega_2\\&=\varepsilon_0\tan\Psi+\tan^2\Psi\delta\Psi+\delta\Psi+\delta\omega_2+\varepsilon_0\omega_2=\frac{\delta\Psi}{\cos^2\Psi}+\delta\omega_2+\varepsilon_0(\omega_2+\tan\Psi)=0.\end{aligned} \tag{F.25}$$

Now it is easy to see that $\omega_2=-\tan\Psi$ solves this equation, and indeed $p_2(Y_o,x)=0$.

The calculation of the X_2 part is completely analogous.

G BDHM relation for $\Delta=d-1$

Although the BDHM relation (5.9) [5] is one of the staring points of our derivation for (5.8), it is instructive to check it directly from the final formula (5.8).

By writing

$$\epsilon:=\frac{1}{\sinh\rho},\;\tanh\rho=\frac{1}{\sqrt{1+\epsilon^2}},\;n_y\cdot n_x:=\cos\gamma,\;\mathcal{R}=\frac{\sqrt{\epsilon^2+\sin^2\gamma}}{\epsilon}, \tag{G.1}$$

we should show

$$O(t,\Omega_y)=\lim_{\epsilon\to0}\frac{\xi_o}{\epsilon^{\Delta-1}}\int d\Omega\,\frac{1}{\sqrt{\epsilon^2+\sin^2\gamma}}\left[O(T_1,\Omega)+O(T_2,\Omega)\right], \tag{G.2}$$

where

$$T_1=t-\frac{\pi}{2}+\Psi,\;T_1=t+\frac{\pi}{2}-\Psi,\;\Psi:=\sin^{-1}\left(\frac{\cos\gamma}{\sqrt{1+\epsilon^2}}\right). \tag{G.3}$$

Since the BDHM relation (5.9) is a linear mapping, it is enough to verify it mode by mode. For the operator $A_{nl\underline{m}}$, what we need to show is

$$Y_{l\underline{m}}(\Omega_y)=\lim_{\epsilon\to0}\frac{\xi_o}{\epsilon^{\Delta-1}}\int d\Omega\,\frac{Y_{l\underline{m}}(\Omega)}{\sqrt{\epsilon^2+\sin^2\gamma}}\left[e^{i\nu_{nl}(\pi/2-\Psi)}+e^{-i\nu_{nl}(\pi/2-\Psi)}\right]. \tag{G.4}$$

JHEP02(2022)015

Without loss of generality, we can take $\Omega_y \sim n_0 := (1, 0, 0, \cdots, 0)$, so that $\gamma = \theta$ and $\Psi = \frac{\pi}{2} - \theta$. From the property of the hyper-spherical harmonics

$$Y_{lm\underline{\hat{m}}}(\Omega) = N^d_{lm\underline{\hat{m}}}(\sin\theta)^m\, C^{\alpha+m}_{l-m}(\cos\theta) Y_{m\underline{\hat{m}}}(\hat{\Omega}), \quad \alpha := \frac{d}{2} - 1,\ \underline{m} := m\underline{\hat{m}}, \tag{G.5}$$

where $\hat{\Omega}$ is a solid angle of the $d-2$ dimensional sphere, we obtain

$$Y_{l\underline{m}}(\Omega_y \sim n_0) = \delta_{\underline{m},\underline{0}} Y_{l\underline{0}}(n_0), \quad Y_{l\underline{0}}(n_0) = N^d_{l\underline{0}} C^\alpha_l(1) a_{d-1} \tag{G.6}$$

since $Y_{\underline{0}}(\hat{\Omega}) = a_{d-1} = 1/\sqrt{\Omega_{d-1}}$. Furthermore, using $d\Omega = \sin^{d-2}\theta\; d\theta\, d\hat{\Omega}$, we have

$$\xi_o \int d\Omega\, Y_{l\underline{m}}(\Omega) F(\theta) = \delta_{\underline{m},\underline{0}} N^d_{l\underline{0}} \xi_o a_{d-1} \Omega_{d-1} \int_0^\pi d\theta \sin^{d-2}\theta F(\theta) C^\alpha_l(\cos\theta), \tag{G.7}$$

where

$$F(\theta) := \frac{1}{\epsilon^{\Delta-1}} \frac{1}{\sqrt{\epsilon^2 + \sin^2\theta}} \left[e^{i\nu_{nl}\theta} + e^{-i\nu_{nl}\theta}\right]. \tag{G.8}$$

We then evaluate the integral given by

$$X := \lim_{\epsilon\to 0} \frac{1}{\epsilon^{\Delta-1}} \int_0^\pi d\theta \frac{\sin^{d-2}\theta}{\sqrt{\epsilon^2 + \sin^2\theta}} C^\alpha_l(\cos\theta) \left[e^{i\nu_{nl}\theta} + e^{-i\nu_{nl}\theta}\right]. \tag{G.9}$$

For odd $d = 2s + 3$ with $\Delta = d - 1$, we have

$$X = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{d-2}} \int_{-\pi}^\pi d\theta \frac{\sin^{d-2}\theta}{\sqrt{\epsilon^2 + \sin^2\theta}} C^\alpha_l(\cos\theta) e^{i\nu_{nl}\theta}. \tag{G.10}$$

By rewriting

$$\frac{1}{\epsilon^{d-2}} \frac{\sin^{d-2}\theta}{\sqrt{\epsilon^2 + \sin^2\theta}} = D_\epsilon(\theta) + \delta D_\epsilon(\theta), \tag{G.11}$$

where

$$D_\epsilon(\theta) := \frac{1}{\epsilon^{d-2}} \frac{\sin^{d-2}\theta}{\sqrt{\epsilon^2 + \sin^2\theta}} - \delta D_\epsilon(\theta), \tag{G.12}$$

$$\delta D_\epsilon(\theta) := \frac{1}{\epsilon} \sum_{k=0}^s \frac{(2k-1)!!}{(2k)!!} (-1)^k \left(\frac{\sin^2\theta}{\epsilon^2}\right)^{s-k}, \tag{G.13}$$

we see that

$$\int_{-\pi}^\pi d\theta\, \delta D_\epsilon(\theta) C^\alpha_l(\cos\theta) e^{i\nu_{nl}\theta} = 0, \tag{G.14}$$

since $\delta D_\epsilon(\theta)$ and $C^\alpha_l(\cos\theta)$ are polynomials of $e^{i\theta}$, $e^{-i\theta}$ of order $(d-3)$ and l, respectively, while $\nu_{nl} = d - 1 + 2n + l$.

Since $D_\epsilon(\theta)$ satisfies

$$\lim_{\epsilon\to 0} D_\epsilon(\theta) = \begin{cases} O(\epsilon) & \to 0, \qquad\qquad\quad \text{for } \sin\theta \neq 0, \\ (-1)^{s+1} O(\epsilon^{-1}) & \to (-1)^{s+1}\infty, \text{ for } \sin\theta = 0, \end{cases} \tag{G.15}$$

and $D_\epsilon(\pi - \theta) = D_\epsilon(\theta)$, we conclude

$$\lim_{\epsilon\to 0} D_\epsilon(\theta) = A(s) \left[\delta(\theta) + \delta(\pi - \theta)\right]. \tag{G.16}$$

G.1 Calculation of A

We write

$$A = \lim_{\epsilon\to 0}(A_0 + A_1), \quad A_0 := \int_0^\pi d\theta \, \frac{1}{\epsilon^{d-2}} \frac{\sin^{d-2}\theta}{\sqrt{\epsilon^2+\sin^2\theta}}, \; A_1 := -\int_0^\pi d\theta \, \delta D_\epsilon(\theta). \tag{G.17}$$

Thus, A_1 becomes

$$A_1 = -\frac{\sqrt{\pi}}{\epsilon^{2s+1}} \sum_{k=0}^{s} \left[\epsilon^{2k}(-1)^k \frac{(2k-1)!!}{(2k)!!} \frac{\Gamma(s-k+\frac{1}{2})}{\Gamma(s-k+1)} \right]. \tag{G.18}$$

The calculation of A_0 is more involved. Making a change of variables as $\cos\theta = \sqrt{1+\epsilon^2}\sin w$, we obtain

$$A_0 = \int_{-a}^{a} dw \, \left[\cos^2 w - \epsilon^2 \sin^2 w \right]^s, \tag{G.19}$$

where

$$\begin{aligned} a &:= \sin^{-1}\frac{1}{\sqrt{1+\epsilon^2}} = \cos^{-1}\frac{\epsilon}{\sqrt{1+\epsilon^2}} = \frac{\pi}{2} - \frac{\epsilon}{1+\epsilon^2} \sum_{n=0}^{\infty} \frac{(2n)!!}{(2n+1)!!} \left(\frac{\epsilon^2}{1+\epsilon^2} \right)^n \\ &= \frac{\pi}{2} - \epsilon \left[1 - \frac{\epsilon^2}{3} + \frac{3\epsilon^4}{15} + O(\epsilon^6) \right]. \end{aligned} \tag{G.20}$$

We calculate A_0 for $s = 0, 1, 2$ as follows.

$$\epsilon A_0(s=0) = 2a, \tag{G.21}$$

$$\epsilon^3 A_0(s=1) = a(1-\epsilon^2) + \sin(2a)\frac{(1+\epsilon^2)}{2} = a(1-\epsilon^2) + \epsilon, \tag{G.22}$$

$$\begin{aligned} \epsilon^5 A_0(s=2) &= a\frac{(3-2\epsilon^2+3\epsilon^4)}{4} + \sin(2a)\frac{(1-\epsilon^4)}{2} + \sin(4a)\frac{(1+2\epsilon^2+\epsilon^4)}{16} \\ &= \frac{1}{4}\left[a(3-2\epsilon^2+3\epsilon^4) + 3\epsilon(1-\epsilon^2) \right], \end{aligned} \tag{G.23}$$

where we use

$$\sin(2a) = \frac{2\epsilon}{1+\epsilon^2}, \quad \sin(4a) = \frac{4\epsilon(\epsilon^2-1)}{(1+\epsilon^2)^2}. \tag{G.24}$$

On the other hands,

$$A_1(s=0) = -\frac{\pi}{\epsilon}, \tag{G.25}$$

$$A_1(s=1) = -\frac{\pi}{2\epsilon^3}(1-\epsilon^2), \tag{G.26}$$

$$A_1(s=2) = -\frac{\pi}{8\epsilon^5}\left(3-2\epsilon^2+3\epsilon^4\right). \tag{G.27}$$

By combining these, we obtain

$$A(s=0) = -2, \qquad A(s=1) = \frac{4}{3}, \qquad A(s=2) = -\frac{16}{15}. \tag{G.28}$$

G.2 The result

The right-hand side of (G.4) now becomes

$$A(s)N^d_{l\underline{0}}\xi_o a_{d-1}\Omega_{d-1}\left[C^\alpha_l(1)+(-1)^l C^\alpha_l(-1)\right]\delta_{\underline{m},\underline{0}} = 2A(s)\xi_o\Omega_{d-1}Y_{l\underline{m}}(n_0), \tag{G.29}$$

where we have used $C^\alpha_l(-1)=(-1)^l C^\alpha_l(1)$. Since

$$2\xi_o\Omega_{d-1} = \frac{(-1)^{s+1}(2s+1)!!}{2^{s+1}s!} = -\frac{1}{2}, \frac{3}{4}, -\frac{15}{16}, \tag{G.30}$$

for $s=0,1,2$, which implies $A(s)2\xi_o\Omega_{d-1}=1$, so that the BDHM relation (G.4) holds for $s=0,1,2$. We expect in general

$$A(s) = (-1)^{s+1}\frac{2^{s+1}s!}{(2s+1)!!}, \tag{G.31}$$

for all non-negative integer s, which we verified up to $s=10$ by Mathematica.

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JHEP02(2022)015