Bound on energy dependence of chaos

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We conjecture a chaos energy bound, an upper bound on the energy dependence of the Lyapunov exponent for any classical/quantum Hamiltonian mechanics and field theories. The conjecture states that the Lyapunov exponent $\lambda(E)$ grows no faster than linearly in the total energy E in the high energy limit. In other words, the exponent c in $\lambda(E) \propto E^c (E \to \infty)$ satisfies $c \leq 1$. This chaos energy bound stems from thermodynamic consistency of out-of-time-order correlators and applies to any classical/quantum system with finite N/large N (N is the number of degrees of freedom) under plausible physical conditions on the Hamiltonians. To the best of our knowledge the chaos energy bound is satisfied by any classically chaotic Hamiltonian system known, and is consistent with the cerebrated chaos bound by Maldacena, Shenker, and Stanford, which is for quantum cases at large N. We provide arguments supporting the conjecture for generic classically chaotic billiards and multiparticle systems. The existence of the chaos energy bound may put a fundamental constraint on physical systems and the Universe.

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I. CONJECTURE

The statement of our conjecture is as follows. For any Hamiltonian system with its Hermitian Hamiltonian made by finite polynomials in coordinate/field variables,[1] the classical/quantum Lyapunov exponent $\lambda(E)$ measured at energy E in the high energy limit satisfies the following upper bound on its power in the energy dependence:

$$c \leq 1 \quad \text{for } \lambda(E) \propto E^c \quad (E \to \infty). \tag{1}$$

More precisely, for a given system there exist $C > 0$ such that $|\lambda(E)| \leq CE$ for any sufficiently large E. For quantum systems, the quantum Lyapunov exponent is measured [1] by out-of-time order correlators (OTOCs) [2]. We call (1) chaos energy bound.

This conjecture[2] is motivated by the well-definedness of the canonical ensemble for chaotic systems. Suppose a quantum (classical) Hamiltonian system has a chaos, as the OTOC $-\langle E|[q(t), p(0)]^2|E\rangle$ [Poisson bracket $\{q(t), p(0)\}_{\mathrm{P}}^2$] grows as $\exp[2\lambda(E)t]$.[3] Then the thermal Lyapunov exponent $\lambda_{\mathrm{th}}(T)$ is defined by [3,4]

$$\lambda_{\mathrm{th}}(T) \equiv \frac{1}{t}\log\left[-\int dE\rho(E)e^{-\beta E}\langle E|[q(t), p(0)]^2|E\rangle\right] \tag{2}$$

for large t (smaller than the Ehrenfest time), where $\rho(E)$ is the density of states and $\beta \equiv 1/T$ where T is the temperature.[4] The well-definedness of the canonical ensemble requires that the energy dependence of $\rho(E)$ should be weaker than that of the Boltzmann factor $e^{-\beta E}$ for the finiteness of the partition function. Thus, it follows that $\rho(E)$ grows no faster than an exponential function of E.[5] Then, the convergence of the

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[1]One can further restrict the domain of the spatial coordinates to some bounded region. We then assume that all components of the extrinsic curvature on the boundary surface are finite.

[2]The conjecture was first mentioned in a footnote in [3].

[3]In the literature, the expectation value of the squared commutator is often called OTOC since it contains out-of-time-ordered terms. We will adopt this definition since the squared-commutator OTOC translates directly into the classical Poisson bracket, offering not only quantum but also classical Lyapunov exponents in the classical limit.

[4]The definition (2) differs from that in [3,4] by a factor of 1/2 since (2) is what was used in the quantum large N bound (4).

[5]For the case of string theory, in spite of $\rho(E)$ being exponentially growing in E, the convergence argument still works as long as the temperature is lower than the Hagedorn temperature.

integral (2) requires the chaos energy bound (1), therefore, the bound allows one to treat the system at finite temperature. This argument for (2) only assumes the finiteness of the partition function and the integral (2) for chaotic systems, and applies no matter whether the system is quantum or classical, at finite N or large N, where N is the number of degrees of freedom of the system.

In the large N limit one can replace E by the temperature T.[6] Hence the chaos energy bound (1) in the large N limit leads to the chaos temperature bound

$$c \leq 1 \quad \text{for } \lambda_{\text{th}}(T) \propto T^c \quad (T \to \infty). \tag{3}$$

Let us remind the readers of the celebrated chaos bound conjectured by Maldacena, Shenker, and Stanford [1] for large N quantum systems,

$$\lambda_{\text{th}}(T) \leq 2\pi T/\hbar, \tag{4}$$

whose saturation is a discrimination diagnosis for existence of a black hole description in gravity dual. We find that the quantum large N bound (4) shows (3), thus the bound (1) (which can be applied to more generic systems[7]) is consistent with (4). Furthermore, since the saturation of (4) needs the saturation of (1), we can further conjecture that any holographic quantum system dual to a black hole should saturate the chaos energy bound (1).

II. EXAMPLES OF SYSTEMS

Any classically chaotic systems studied in literature satisfy the chaos energy bound (1), as far as we have checked. Here we list some for the readers' reference.

First we note that an ordered phase in the high energy limit is allowed in many chaotic models including double pendulum and sigma models [3], meaning $c < 0$ satisfying the chaos energy bound (1).[8]

For many-body systems, we are concerned with their largest Lyapunov exponent. For the Fermi-Pasta-Uram β-model, an analytic formula for the largest Lyapunov exponent [9] gives $c \sim 1/4$ and the bound (1) is satisfied. For a large number of coupled rotors, the formula gives $c = -1/6$ [9], which satisfies the bound (1).

As a field theoretic example, a chaotic string in anti–de Sitter soliton geometry [10] shows $\lambda(E) \sim \log E$, consistent with the bound (1). Thermalized fluids show $c = 1/2$ [11], satisfying (1). Homogeneous Yang-Mills mechanics [12] gives $c = 1/4$ [13,14], which is determined by the scaling. In Yang-Mills theories on a lattice [15–17] (see also [18–20]), the largest Lyapunov exponent $\lambda(E) \propto g^2 E$ (up to dimensionless coefficients where g is the coupling constant) saturates the bound (1).

III. GENERAL PARTICLE MOTION AND BILLIARDS

We show that billiards and their generalization satisfy the bound (1). Classical billiards with a standard kinetic Hamiltonian $H = p^2/2m$ allows a particle motion with the velocity $\dot{x} \propto \sqrt{E}$, thus any Lyapunov exponent of billiards, which is proportional to the inverse duration of hitting the boundary wall, satisfies $\lambda(E) \propto \dot{x} \propto \sqrt{E}$. This exponent of the billiard,

$$c = 1/2, \tag{5}$$

is subject to the bound (1).

For a generalized billiard with a kinetic Hamiltonian $H = p^\gamma$, Hamilton's equation is $\dot{x} = \gamma p^{\gamma-1} \propto E^{(\gamma-1)/\gamma}$, thus the Lyapunov exponent has a power

$$c = 1 - 1/\gamma, \tag{6}$$

which is less than 1 for any positive and finite γ. Therefore, the bound (1) is always satisfied.[9,10]

The argument above is expected to apply to any sparse many-body system of N particles with a finite-range interaction, as any potential boils down to a contact scattering at $E \to \infty$. Then, the typical velocity of the particle is $v \sim \sqrt{E/N}$ for $\gamma = 2$. The scattering rate is proportional to v and, thus, $\lambda(E) \propto \sqrt{E}$.

We can also show that billiards with softened walls, which are particles in generic potentials, may obey the bound (1). Consider a two-dimensional system approximated by $H = p_1^2/2 + p_2^2/2 + x_1^m x_2^n = \dot{x}_1^2/2 + \dot{x}_2^2/2 + x_1^m x_2^n$, where the last term is a dominant term in a generic potential $V(x)$ at $E \to \infty$. Here m and n are positive because the orbits for defining the chaos need to be bounded.[11] Then the following scaling symmetry[12] $t \to \alpha t$, $x_i \to \alpha^{2/(2-m-n)} x_i$ and $H \to \alpha^{2(m+n)/(2-m-n)} H$ leads to an equation $\lambda(E) t = \lambda(\alpha^{2(m+n)/(2-m-n)} E) \cdot \alpha t$, which is solved to give the exponent as

[6] See Sec. I of the Supplemental Material [5] for the large N saddle point approximation.

[7] Note that the classical limit $\hbar \to 0$ of (4) does not lead to any bound.

[8] The Hénon-Heiles system and particle/string motion around black holes [6–8] may not allow the high energy limit.

[9] Note that any negative γ does not allow chaos to be defined, because the motion stops asymptotically.

[10] An example $H \sim e^{p^2}$ can violate the bound but does not satisfy our polynomial assumption for H. In fact, in quantum mechanics, $H \sim e^p, e^{p^2}$ are nonlocal and thus physically deserted. See Sec. II of the Supplemental Material [5] for details about our assumptions on the physicality of the Hamiltonians.

[11] A class of hard-wall billiards is described by a limit $m, n \to \infty$.

[12] This scaling is a generalization of what is described in [13,21,22]. A general argument for the determination of the energy dependence of the Lyapunov exponent by the scaling symmetry is given in Sec. III of the Supplemental Material [5].

$$c = 1/2 - 1/(m+n). \quad (7)$$

Thus the general two-dimensional classical mechanics satisfies the chaos energy bound (1).[13]

IV. SPECULATIONS

The chaos energy bound (1) in quantum field theories (QFTs) is naturally understood as follows. First, notice that, although there can be many coupling constants in the theory, the one with the smallest mass dimension will be dominant at high energy. Let g be such a coupling,[14] and denote its mass dimension as d_g. In other words, at high energy, the only dimensionful parameters at hand are E and g. Then, using some dimensionless constants a, b, and c, the Lyapunov exponent should be written as $\lambda = bE^c g^a$ with $a > 0$ since the chaos should vanish when the nonlinearity goes away at $g = 0$. Since the Lyapunov exponent has mass dimension 1, the dimensional analysis determines c as

$$c = 1 - ad_g. \quad (8)$$

Assuming that the QFT is consistent at high energy, the perturbative renormalizability requires $d_g \geq 0$, which means the chaos energy bound (1). The renormalizability makes sure that no new structure emerges at higher energy scales, which is in accord with the spirit of our polynomial assumption made for Hamiltonians.

Conversely, as we can see from the above argument, if a given field theory allows a particle picture (i.e., the perturbation theory applies), the chaos energy bound ensures the perturbative renormalizability.[15] Thus, requiring the perturbative renormalizability is equivalent to requiring the chaos energy bound. Moreover, perturbatively renormalizable theories have been found only when the spacetime dimensions are less than or equal to four. Therefore, we can even argue that the chaos energy bound implies a restriction of the spacetime dimensions.

Finally, recall that the chaos energy bound follows just from the well-definedness of the canonical ensemble, namely, the finiteness of the canonical partition function and the integral (2) for chaotic systems. In particular, no matter whether the system is quantum or classical, at finite N or large N, the chaos energy bound (1) applies. This universality may put a novel constraint on physical theories and even the chaos of the universe. For example, remember that the sum of the positive Lyapunov exponents is the Kolmogolov-Sinai entropy growth rate. Since naively a subsystem with the dominant entropy production may dominate the whole system, the fundamental theory of the Universe may need to saturate the chaos energy bound, which could be a selection principle of a QFT dictating the Universe. According to (8), the bound is saturated by QFTs with $d_g = 0$, which are classically conformal theories. Interestingly, the standard model of elementary particles is almost classically conformal [23]. Thus, the classical conformality as a principle of particle phenomenology [24,25] can be motivated also from the saturation of the chaos energy bound (1).

[13] In Sec. IV of the Supplemental Material [5] we provide calculations of the exponent c for more generic Hamiltonians.

[14] Even if there are multiple coupling constants with the smallest mass dimension, our argument still holds.

[15] Here, the chaos energy bound plays a crucial role. The existence of particle description is not sufficient for the perturbative renormalizability. For example, the four-Fermi theory allows particle description while it is nonrenormalizable in four dimensions.

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