Linear Theory of Tearing Instability with Viscosity, Hyper-Resistivity, and improved WKB-approximation

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研究目的 (Research Objective):

In past years, the linear perturbation equations of tearing instability derived by Loureiro (Loureiro,PoP2007 named as LSC theory)) has been deeply and widely explored by numerically solving as initial value problem (IVP) (Shimizu, AAPPS-DPP2018, KDK Research Report2018, Shimizu&Kondoh, arXiv4472111). The Loureiro's equations are based on Non-Viscous case. In this paper, Viscosity and Hyper resistivity are introduced to the equations (Shimizu, KDK Research Report2020,21&22, Shimizu&Fujimoto, AOGS2021&22). Then, Non-Uniform viscosity and improved WKB-approximation are also studied (Shimizu, AAPPS-DPP2021&22 and ICNSP2022). This paper summaries those variations of the perturbation equations with some highlighted numerical results.

1. Introduction:

This paper starts from the linear perturbation equations of tearing instability shown next, which were derived by Loureiro,et.al. (PoP2007).

$$\phi'' - \kappa^2\epsilon^2\phi = -f(\xi)(\psi'' - \kappa^2\epsilon^2\psi)/\lambda + f''(\xi)\psi/\lambda, \qquad (1\text{-}1)$$

$$\psi'' - \kappa^2\epsilon^2\psi = \kappa\lambda\psi - \kappa f(\xi)\phi. \qquad (1\text{-}2)$$

$$f(\xi) = \xi_0 e^{-\xi^2/2}\int_0^{\xi} dz e^{z^2/2}, \qquad (1\text{-}3)$$

Every notation in this paper is based on the Loureiro's definitions, where ϕ and ψ are respectively perturbed potential functions of flow and magnetic fields. The prime is the derivative for the direction normal to the current sheet, where f(ξ) is the equilibrium function of magnetic field Bxo, as shown below.

$$B_{x0}(\xi) = V_A f(\xi), \quad u_{y0} = -\Gamma_0 y, \qquad (1\text{-}4,\ 1\text{-}5)$$

Eq.(1-4) is based on the equilibrium linear flow field of Eq.(1-5), where y is translated to ξ with y=1.307 ξ . In Eqs.(1-1&2), λ is the linear growth rate and κ is the wave number of ϕ and ψ along the current sheet. In the original LSC theory (PoP2007) and most of my previous studies, Eqs.(1-4&5) are applied only for ξ<1.307, i.e., inside of the current sheet. However, in this paper, Eqs.(1-4&5) are applied also for ξ>1.307 to rigorously keep the equilibrium even in the introduction of viscosity. Resistivity ε and Lundquist number S are defined below, with the sheet thickness δ cs and sheet length Lcs of Sweet-Parker model.

$$\epsilon = 2\delta_{cs}/L_{cs} = 2/\sqrt{S} \tag{1-6}$$

Loureiro analytically solved Eqs.(1-1)~(1-6) under the upstream condition of $\phi = \psi$=0 at ξ=+∞ (PoP2007). To do so, the traditional approximation introduced by FKR theory (Fruth,PhysFluids1963) was employed, where the outer region of the current sheet was assumed to be ideal-MHD, and hence, the inner region is only solved in resistive-MHD. In contrast, Shimizu solved Eqs.(1-1)~(1-6) without the assumption of the ideal-MHD, i.e., every region was seamlessly solved in resistive-MHD. To do so, Shimizu did not consider the upstream condition of $\phi = \psi$=0 at ξ=+∞. Instead, $\phi = \psi$=0 at ξc<+∞, i.e., a finite point ξc, was studied. The condition can be considered to be open boundary condition at finite upstream point ξc. The concept of the open boundary may be close to what is widely employed in MHD simulations. Then, Eqs.(1-1)~(1-6) were numerically solved as initial value problem (IVP) from ξ=0 to ξc (Shimizu, AAPPS-DPP2018, KDK Research Report2018, Shimizu&Kondoh, arXiv4472111). The LSC theory modified by Shimizu was named as modified-LSC theory. The modified-LSC can explore the case of λ=0, which gives the critical (marginal) unstable condition of tearing instability, as shown in the ν=0 line of Figs.1 and 2.

2. Uniform Viscosity:

When viscosity is added to Eqs(1-1&2), the equations to be solved are as below, where ν is the viscosity coefficient in isotropic viscosity.

$$\nu\phi'''' = \kappa\epsilon^2((\lambda + 2\kappa\nu)\phi'' - (\lambda + \kappa\nu)\kappa^2\epsilon^2\phi + f(\xi)(\psi'' - \kappa^2\epsilon^2\psi) - f''(\xi)\psi) \tag{2-1}$$

$$\psi'' - \kappa^2\epsilon^2\psi = \kappa\lambda\psi - \kappa f(\xi)\phi. \tag{2-2}$$

These equations can be also numerically solved as IVP from ξ=0 with initial values of ϕ'(0) and ϕ'''(0), where ϕ(0)=ϕ''(0)=ψ'(0)=0 and ψ(0)=1 are fixed for the symmetric current sheet. In addition to ϕ'(0) and ϕ'''(0), changing λ, κ, ε and ν, ϕ and ψ can be solved as IVP so that $\phi = \phi' = \psi$=0 are satisfied at ξc. Let us call it Zero-Contact solution, where "Zero-Contact" means $\phi = \phi'$=0. Hence, once a set of λ, κ, ε and ν is specified, ξc is determined. It means that the linear growth rate λ depends on the location ξc of upstream boundary. Then, Zero-Converging solution which satisfies ϕ(+∞)=ψ(+∞)=0 may be deduced by examining larger ξ c (see Fig.1). Then, if λ =0 is set, the critical unstable condition is found, as shown in ν >0 in Fig.2.

3. Non-Uniform Viscosity:

In traditional studies such as FKR(1963) and LSC(2007) theory, the outer region (ξ >1.307) is assumed to be ideal-MHD which means $\varepsilon = \nu$ =0. It seems that they expect that the assumption is also applicable for when the outer region is solved in resistive-MHD. In other words, they expect that whether the outer region is ideal-MHD or resistive-MHD is not essential to study the linear growth of tearing instability. However, such an expectation fails at some points.

In this section, Eqs.(2-1&2) are solved only in the inner region of the current sheet (ξ <1.307). Meanwhile, ν =0 is assumed in the outer region (1.307< ξ), where Eqs.(1-1&2) is solved. Hence, viscosity works only in the current sheet. Note that resistivity ε (>0) is uniformly kept even in the outer region. In this case, to compensate the discontinuity of ν at ξ =1.307, there are two strategies for the combination of the differential continuity of ϕ at ξ =1.307 and the upstream boundary condition at ξ c.

Strategy 1:

To keep the continuity of ϕ" at ξ=1.307, the next equation must be satisfied at ξ =1.307-0, where is close to ξ =1.307 from ξ <1.307.

$$\phi'''' = 2\kappa^2\epsilon^2\phi'' - \kappa^4\epsilon^4\phi \qquad (3\text{-}1)$$

Then, find ϕ and ψ which satisfies $\phi = \psi$ =0 at ξ c. In this strategy, ϕ'=0 is not required at ξc. Hence, that is not Zero-Contact solution. Rather, that is called as Zero-Crossing solution. This will be the most rigorous solution. Mathematically, this may be a kind of "strong" solution. To study the case of λ =0, Eq.(1-1&2) for ξ >1.307 is replaced by below. Fig.1 includes this numerical result, and Fig.3 shows the highlighted summary of λ =0.

$$0 = \kappa f(\xi)^2\phi + f''(\xi)\psi, \qquad (3\text{-}2)$$

$$\psi'' - \kappa^2\epsilon^2\psi = -\kappa f(\xi)\phi = f''(\xi)\psi/f(\xi). \qquad (3\text{-}3)$$

Strategy 2:

In this strategy, Eq.(3-1) is ignored. Hence, the continuity of ϕ" at ξ=1.307 is not satisfied but ϕ' is still continuous. Then, find ϕ and ψ which satisfies ϕ = ϕ'= ψ =0 at ξ c. Hence, this is Zero-Contact solution. Mathematically, this may be a kind of "weak" solution.

Physically, Non-Uniform viscosity may be considered to be "anomalous" viscosity. That is physics. On the other hand, most numerical simulations show not only strong solutions but often also weak solutions, depending on the employed numerical scheme. That is not physics but numerical error. In actual, when Non-Uniform viscosity is steadily included in numerical dissipation to numerically stabilize MHD simulations. For example, in the shock-capturing schemes such as TVD and HLLD, higher-order differential continuity of the solutions may not necessarily be kept around the extremely thin current sheet. Strategy 2 may be able to examine how tearing instability is disturbed by such numerical Non-Uniform viscosity. Fig.1 includes this numerical result.

4. Hyper Resistivity (electron viscosity):

Hyper resistivity means the fourth-order differential magnetic diffusion, while usual resistivity is the second-order magnetic diffusion. Some kinematic full-particle simulations of the magnetic reconnection process predict that such higher-order magnetic diffusion is dominant rather than the second-order. That is physics. In another viewpoint, every finite-differential MHD simulations have numerical dissipations of such higher-order diffusion to stably simulate extremely thin current sheets. That is not physics. Such numerical diffusive error must be examined for how tearing instability is affected or not. For these reasons, hyper resistivity is worth to be studied in comparison with usual resistivity.

For simplicity, the viscosity ν examined in the preceding section is ignored in this section. First, equilibriums f(ξ) are studied in the mixture of hyper and usual resistivities. Second, the perturbed solutions are studied on the basis of the equilibrium.

Equilibrium 1:

Note that Eq.(1-3) is applicable only for usual resistivity. When hyper resistivity effect is added to the usual resistivity effect, f(ξ) must satisfy the following equation.

$$(1/S_{Hi})f'''(\xi) = (1/S_i)f'(\xi) + \xi f(\xi) + c \qquad (4\text{-}1)$$

where 1/Si and 1/SHi are respectively the intensity of usual resistivity and hyper resistivity. Si and SHi are "each" Lundquist number for inflow speed to neutral sheet (Shimizu&Fujimoto, AOGS2022). At this point, ε defined in Eq.(1-6) remains as "total" Lundquist number on the basis of Sweet-Parker model. In other words, Lundquist number referred in this section consists of ε (1st step) and either of Si and SHi (2nd step).

To appropriately normalize Eq.(4-1), let us fix the inflow speed uyo=-1.307 at ξ=1.307 in Eq.(1-5). In addition, let us fix f(1.307)=1.0 and f'(1.307)=0. With these setups, the second term of the rhs of Eq.(4-1), i.e., the convection electric field (V×B) is fixed at -1.307f(1.307)=-1.307, where ξ=1.307 is the boundary point of the inner and outer region of the current sheet. This normalization is an extension of the concept employed in the original LSC (PoP2007).

Eq.(4-1) can be numerically solved as IVP from ξ=0 with initial values f(0)=f'''(0)=0 and f'(0). Note that f(0)=f'''(0)=0 is fixed for the symmetric current sheet. Eventually, Si, SHi, f'(0) and c are the control parameters to numerically find the solution of f(ξ) which satisfies f(1.307)=1, f'(1.307)=0 and f(ξ=+∞)=0.

As for the case of 1/SHi=0, f'(0) is not needed to solve Eq.(4-1). At this time, f(ξ) obtained for 1/Si=1.0 and c=-1.307 coincides with Eq.(1-3). Then, as 1/Si decreases from 1.0 to 0.0, f(ξ) gradually changes, where 1/SHi increases from 0.0 to 0.43, which is shown as the line of white square boxes of Fig.4a and 4b.

Equilibrium 2:

Note that Eq.(1-3) is derived for Eq.(1-5). If Eq.(1-5) is changed, Eq.(1-3) is changed. If Eq.(1-5) is replaced by uyo=g tanh(a ξ), Eq.(4-1) is replaced by below.

$$(1/S_{Hi})f'''(\xi) = (1/S_i)f'(\xi) + g\tanh(a\xi)f(\xi) + c \qquad (4\text{-}2)$$

where *g* and *a* are free parameters to adjust the scaling of the equilibrium flow field. In the same manner as Equilibrium 1, let us fix the convection electric field (V×B)=-1.307. At the time, *g* and *a* are mutually related. For example, *a*=1.0 results in *g*=-1.307/0.8635. Otherwise, *a*=0.5 results in *g*=-1.307/0.5740, where *g* tanh(1.307*a*)=1.307 is always kept. It may be noted that, if *a*=1 and SHi=0 are set, Eq.(4-2) analytically results in f(ξ)=tanh(ξ) which is the well-known Harris sheet. On the other hand, in the *a*=0 limit, f(ξ) in Eq.(4-2) becomes that of Eq.(4-1), resulting in Eq.(1-3). Meanwhile, if a≠1, such analytical solutions are not found but Eq.(4-2) is numerically solved as IVP, so that f(ξ) satisfies f(1.307)=1, f'(1.307)=0 and f(ξ=+∞)=0. The numerical result is shown in Fig.4a.

Perturbation equations：

The perturbed equations to be solved for Equilibrium1 are shown below. For simplicity, Equilibrium 2 is not studied in this paper.

$$(1/S_{Hi})\psi'''' = -\lambda\kappa\psi + \kappa f(\xi)\phi + (1/S_i)(\psi'' - \kappa^2\epsilon^2\psi)) - (1/S_{Hi})\kappa^4\epsilon^4\psi \qquad (4\text{-}3)$$

$$\phi'' - \kappa^2\epsilon^2\phi = -f(\xi)(\psi'' - \kappa^2\epsilon^2\psi)/\lambda + f''(\xi)\psi/\lambda \qquad (4\text{-}4)$$

Also, these equations can be numerically solved as IVP, where Si, SHi, ε, λ, κ, $\phi'(0)$, and $\phi''(0)$ are the control parameters to find Zero-Contact solutions which satisfy $\phi=\psi=0$ and $\phi'=0$ at ξc. The numerical results of IVP are shown in $\varepsilon=0.1$ and 0.5 of Fig.4b.

5. Improvement of WKB Approximation:

Rigorously, Eqs.(1-1)-(4-4) are inapplicable for $\kappa\sim0$ range because the WKB approximation is the zero-order. In other words, κ is assumed to be constant in time. To explore the $\kappa\sim0$ range, Eqs.(6)-(7) shown in Loureiro,PoP2007 must be solved, which has the first-order, and hence, the time variation of κ is considered. However, Eqs.(6)-(7) cannot be directly solved as IVP because some terms (e.g., $-\xi\phi'''/\kappa$ term in Eq.(5-1)) for WKB become zero at $\xi=0$. However, the IVP can be solved with viscosity terms. Eventually, Eqs.(2-1&2) is replaced by the next equations.

$$\nu\phi'''' = \kappa\epsilon^2((\lambda+2\kappa\nu)\phi''-(\lambda+\kappa\nu)\kappa^2\epsilon^2\phi+f(\xi)(\psi''-\kappa^2\epsilon^2\psi)-\xi\phi'''/\kappa+\kappa\epsilon^2\xi\phi'+2\kappa\epsilon^2\phi-f''(\xi)\psi) \quad (5\text{-}1)$$

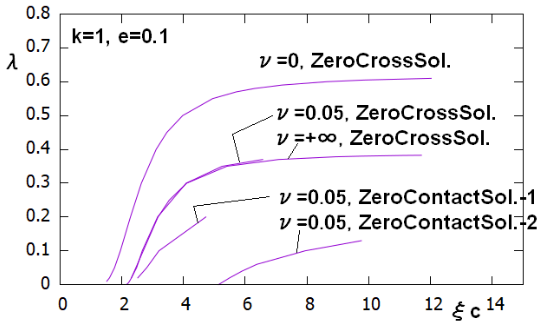
$$\psi''-\kappa^2\epsilon^2\psi=\kappa\lambda\psi-\xi\psi'-\kappa f(\xi)\phi \quad (5\text{-}2)$$

How to solve this IVP is basically the same as Eqs.(2-1)-(2-2). Until last year, the Zero-Contact solutions of Eqs.(5-1&2) could not be found but, eventually, have been found by improving the IVP technique. The Zero-Contact solutions of Uniform viscosity for $\lambda=0$ and $\varepsilon=0.1$ are summarized in Fig.4.

6. Numerical Results of IVP:

Fig1 shows how the linear growth rate λ depends on ξc which is the location of the upstream open boundary. The current sheet is located in $0<\xi<1.307$. Every solid line shows that λ tends to be higher as ξc is separated from the current sheet. However, it seems that λ deduced at $\xi c=+\infty$ (right outside of figure) does not exceed unity, i.e. Alfven speed unit time. It suggests that the linear growth of tearing instability cannot be fast beyond the Alfven speed measured in the upstream magnetic field region.

The solid line of $\nu=0$ obtained for Eqs.(1-1&2) (i.e., Non-Viscous case) takes the highest growth rate in this figure. Inversely, the solid line of $\nu=0.05$ labeled as "ZeroContactSol-2" takes the lowest rate, which is obtained for Uniform viscosity, i.e., Eqs(2-1&2). The other three lines are for Non-Uniform viscosity, which are obtained for Strategy1 (ZeroCrossSol.) and 2 (ZeroContactSol.-1).



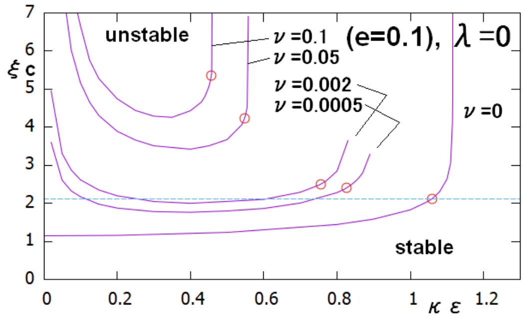
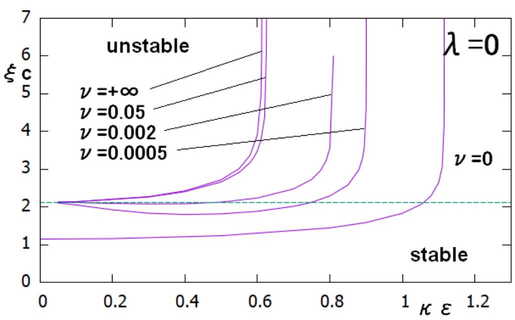


Fig.1: λ and ξ c relations　Fig.2: Critical unstable condition (λ =0). The line of ν =0 is for Eqs.(1-1&2) and the lines of ν >0 are for Uniform case of Eqs.(2-1&2).

It is remarkable that the line of ν =+∞ for Non-Uniform viscosity gives a finite growth rate. Similar results have been reported in Fig.1b of Shimizu,KDK Research Report2022. In contrast, for Uniform viscosity, Zero-Contact solutions in ν =+∞ cannot be found. It suggests that Uniform and Non-Uniform cases have essentially different characteristic.

Fig.2 shows how ξ c obtained for λ =0 depends on wave number κ, where Non-Viscous and Uniform viscosity cases are studied for ε (=e)=0.1. Hence, Fig.2 indicates the foot points (λ =0) of the solid lines of Fig.1. Under each solid line is stable (λ <0). Hence, when ξ c<1.307, tearing instability is completely stabilized for all κ range. Then, as ν increases from 0.0, the stable region spreads upward, i.e., to larger ξ c. The solid line of ν =0.0 completely becomes vertical around ke=1.15 and higher ξ c. This corresponds to the critical condition of the positive prime index Δ'>0 in FKR (see Appendix C in Shimizu&Kondoh, arXiv4472111 but it was when Eqs.(1-4&5) is applied only in ξ <1.3), which is observed also in Figs.3 and 4. Since the vertical solid lines observed in κ ε >0.4 shifts to lower κ, as ν increases. It means that the critical condition depends on viscosity.



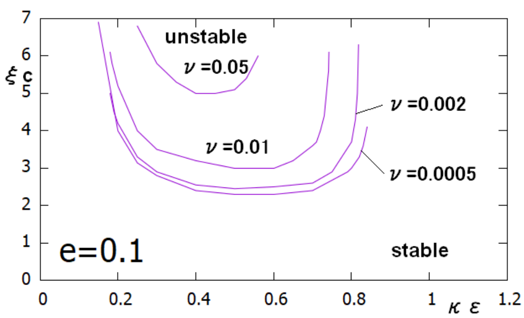


Fig.3(Non-Uniform) and 4(improved-WKB-Uniform): Critical unstable conditions (λ =0 & ε (=e)=0.1).

Figs.3 and 4 respectively show the cases of Non-Uniform viscosity (Eqs. (2-1&2) &(3-1,2&3)) and improved-WKB-Uniform viscosity (Eqs.(5-1&2)). In comparison with Fig.2, the unstable area in Fig.3 is wider in κ ~0 range. As shown in Fig.1, it is remarkable that unstable region even in ν=+∞ appears in ξc>2.1 and $\kappa\varepsilon$ <0.6. Inversely, the stable area in Fig.4 is wider. It means that viscosity can steadily stabilize tearing instability.

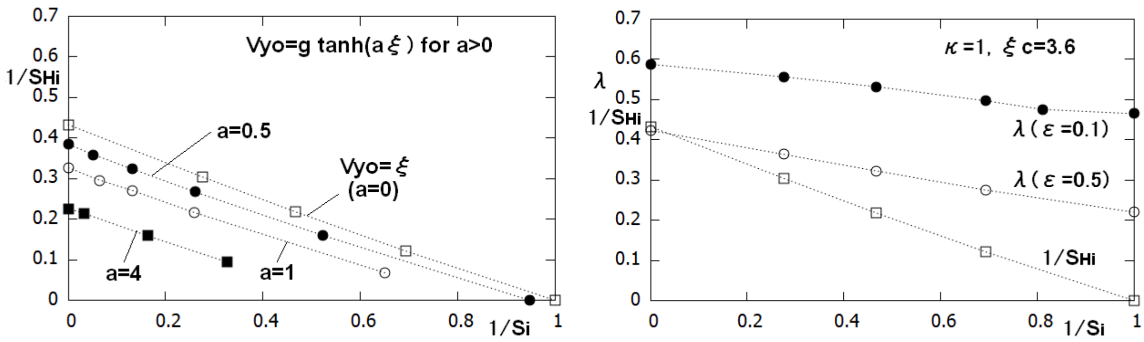


Fig.4a(Equilibriums) and 4b(Perturbed solutions): Hyper Resistivity.

Fig.4a shows the relation of 1/Si and 1/SHi established in the equilibrium variations obtained for Eqs.(4-1 or 2). Vyo in this figure is -uyo in Eq.(1-5). For Eq.(4-2), the variations of 0.5<a<4 is shown. In the a=0 limit, Eq.(4-2) coincides with Eq.(4-1). Since the convection electric field v×B at ξ=1.307 is fixed at -1.307, the magnetic flux conveyed through ξ=1.307 every unit time is fixed. It means that 1/Si and 1/SHi are complimentarily balanced. In other words, the increase (decrease) of 1/Si results in decrease (increase) of 1/SHi, as shown in Fig.4a.

Fig.4b shows how λ depends on 1/Si. The line of 1/SHi is the same as that of Fig.4a. The most right side (i.e., 1/Si=1) of this figure corresponds to Non-Viscous case, i.e., Eqs.(1-1&2). As hyper resistivity 1/SHi is strengthen (i.e., 1/Si is weakened), λ becomes higher. It means that the tearing instability caused by hyper resistivity grows faster than that of usual resistivity.

公表状況（Publications and Presentations）：

1. Tohru Shimizu and K.Kondoh, A New Approach of Linear Theory of Tearing Instability in Uniform Resistivity, physics.plasma-ph, http://arxiv.org/abs/2209.00149
2. Tohru Shimizu and K. Fujimoto, MHD Linear Theory of Tearing Instability for Fourth-Order Differential Diffusion (Hyper Resistivity) Effect, ST03-A005, proc. of AOGS2022, (Singapore, Remote, 2022Aug.)
3. Tohru Shimizu, Linear Theory of Tearing Instability with Open Boundary Conditions, P12,1F-12, proc. of ICNSP2022 (Remote, 2022Aug.-Sept.)
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5. 清水徹、開放(自由)境界条件におけるテアリング不安定性の線形理論、SGEPSS2022, (Kanagawa, Japanese domestic meeting, 2022Nov.).