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Low regularity a priori estimate for KDNLS via the short-time Fourier restriction method

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Dedicated to the memory of Professor Jean Ginibre

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Abstract

In this article, we consider the kinetic derivative nonlinear Schrödinger equation (KDNLS), which is a one-dimensional nonlinear Schrödinger equation with a cubic derivative nonlinear term containing the Hilbert transformation. For the Cauchy problem, both on the real line and on the circle, we apply the short-time Fourier restriction method to establish *a priori* estimate for small and smooth solutions in Sobolev spaces H^s with $s > 1/4$.

Keywords: Kinetic derivative nonlinear Schrödinger equation; Short-time Fourier restriction method; A priori estimate; Well-posedness

1 Introduction

In the present article, we continue our study in [1, 2] and consider the kinetic derivative nonlinear Schrödinger equation (KDNLS) on \mathbb{R} and on $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$:

$$\partial_t u = i\partial_x^2 u + \alpha \partial_x (|u|^2 u) + \beta \partial_x (H(|u|^2)u), \quad t \in (0, T), x \in \mathbb{R} \text{ or } \mathbb{T}, \quad (1.1)$$

where α, β are real constants and H is the Hilbert transformation. We assume $\beta < 0$ throughout this article.

In the periodic case, we proved in [2] that the Cauchy problem has a (forward-in-time) global solution for any initial data in $H^s(\mathbb{T})$ if $s > 1/4$, with the solution map $u(0) = u_0 \mapsto u(\cdot)$ being (locally-in-time) continuous in the H^s topology away from the origin $u_0 = 0$. More precisely, we proved the following claims:

- (i) For any $s > 1/4$ and any $R \geq r > 0$, there exist $T > 0$ and a solution map $u_0 \mapsto u$ on the set $\{u_0 \in H^s(\mathbb{T}) : \|u_0\|_{H^s} \leq R, \|u_0\|_{L^2} \geq r\}$, which gives a solution $u \in C([0, T]; H^s(\mathbb{T}))$ to (1.1) on $[0, T]$ with $u(0) = u_0$ and is continuous in the H^s topology.
- (ii) The above (nontrivial) solution $u(t)$ is smooth (especially in $H^1(\mathbb{T})$) for $t > 0$, and then it extends to a global solution by means of the H^1 -upper and L^2 -lower *a priori* bounds, which are obtained for H^1 solutions of arbitrary size.

Note that the trivial solution $u \equiv 0$ is a global solution for $u_0 = 0$. The continuity of the solution map at the origin can be verified if $s > 1/2$ ([1]), but it is open for $1/2 \geq s > 1/4$.

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This is because *a priori* estimates and the local existence time given by the contraction argument depend on the reciprocal of the L^2 norm of solution for $1/2 \geq s > 1/4$. In the nonperiodic case, local well-posedness of the Cauchy problem in $H^s(\mathbb{R})$ can be proved for $s > 3/2$ by the energy method, but no result seems to be currently available below $H^{3/2}$. To summarize, on \mathbb{T} we have a global solution for $s > 1/4$, while on \mathbb{R} we only have a local solution for $s > 3/2$. We also note that these solutions to the Cauchy problem are unique in $C_t H_x^s$ if $s > 3/2$.

The goal of this article is to prove an *a priori* H^s estimate for small and smooth solutions to (1.1) in the regularity range $1/2 \geq s > 1/4$. In the periodic case, this and an approximation argument would imply the same estimate for the small (rough) H^s solutions constructed in [2], thus verifying the continuity of the solution map at the origin. Although our argument in the present paper is applicable to both periodic and nonperiodic problems, we will mainly consider the periodic case, which seems technically more complicated. (See Remark 2.9 for a comment on the nonperiodic case.)

Theorem 1.1 *Let $\mathcal{M} = \mathbb{R}$ or \mathbb{T} and $s > 1/4$. Then there exist $\delta > 0$ and $C > 0$ such that if $0 < T \leq 1$ and $u \in C([0, T]; H^\infty(\mathcal{M}))$ is a smooth solution to (1.1) on \mathcal{M} satisfying $\|u(0)\|_{H^s} \leq \delta$, then it holds that*

$$\|u\|_{L^\infty([0, T]; H^s)} \leq C \|u(0)\|_{H^s}. \tag{1.2}$$

To establish the H^s *a priori* bound (1.2), we shall employ the short-time Fourier restriction method. The short-time $X^{s,b}$ norms were introduced by Ionescu, Kenig, and Tataru [3]; the idea is to combine the $X^{s,b}$ analysis implemented in frequency-dependent small time intervals with an energy-type argument recovering the estimate on the whole interval. The method has been applied to the modified Benjamin–Ono and the derivative NLS equations by Guo [4] in the nonperiodic case, and in the periodic case by Schippa [5], who used the U^p-V^p type spaces instead of $X^{s,b}$. The $X^{s,b}$ type spaces are suitable for detailed analysis on the resonance structure, while the U^p-V^p type spaces work well with sharp cut-off functions in time. For our purpose, the U^p-V^p type spaces seem to be more convenient. In our argument with the short-time Fourier restriction method, the modified energy plays a crucial role. Our way of constructing the modified energy is slightly different from that in [4, 6], and [5] because of the presence of the Hilbert transformation in the cubic nonlinearity. To be specific, (1.1) has less symmetry than the DNLS, the cubic NLS, and the modified Benjamin–Ono equations. Moreover, it is known that the kinetic term $\beta \partial_x(H(|u|^2)u)$ in (1.1) exhibits a kind of dissipation when $\beta < 0$ (e.g., we have $\partial_t \|u(t)\|_{L^2}^2 \leq 0$ for (smooth) solutions of (1.1), while the L^2 norm is conserved for the DNLS equation). This dissipative nature has to be taken into account in the construction and the estimate of the modified energy since otherwise there would remain some uncanceled terms with higher order derivatives compared to the corresponding estimate for the nonlinearity $\alpha \partial_x(|u|^2 u)$. Here, we do not have to estimate the difference of two solutions, since we only consider the continuity of the solution map at the origin. So, we do not have to consider the modified energy for the difference of two solutions, either.

Remark 1.2 (i) In the case of DNLS, a similar *a priori* H^s estimate was obtained in [4, 5] for solutions of arbitrary size by using a rescaling argument. Although the same idea may work

for our problem (1.1), to remove the smallness condition in Theorem 1.1, we will focus on small solutions in order to keep the argument not too complicated, and also because of our particular interest in the continuity of the solution map at the origin.

(ii) An adaptation of the theory of low-regularity conservation laws for integrable PDEs by Killip, Viřan, and Zhang [7] might be another possible approach. For the derivative NLS on \mathbb{R} and on \mathbb{T} , the H^s *a priori* estimate for $s > 0$ was established in [8] by this method. Of course, KDNLS is not known to be completely integrable, but the method seems also useful to some dissipative perturbations of the integrable dispersive equations (e.g., the KdV–Burgers equation). Unfortunately, this approach has not been successful for KDNLS up to now.

The plan of the present paper is as follows. In Sect. 2, we describe the definition of function spaces we work with, the short-time Strichartz estimates and the short-time bilinear Strichartz estimates. Assuming the trilinear estimates and the modified energy estimate, which are proved in later sections, we give the proof of our main Theorem 1.1. In Sect. 3, we give the trilinear estimate on the cubic nonlinearity in terms of short-time norms. In Sect. 4, we define the modified energy and prove its estimates, which are helpful for the short-time argument.

We would like to conclude this section with a couple of comments on Jean Ginibre’s work about nonlinear wave and dispersive equations. Ginibre started to study the scattering theory in the finite energy class for nonlinear Klein–Gordon and Schrödinger equations in late 1970s with Giorgio Velo. Since then, he has made the great contribution to nonlinear partial differential equations, specifically nonlinear wave and dispersive equations. In early 1990s, Bourgain presented the so-called Fourier restriction method to study the well-posedness of the Cauchy problem for nonlinear dispersive equations such as nonlinear Schrödinger equations and the KdV equation. The Fourier restriction method is very powerful, but it is rather complicated. In fact, Bourgain’s papers were not very easy to read. Many people hoped the readable exposition on Bourgain’s work about the Fourier restriction method. In 1996, Ginibre wrote the nice exposition [9] on the Fourier restriction method, which contained several new and important observations, for example, the relation between the Fourier restriction norm and the interaction representation in quantum physics. This helped the Fourier restriction method to prevail among the community of nonlinear wave and dispersive equations.

2 Function spaces, Strichartz estimates

2.1 Definition of function spaces

For $1 \leq p < \infty$ and an interval $I = (a, b)$, $-\infty \leq a < b \leq \infty$, let $U^p(I)$, $V^p(I)$ be the (L_x^2 -valued) ℓ^p -atomic space and the space of functions of bounded p -variation, respectively, on I . For the precise definition of these spaces, see [10] (also [11]) and [12]. Recall that $U^p(I)$, $V^p(I)$ are Banach spaces, and their elements are bounded functions from I to L_x^2 that have one-sided limits at every point in $[a, b]$. Moreover, $u \in U^p(I)$ is right continuous and satisfies $\lim_{t \rightarrow a} u(t) = 0$. As usual, we write $V_{rc}^p(I) := \{v \in V^p(I) : v \text{ is right continuous}\}$ and $V_{-,rc}^p(I) := \{v \in V_{rc}^p(I) : \lim_{t \rightarrow a} v(t) = 0\}$, which are closed subspaces of $V^p(I)$. We have $U^p(I) \subset U^q(I)$ for $p < q$ with continuous inclusion, and similarly for $V^p(I)$, $V_{rc}^p(I)$, $V_{-,rc}^p(I)$. Following [13] (see also [6]), we consider the space $DU^p(I) := \{u' : u \in U^p(I)\}$, where the derivative is taken in the sense of L_x^2 -valued distributions on I . For each $f \in DU^p(I)$,

there is unique $u \in U^p(I)$ satisfying $f = u'$, and hence $DU^p(I)$ is a Banach space equipped with the norm $\|f\|_{DU^p(I)} = \|u\|_{U^p(I)}$. Finally, we write $U_\Delta^p(I) := \{v : e^{i(-)\partial_x^2} v(\cdot) \in U^p(I)\}$ with $\|v\|_{U_\Delta^p(I)} := \|e^{i(-)\partial_x^2} v(\cdot)\|_{U^p(I)}$, and similarly for $V_\Delta^p(I)$, $V_{rc,\Delta}^p(I)$, $V_{-rc,\Delta}^p(I)$, and $DU_\Delta^p(I)$. Note that $DU_\Delta^p(I) = \{(\partial_t - i\partial_x^2)u : u \in U_\Delta^p(I)\}$.

We collect some basic properties of these spaces.

Lemma 2.1 *Let $I = (a, b)$ be any interval.*

(i) [Continuous embeddings] *For any $1 \leq p < q < \infty$, we have*

$$U_\Delta^p(I) \hookrightarrow V_{-rc,\Delta}^p(I) \hookrightarrow U_\Delta^q(I) \hookrightarrow L^\infty(I; L^2), \quad V_{rc,\Delta}^p(I) \hookrightarrow L^\infty(I; L^2).$$

(ii) [Duality] *For $1 < p < \infty$, we have $L^1(I; L^2) \hookrightarrow DU_\Delta^p(I)$ and*

$$\begin{aligned} \|f\|_{DU_\Delta^p(I)} &= \left\| \int_a^t e^{i(t-t')\partial_x^2} f(t') dt' \right\|_{U_\Delta^p(I)} \\ &= \sup \left\{ \left| \int_I \int f \bar{v} dx dt \right| : v \in V_{rc,\Delta}^{p'}(I), \|v\|_{V_{rc,\Delta}^{p'}(I)} \leq 1 \right\} \\ &\lesssim \|f\|_{L^1(I; L^2)} \end{aligned}$$

for $f \in L^1(I; L^2)$.

(iii) [Extension] *Let $n \geq 1$ and $T : (L_x^2)^n \rightarrow L_{loc,x}^1$ be an operator such that it is either linear or conjugate linear in each variable. Let $1 \leq p, q \leq \infty$, and assume that the map $(\phi_1, \dots, \phi_n) \mapsto [t \mapsto T(e^{it\partial_x^2} \phi_1, \dots, e^{it\partial_x^2} \phi_n)]$ is bounded from $(L_x^2)^n$ to $L_t^p(I; L_x^q)$:*

$$\|T(e^{it\partial_x^2} \phi_1, \dots, e^{it\partial_x^2} \phi_n)\|_{L_t^p(I; L_x^q)} \leq A \prod_{j=1}^n \|\phi_j\|_{L_x^2}$$

for some $A > 0$. Then T can be regarded as a map from $(U_\Delta^p(\mathbb{R}))^n$ to $L_t^p(I; L_x^q)$ by $(u_1, \dots, u_n) \mapsto [t \mapsto T(u_1(t), \dots, u_n(t))]$, and it is bounded:

$$\|T(u_1, \dots, u_n)\|_{L_t^p(I; L_x^q)} \leq A \prod_{j=1}^n \|u_j\|_{U_\Delta^p(\mathbb{R})}.$$

Here, $U_\Delta^p(\mathbb{R})$ is replaced by $L_t^\infty(\mathbb{R}; L_x^2)$ if $p = \infty$.

(iv) [Interpolation] *Let $1 \leq p < q < \infty$, E be a Banach space, and $T : U_\Delta^q(I) \rightarrow E$ be a bounded, linear, or conjugate linear operator such that $\|T\|_{U_\Delta^q(I) \rightarrow E} \leq C_q$,*

$\|T\|_{U_\Delta^p(I) \rightarrow E} \leq C_p$ for some $0 < C_p \leq C_q < \infty$. Then we have

$$\|T\|_{V_{-rc,\Delta}^p(I) \rightarrow E} \lesssim \left(1 + \log \frac{C_q}{C_p}\right) C_p.$$

Proof (i) See, e.g., [10, Propositions 2.2, 2.4, Corollary 2.6].

(ii) The first equality holds by definition. If $f \in L^1(I, L^2)$, the function $t \mapsto \int_a^t e^{-it'\partial_x^2} f(t') dt' \in L^2$ is absolutely continuous and of bounded variation on \bar{I} . Then the second equality follows, e.g., from [10, Theorem 2.8, Propositions 2.9, 2.10]. The last inequality follows from the Hölder inequality and the embedding $V_{rc,\Delta}^{p'}(I) \hookrightarrow L^\infty(I; L^2)$.

(iii) See, e.g., [10, Proposition 2.19].

(iv) See, e.g., [10, Proposition 2.20]. □

Now, we define the short-time norms. In this article, we use capital letters N, K, \dots for dyadic integers $1, 2, 4, 8, \dots$

Definition 2.2 First of all, we fix a bump function

$$\eta \in C_0^\infty(\mathbb{R}) \quad \text{even, monotone on } [0, \infty), \text{ and } \chi_{[-4/3, 4/3]} \leq \eta \leq \chi_{[-5/3, 5/3]},$$

where χ_Ω denotes the characteristic function of a set Ω . Define $\{\psi_N\}_{N \geq 1} \subset C_0^\infty(\mathbb{R})$ by

$$\psi_1(\xi) := \eta(\xi), \quad \psi_N(\xi) := \eta(\xi/N) - \eta(2\xi/N) \quad \text{for } N \geq 2,$$

so that $1 = \sum_{N \geq 1} \psi_N(\xi)$ and $\text{supp}(\psi_N) \subset \mathcal{I}_N$, where

$$\mathcal{I}_1 := [-2, 2], \quad \mathcal{I}_N := [-2N, 2N] \setminus (-N/2, N/2) \quad \text{for } N \geq 2.$$

We define the corresponding Littlewood–Paley projections $P_N := \mathcal{F}_\xi^{-1} \psi_N \mathcal{F}_x$.

Next, we define frequency-localized short-time norms $F_N(T), G_N(T)$ for functions $u : [0, T] \rightarrow L^2$ with $\text{supp}(\hat{u}(t, \xi)) \subset [0, T] \times \mathcal{I}_N$ by

$$\begin{aligned} \|u\|_{F_N(T)} &:= \sup_{I=[a,b] \subset [0,T], |I| \leq N^{-1}} \|\chi_I u\|_{U_\Delta^2(\mathbb{R})}, \\ \|u\|_{G_N(T)} &:= \sup_{I=(a,b) \subset [0,T], |I| \leq N^{-1}} \|u|_I\|_{DU_\Delta^2(I)}. \end{aligned}$$

In the definition of the $F_N(T)$ norm, we regard $\chi_I u$ as a function on \mathbb{R} by extending it by zero outside I . Here, we consider half-open intervals $I = [a, b)$ so that $\chi_I u$ can be right continuous, and we avoid writing the norm $\|\chi_I u\|_{U_\Delta^2(\mathbb{R})}$ as $\|u\|_{U_\Delta^2((a,b))}$ since the $U_\Delta^2((a, b))$ norm can be defined only for functions satisfying $\lim_{t \rightarrow a+0} u(t) = 0$. The short-time U_Δ^2 -type space on $[0, T]$ with spatial regularity $s \in \mathbb{R}$ is defined by

$$F^s(T) := \{u \in C([0, T]; H^s) : \|u\|_{F^s(T)} := \|N^s \|P_N u\|_{F_N(T)}\|_{\ell_N^2} < \infty\}.$$

To measure the nonlinearity, the following short-time norm is used:

$$\|u\|_{G^s(T)} := \|N^s \|P_N u\|_{G_N(T)}\|_{\ell_N^2}.$$

We also need the following energy norm:

$$\|u\|_{E^s(T)} := \|N^s \|P_N u\|_{L^\infty([0,T], L^2)}\|_{\ell_N^2}.$$

2.2 Proof of the main theorem

It is known (e.g., [13, Lemma 3.1]) that the norms defined above satisfy the basic linear estimate

$$\|u\|_{F^s(T)} \lesssim \|u\|_{E^s(T)} + \|(\partial_t - i\partial_x^2)u\|_{G^s(T)}$$

for any $s \in \mathbb{R}$. Then, what we need to show is the trilinear estimate

$$\|\partial_x(|u|^2u)\|_{G^s(T)} + \|\partial_x(H(|u|^2)u)\|_{G^s(T)} \lesssim \|u\|_{F^s(T)}^3$$

and the energy estimate

$$\|u\|_{E^s(T)} \lesssim \|u(0)\|_{H^s} + \|u\|_{F^s(T)}^3.$$

We will prove the trilinear estimate for general functions $u \in F^s(T)$ in Sect. 3, and the energy estimate for smooth solutions of (1.1) with small initial data in Sect. 4. Both of these estimates require $s > 1/4$ and also have the constants uniform for $T \in (0, 1]$ but growing for $T > 1$.

Let us admit these estimates and prove Theorem 1.1. For $0 < T' \leq T$ and a smooth solution u with initial data small in H^s , define

$$X_s(T') := \|u\|_{E^s(T')} + \|\partial_x(|u|^2u)\|_{G^s(T')} + \|\partial_x(H(|u|^2)u)\|_{G^s(T')}.$$

The above three estimates show that

$$X_s(T') \lesssim \|u(0)\|_{H^s} + X_s(T')^3.$$

On the other hand, it is easy to show (e.g., for $u \in C([0, T]; H^{s+1+})$) that $X_s(T')$ is continuous in T' and

$$\limsup_{T' \rightarrow +0} X_s(T') \lesssim \|u(0)\|_{H^s}.$$

Hence, by a bootstrap argument, we have

$$X_s(T') \lesssim \|u(0)\|_{H^s}, \quad 0 < T' \leq T.$$

Since $\|u\|_{L^\infty([0, T]; H^s)} \lesssim \|u\|_{E^s(T)}$, this concludes the proof of Theorem 1.1.

2.3 Short-time L^6 and bilinear Strichartz estimates

Most of Strichartz-type estimates for the nonperiodic Schrödinger equation are known to hold for the periodic problem in the short-time setting, and these estimates will be used as basic tools to prove the trilinear and energy estimates. We begin with the following L^6 Strichartz estimate.

Lemma 2.3 *For $N \geq 1$ and $0 < \delta \lesssim N^{-1}$, we have*

$$\|P_{\leq N} e^{it\partial_x^2} \phi\|_{L^6([0, \delta]; L^6(\mathbb{T}))} \lesssim \|\phi\|_{L^2}.$$

As a consequence, we have

$$\|P_{\leq N} u\|_{L^6(I; L^6(\mathbb{T}))} \lesssim \|\chi_I u\|_{U_\Delta^6(\mathbb{R})}$$

for any interval $I = [a, b) \subset \mathbb{R}$ with $|I| \lesssim N^{-1}$ and any $u : I \rightarrow L^2$ such that the zero extension $\chi_I u$ belongs to $U_\Delta^6(\mathbb{R})$.

Proof The first estimate was shown in [14, Proposition 2.9]. To obtain the second claim, we use Lemma 2.1(iii) with the operator $T : \phi \mapsto P_{\leq N}\phi$ and apply the resulting estimate to $\chi_I u \in U_{\Delta}^6(\mathbb{R})$. \square

As a counterpart of the bilinear Strichartz estimate of Ozawa and Tsutsumi [15, Theorem 2 (1)], we have the following short-time bilinear Strichartz estimate on \mathbb{T} . A Fourier analytic proof was given in [16], which we will recall below for completeness.

Lemma 2.4 *For $K \geq 1$ and $\delta > 0$, we have*

$$\|P_K(e^{it\partial_x^2} \phi_1 \overline{e^{it\partial_x^2} \phi_2})\|_{L^2([0,\delta];L^2(\mathbb{T}))} \lesssim \left(\frac{1+K\delta}{K}\right)^{1/2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2}. \tag{2.1}$$

In particular, if $N_1 \gg N_2 \geq 1$ and ϕ_1, ϕ_2 satisfy $\text{supp}(\hat{\phi}_j) \subset \mathcal{I}_{N_j}$, then for $0 < \delta \lesssim N_1^{-1}$ we have

$$\|e^{it\partial_x^2} \phi_1 \overline{e^{it\partial_x^2} \phi_2}\|_{L^2([0,\delta];L^2(\mathbb{T}))} \lesssim N_1^{-1/2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2}. \tag{2.2}$$

Remark 2.5 The latter estimate (2.2) clearly holds regardless of the complex conjugation, while this is not the case for the former estimate (2.1). For the product without conjugation of two functions of comparable frequencies, we can deduce, for instance, the following result from (2.1): if $|\xi_1 - \xi_2| \sim K$ for any $\xi_j \in \text{supp}(\hat{\phi}_j)$, $j = 1, 2$, then

$$\|e^{it\partial_x^2} \phi_1 e^{it\partial_x^2} \phi_2\|_{L^2([0,\delta];L^2(\mathbb{T}))} \lesssim \left(\frac{1+K\delta}{K}\right)^{1/2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2}.$$

Remark 2.6 As for the L^6 Strichartz estimate, from (2.1) and Lemma 2.1(iii) we immediately obtain the corresponding bilinear estimates in U_{Δ}^2 : for $I = [a, b]$ with $|I| \lesssim K^{-1}$, we have

$$\|P_K(u\bar{v})\|_{L^2(I;L^2(\mathbb{T}))} \lesssim K^{-1/2} \|\chi_I u\|_{U_{\Delta}^2(\mathbb{R})} \|\chi_I v\|_{U_{\Delta}^2(\mathbb{R})}.$$

A similar extension is valid also for Lemmas 2.7 and 2.8. On the other hand, by the Bernstein and Hölder inequalities and the assumption $|I| \lesssim K^{-1}$, together with the embedding $U_{\Delta}^p \hookrightarrow L^{\infty}L^2$, we have

$$\|P_K(u\bar{v})\|_{L^2(I;L^2(\mathbb{T}))} \lesssim K^{1/2}|I|^{1/2} \|u\bar{v}\|_{L^{\infty}(\mathbb{R};L^1(\mathbb{T}))} \lesssim \|u\|_{U_{\Delta}^2(\mathbb{R})} \|v\|_{U_{\Delta}^4(\mathbb{R})}.$$

By applying Lemma 2.1(iv) to the operator $v \mapsto P_K(u\bar{v})$ with these estimates, we have

$$\|P_K(u\bar{v})\|_{L^2(I;L^2(\mathbb{T}))} \lesssim K^{-1/2}(1 + \log K) \|\chi_I u\|_{U_{\Delta}^2(\mathbb{R})} \|\chi_I v\|_{V_{\Delta}^2(\mathbb{R})}$$

for $u, v : I \rightarrow L^2$ such that $\chi_I u \in U_{\Delta}^2(\mathbb{R})$ and $\chi_I v \in V_{-,rc,\Delta}^2(\mathbb{R})$.

Proof of Lemma 2.4 If $K = 1$, the claim follows from the Hölder inequality in t and the Bernstein inequality in x .

Assume $K > 1$. We observe that

$$\begin{aligned} P_K(e^{it\partial_x^2} \phi_1 \overline{e^{it\partial_x^2} \phi_2}) &= \sum_{n_1, n_2} e^{i(n_1 - n_2)x} e^{i(-n_1^2 + n_2^2)t} \psi_K(n_1 - n_2) \hat{\phi}_1(n_1) \overline{\hat{\phi}_2(n_2)} \\ &= \sum_n e^{inx} \psi_K(n) e^{-in^2 t} \sum_{n_2} \hat{\phi}_1(n + n_2) \overline{\hat{\phi}_2(n_2)} e^{-2inn_2 t}. \end{aligned}$$

By the Plancherel theorem and the change of variable $t' = -2nt$, we have

$$\begin{aligned} &\|P_K(e^{it\partial_x^2} \phi_1 \overline{e^{it\partial_x^2} \phi_2})\|_{L^2([0, \delta]; L^2(\mathbb{T}))}^2 \\ &= \int_0^\delta \sum_n |\psi_K(n)|^2 \left| \sum_{n_2} \hat{\phi}_1(n + n_2) \overline{\hat{\phi}_2(n_2)} e^{-2inn_2 t} \right|^2 dt \\ &= \sum_n |\psi_K(n)|^2 \frac{1}{-2n} \int_0^{-2n\delta} \left| \sum_{n_2} \hat{\phi}_1(n + n_2) \overline{\hat{\phi}_2(n_2)} e^{in_2 t'} \right|^2 dt' \\ &\lesssim \sum_n \frac{1 + K\delta}{K} \int_0^{2\pi} \left| \sum_{n_2} \hat{\phi}_1(n + n_2) \overline{\hat{\phi}_2(n_2)} e^{in_2 t'} \right|^2 dt'. \end{aligned}$$

Since the last term is equal to $\frac{1+K\delta}{K} \|\phi_1\|_{L^2}^2 \|\phi_2\|_{L^2}^2$ by the Plancherel theorem, the claimed estimate follows. \square

To deal with the nonlinearity of (1.1) including the Hilbert transformation, we prepare the next two lemmas. These estimates can be shown in the same manner as Lemma 2.4.

Lemma 2.7 *Let $\phi_1, \phi_2, \phi_3 \in L^2(\mathbb{T})$ satisfy $\text{supp}(\hat{\phi}_j) \subset \mathcal{I}_{N_j}$, and assume that $N_1 \gg N_2, N_3$. Then, for $0 < \delta \lesssim N_1^{-1}$, we have*

$$\|e^{it\partial_x^2} \phi_1 \cdot H(e^{it\partial_x^2} \phi_2 \overline{e^{it\partial_x^2} \phi_3})\|_{L^2([0, \delta]; L^2(\mathbb{T}))} \lesssim \left(\frac{N_2 \wedge N_3}{N_1}\right)^{1/2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2} \|\phi_3\|_{L^2}.$$

The same estimate holds if $e^{it\partial_x^2} \phi_1$ is replaced by $\overline{e^{it\partial_x^2} \phi_1}$ and also if H is replaced by any Fourier multiplier with bounded symbol (such as $P_{\leq N}$).

Proof Since $\overline{H(u\bar{v})} = H(\bar{u}v)$, we may assume $N_2 \leq N_3$. We observe that

$$\begin{aligned} &e^{it\partial_x^2} \phi_1 \cdot H(e^{it\partial_x^2} \phi_2 \overline{e^{it\partial_x^2} \phi_3}) \\ &= \sum_{n_1, n_2, n_3} e^{i(n_1 + n_2 - n_3)x} e^{i(-n_1^2 - n_2^2 + n_3^2)t} (-i) \text{sgn}(n_2 - n_3) \hat{\phi}_1(n_1) \hat{\phi}_2(n_2) \overline{\hat{\phi}_3(n_3)} \\ &= -i \sum_n e^{inx} \sum_{n_2} e^{-i(n_2^2 + (n - n_2)^2)t} \hat{\phi}_2(n_2) \\ &\quad \times \sum_{n_3} \text{sgn}(n_2 - n_3) \hat{\phi}_1(n - n_2 + n_3) \overline{\hat{\phi}_3(n_3)} e^{-2in_3(n - n_2)t}, \end{aligned}$$

and hence

$$\|e^{it\partial_x^2} \phi_1 \cdot H(e^{it\partial_x^2} \phi_2 \overline{e^{it\partial_x^2} \phi_3})\|_{L^2([0, \delta]; L^2(\mathbb{T}))}^2$$

$$\begin{aligned}
 &= \int_0^\delta \sum_n \left| \sum_{n_2} e^{-i(n_2^2+(n-n_2)^2)t} \hat{\phi}_2(n_2) \right. \\
 &\quad \times \left. \sum_{n_3} \operatorname{sgn}(n_2 - n_3) \hat{\phi}_1(n - n_2 + n_3) \overline{\hat{\phi}_3(n_3)} e^{-2in_3(n-n_2)t} \right|^2 dt \\
 &\leq \int_0^\delta \sum_n \left[\sum_{n_2} |\hat{\phi}_2(n_2)| \left| \sum_{n_3} \operatorname{sgn}(n_2 - n_3) \hat{\phi}_1(n - n_2 + n_3) \overline{\hat{\phi}_3(n_3)} e^{-2in_3(n-n_2)t} \right| \right]^2 dt \\
 &\leq \left[\sum_{n_2} |\hat{\phi}_2(n_2)| \left(\sum_n \int_0^\delta \left| \sum_{n_3} \operatorname{sgn}(n_2 - n_3) \hat{\phi}_1(n - n_2 + n_3) \right. \right. \right. \\
 &\quad \times \left. \left. \overline{\hat{\phi}_3(n_3)} e^{-2in_3(n-n_2)t} \right|^2 dt \right)^{1/2} \right]^2,
 \end{aligned}$$

where in the last step we have used the Minkowski inequality to replace the $L_t^2 \ell_n^2 \ell_{n_2}^1$ norm by $\ell_{n_2}^1 \ell_n^2 L_t^2$. Now, for fixed $|n| \sim N_1$ and $n_2 \in \mathcal{I}_{N_2}$, we have $|2(n - n_2)\delta| \lesssim 2\pi$, and thus

$$\begin{aligned}
 &\int_0^\delta \left| \sum_{n_3} \operatorname{sgn}(n_2 - n_3) \hat{\phi}_1(n - n_2 + n_3) \overline{\hat{\phi}_3(n_3)} e^{-2in_3(n-n_2)t} \right|^2 dt \\
 &= \frac{1}{-2(n - n_2)} \int_0^{-2(n-n_2)\delta} \left| \sum_{n_3} \operatorname{sgn}(n_2 - n_3) \hat{\phi}_1(n - n_2 + n_3) \overline{\hat{\phi}_3(n_3)} e^{in_3 t'} \right|^2 dt' \\
 &\lesssim N_1^{-1} \int_0^{2\pi} \left| \sum_{n_3} \operatorname{sgn}(n_2 - n_3) \hat{\phi}_1(n - n_2 + n_3) \overline{\hat{\phi}_3(n_3)} e^{in_3 t'} \right|^2 dt' \\
 &\lesssim N_1^{-1} \sum_{n_3} |\hat{\phi}_1(n - n_2 + n_3)|^2 |\hat{\phi}_3(n_3)|^2.
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 &\| e^{it\partial_x^2} \phi_1 \cdot H(e^{it\partial_x^2} \phi_2 \overline{e^{it\partial_x^2} \phi_3}) \|_{L^2([0,\delta];L^2(\mathbb{T}))}^2 \\
 &\lesssim \left[\sum_{n_2} |\hat{\phi}_2(n_2)| \left(\sum_n N_1^{-1} \sum_{n_3} |\hat{\phi}_1(n - n_2 + n_3)|^2 |\hat{\phi}_3(n_3)|^2 \right)^{1/2} \right]^2 \\
 &\lesssim N_2 \sum_{n_2} |\hat{\phi}_2(n_2)|^2 \sum_n N_1^{-1} \sum_{n_3} |\hat{\phi}_1(n - n_2 + n_3)|^2 |\hat{\phi}_3(n_3)|^2 \\
 &\lesssim N_1^{-1} N_2 \|\phi_1\|_{L^2}^2 \|\phi_2\|_{L^2}^2 \|\phi_3\|_{L^2}^2,
 \end{aligned}$$

as desired. □

Lemma 2.8 *Let $\phi_1, \phi_2, \phi_3 \in L^2(\mathbb{T})$ satisfy $\operatorname{supp}(\hat{\phi}_j) \subset \mathcal{I}_{N_j}$, and assume that $N_1 \sim N_2 \gg N_3$. Further, assume $K \ll N_1$. Then, for $0 < \delta \lesssim N_1^{-1}$, we have*

$$\|HP_{\leq K}(e^{it\partial_x^2} \phi_1 \overline{e^{it\partial_x^2} \phi_2}) e^{it\partial_x^2} \phi_3 \|_{L^2([0,\delta];L^2(\mathbb{T}))} \lesssim \left(\frac{K}{N_1}\right)^{1/2} \|\phi_1\|_{L^2} \|\phi_2\|_{L^2} \|\phi_3\|_{L^2}.$$

The same estimate holds if $e^{it\partial_x^2} \phi_3$ is replaced by $\overline{e^{it\partial_x^2} \phi_3}$ and also if H is replaced by any Fourier multiplier with bounded symbol.

Proof By an almost orthogonality argument, we can restrict the frequencies of ϕ_1 and ϕ_2 onto intervals of length K . Then the same argument as for the preceding lemma can be used. \square

Remark 2.9 We note that all the above short-time L^6 and bilinear Strichartz estimates (Lemmas 2.3, 2.4, 2.7, and 2.8) are true in the nonperiodic case as well. In fact, these estimates hold on \mathbb{R} without restricting to a frequency-dependent short time interval (i.e., with the $L^6_{t,x}$ or $L^2_{t,x}$ norm over $\mathbb{R} \times \mathbb{R}$ on the left-hand side). Concerning Lemmas 2.7 and 2.8, this can be shown by a slight modification of the proofs for the periodic estimates given above.

3 Trilinear estimate in the short-time norms

In this section, we shall prove the following trilinear estimate in the $G^s(T)$ norm.

Proposition 3.1 *For $s > 1/4$ and $0 < T \leq 1$, we have*

$$\|\partial_x(u_1 \bar{u}_2 u_3)\|_{G^s(T)} + \|\partial_x(H(u_1 \bar{u}_2)u_3)\|_{G^s(T)} \lesssim \|u_1\|_{F^s(T)} \|u_2\|_{F^s(T)} \|u_3\|_{F^s(T)}.$$

Proof We only consider the second term on the left-hand side with the Hilbert transformation. The first term (for DNLS) was treated in [4, 5]; in fact, it can be dealt with in a similar manner but more easily.

We apply dyadic decompositions as

$$H(u_1 \bar{u}_2)u_3 = \sum_{\substack{N_1, \dots, N_4 \geq 1 \\ N_1^* \sim N_2^*}} P_{N_4} (H(P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3),$$

where we write N_1^*, \dots, N_4^* to denote the numbers N_1, \dots, N_4 rearranged in decreasing order. It then suffices to show for each $\underline{N} = (N_1, \dots, N_4)$ the localized estimate

$$\begin{aligned} & \|\partial_x P_{N_4} (H(P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3)\|_{G_{N_4}(T)} \\ & \lesssim C(\underline{N}) \|P_{N_1} u_1\|_{F_{N_1}(T)} \|P_{N_2} u_2\|_{F_{N_2}(T)} \|P_{N_3} u_3\|_{F_{N_3}(T)} \end{aligned} \tag{3.1}$$

with some $C(\underline{N})$ satisfying

$$C(\underline{N}) \lesssim \frac{N_1^s N_2^s N_3^s}{N_4^s} (N_3^*)^{0-}.$$

(Since $N_1^* \sim N_2^*$, the factor $(N_3^*)^{0-}$ allows us to restore the claimed estimate by summing up (3.1) in \underline{N} .) We will actually obtain (3.1) with smaller $C(\underline{N})$, which satisfies

$$C(\underline{N}) \lesssim \left(\frac{N_1 N_2 N_3}{N_1^*} \right)^{(1/4)+}. \tag{3.2}$$

From the definition of the $F_N(T)$, $G_N(T)$ norms, we need to prove

$$\sup_{I_4=(a,b) \subset [0,T], |I_4| \leq N_4^{-1}} \|\partial_x P_{N_4} (H(P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3)\|_{DU^2_{\Delta}(I_4)}$$

$$\lesssim C(\underline{N}) \prod_{j=1}^3 \sup_{I_j=[a,b] \subset [0,T], |I_j| \leq N_j^{-1}} \|\chi_{I_j} P_{N_j} u_j\|_{U_{\Delta}^2(\mathbb{R})}.$$

Since $\partial_x P_{N_4}(H(P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3) \in L^1([0, T]; L^2)$ for $u_1, u_2, u_3 \in F^s(T) \subset C([0, T]; H^s)$, by Lemma 2.1(ii) it suffices to prove either

$$\begin{aligned} & \|\partial_x P_{N_4}(H(P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3)\|_{L^1(I_4; L^2)} \\ & \lesssim C(\underline{N}) \prod_{j=1}^3 \sup_{I_j=[a,b] \subset [0,T], |I_j| \leq N_j^{-1}} \|\chi_{I_j} P_{N_j} u_j\|_{U_{\Delta}^2(\mathbb{R})} \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} & \left| \int_{I_4} \int H(P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3 \cdot \partial_x P_{N_4} \bar{u}_4 \, dx \, dt \right| \\ & \lesssim C(\underline{N}) \|u_4\|_{V_{\Delta}^2(I_4)} \prod_{j=1}^3 \sup_{I_j=[a,b] \subset [0,T], |I_j| \leq N_j^{-1}} \|\chi_{I_j} P_{N_j} u_j\|_{U_{\Delta}^2(\mathbb{R})} \end{aligned} \tag{3.4}$$

for any $I_4 = (a, b) \subset [0, T]$, $|I_4| \leq N_4^{-1}$ and any $u_4 \in V_{rc, \Delta}^2(I_4)$.

When $N_4 \ll N_1^*$, the time scale on the right-hand side is finer than that on the left-hand side, and therefore we need to first divide I_4 into subintervals of size $\leq (N_1^*)^{-1}$, the number of which is $O(N_1^*/N_4)$. Then, to verify (3.3), we need to show

$$\begin{aligned} & (N_1^*)^{1/2} \|P_{N_4}(H(P_{N_1} u_1 P_{N_2} \bar{u}_2) P_{N_3} u_3)\|_{L^2(I; L^2)} \\ & \lesssim C(\underline{N}) \prod_{j=1}^3 \|\chi_I P_{N_j} u_j\|_{U_{\Delta}^2(\mathbb{R})} \end{aligned} \tag{3.5}$$

for any interval I with

$$I = [a, b] \subset [0, T], \quad |I| \leq (N_1^*)^{-1}.$$

In fact, (3.5) implies (3.3) by the Schwarz inequality in t and the Bernstein inequality in x . From now on, we write simply u_j for $P_{N_j} u_j$.

Case (I) $N_4 \sim N_1^*$.

(1a) [high \times high \times high \rightarrow high] $N_1 \sim N_2 \sim N_3 \sim N_4$.

We simply use the L^6 Strichartz estimate (Lemma 2.3) for each function:

$$(N_1^*)^{1/2} \|H(u_1 \bar{u}_2) u_3\|_{L^2(I; L^2)} \lesssim (N_1^*)^{1/2} \prod_{j=1}^3 \|u_j\|_{L^6(I; L^6)} \lesssim (N_1^*)^{1/2} \prod_{j=1}^3 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})}.$$

This shows (3.5) with $C(\underline{N}) = (N_1^*)^{1/2}$, which satisfies (3.2).

(1b) [high \times high \times low \rightarrow high] $N_1^* \sim N_3^* \gg N_4^*$.

(i) $N_1 \sim N_2 \gg N_3$. In this case, we apply the standard bilinear Strichartz estimate (Lemma 2.4) to the product $u_1 \bar{u}_2$, on which we may put $P_{\sim N_1^*}$. Using the L^∞ embedding

$$\|u_3\|_{L^\infty(I; L^\infty)} \lesssim N_3^{1/2} \|u_3\|_{L^\infty(I; L^2)} \lesssim N_3^{1/2} \|\chi_I u_3\|_{U_{\Delta}^2(\mathbb{R})},$$

we have (3.5) with $C(\underline{N}) = N_3^{1/2}$, which satisfies (3.2).

(ii) $N_2 \sim N_3 \gg N_1$. Noticing that $H(u_1 \bar{u}_2) = u_1 H(\bar{u}_2)$, we apply Lemma 2.4 to the product $H(\bar{u}_2)u_3$ and follow the argument in the preceding case to obtain (3.5) with $C(\underline{N}) = N_1^{1/2}$, which again satisfies (3.2).

(iii) $N_1 \sim N_3 \gg N_2$. In this case, we need to consider the dual estimate (3.4), because we cannot use Lemma 2.4 to the product $H(u_1)u_3$ (in the form of Remark 2.5) when the Fourier supports of u_1 and u_3 are overlapping. We first replace $u_4 \in V_{rc,\Delta}^2(I_4)$ with its extension $\tilde{u}_4 \in V_{-rc,\Delta}^2(\mathbb{R})$ defined by $\tilde{u}_4(a) := \lim_{t \rightarrow a+0} u_4(t)$ and $\tilde{u}_4(t) := 0$ for $t \notin [a, b)$ (recall that $I_4 = (a, b)$). Next, we decompose I_4 into subintervals of length $\leq (N_1^*)^{-1}$, the number of which is $O(1)$. Then, for each integral on a subinterval $I = [a', b')$, we apply Lemma 2.4 (in the form obtained in Remark 2.6) to the product $u_3 \partial_x \bar{u}_4$ (on which we may put $P_{\sim N_1^*}$), bound the remaining functions $H(u_1)$, \bar{u}_2 in the $L^\infty(I; L^2)$ and the $L^2(I; L^\infty)$ norms respectively, and finally derive the factor $(N_1^*)^{-1/2} N_2^{1/2}$ from the last one by the Hölder inequality in t and the Bernstein inequality in x . The resulting bound is

$$\left| \int_I \int H(u_1 \bar{u}_2) u_3 \partial_x \bar{u}_4 \, dx \, dt \right| \lesssim (N_1^*)^{-(1/2)+} N_4 (N_1^*)^{-1/2} N_2^{1/2} \|\chi_I \tilde{u}_4\|_{V_\Delta^2(\mathbb{R})} \prod_{j=1}^3 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})}.$$

(Since we have to bound \tilde{u}_4 in V_Δ^2 , the bilinear Strichartz estimate is accompanied by a factor $(N_1^*)^{0+}$.) Now, it is verified directly from the definition of the V_Δ^2 norm that

$$\|\chi_I \tilde{u}_4\|_{V_\Delta^2(\mathbb{R})} \leq \|\tilde{u}_4\|_{V_\Delta^2(\mathbb{R})} \leq \sqrt{2} \|u_4\|_{V_\Delta^2(I_4)}.$$

As a result, we obtain (3.4) with $C(\underline{N}) = (N_1^*)^{0+} N_2^{1/2}$, which satisfies (3.2).

(Ic) [high \times low \times low \rightarrow high] $N_1^* \sim N_2^* \gg N_3^*$.

We show (3.5) with $C(\underline{N}) \lesssim (N_4^*)^{1/2}$. If N_1 or $N_2 \sim N_1^*$ (so that $N_1 \approx N_2$), we can put H on a single function. Then, similarly to the case (Ib-i), we apply Lemma 2.4 to the product of functions corresponding to N_1^* and N_3^* and use the L^∞ embedding for the other one corresponding to N_4^* to obtain the desired bound. In the remaining case, i.e., if $N_3 \sim N_4 \gg N_1, N_2$, we apply the first modified bilinear Strichartz estimate (Lemma 2.7) to the left-hand side of (3.5), which gives the same bound.

Case (II) $N_4 \ll N_1^*$.

(IIa) [high \times high \times high \rightarrow low] $N_1 \sim N_2 \sim N_3 \gg N_4$.

We follow the argument in the case (Ia) to obtain (3.5) with $C(\underline{N}) = (N_1^*)^{1/2}$, which satisfies (3.2).

(IIb) [high \times high \times low \rightarrow low] $N_1^* \gg N_3^*$.

(i) If $N_4 \lesssim \min\{N_1, N_2, N_3\}$, then we show (3.5) with $C(\underline{N}) \lesssim \min\{N_1, N_2, N_3\}^{1/2}$ by considering the following two cases separately.

- If $N_3 \sim N_1^*$ (which implies $N_1 \approx N_2$), then we first bound the left-hand side of (3.5) by

$$(N_1^*)^{1/2} N_4^{1/2} \|H(u_1 \bar{u}_2) u_3\|_{L^2(I; L^1)}$$

and then apply Lemma 2.4 to $u_1 \bar{u}_2$ (on which we may put $P_{\sim N_1^*}$). This implies (3.5) with $C(\underline{N}) = N_4^{1/2}$.

- If $N_3 \ll N_1^*$ (which implies $N_1 \sim N_2 \sim N_1^*$), then we may put $P_{\lesssim N_3}$ on $u_1 \bar{u}_2$. Using the second modified bilinear Strichartz estimate (Lemma 2.8), we obtain (3.5) with $C(\underline{N}) = N_3^{1/2}$.
- (ii) If $N_4 \gg \min\{N_1, N_2, N_3\}$, then we consider the dual estimate (3.4). Note that we can always put H on a single function since $N_3 \approx N_4$ and

$$\int_{I_4} \int H(u_1 \bar{u}_2) u_3 \partial_x \bar{u}_4 \, dx \, dt = - \int_{I_4} \int u_1 \bar{u}_2 H(u_3 \partial_x \bar{u}_4) \, dx \, dt.$$

Then the argument is parallel to the case (Ib-iii). This time we decompose I_4 into subintervals of length $\leq (N_1^*)^{-1}$, the number of which is $O(N_1^*/N_4)$, and apply Lemma 2.4 to the product of functions corresponding to N_1^* and $N_3^*(= N_4)$. Further, we bound the remaining functions corresponding to N_2^* and N_4^* in the $L^\infty(I; L^2)$ and the $L^2(I; L^\infty)$ norms, respectively. We then obtain (3.4) with

$$C(\underline{N}) \lesssim \frac{N_1^*}{N_4} \cdot (N_1^*)^{-(1/2)+} N_4 (N_1^*)^{-1/2} (N_4^*)^{1/2} \lesssim (N_1^*)^{0+} (N_4^*)^{1/2},$$

which satisfies (3.2).

We have seen all the possible cases, and this completes the proof of the localized estimate (3.1) with (3.2). □

4 Energy estimate

In this section, we shall prove the following *a priori* estimate.

Proposition 4.1 *Assume $0 < T \leq 1$ and that $u \in C([0, T]; H^\infty)$ is a solution to (1.1) on the time interval $[0, T]$. Then, for $s > 1/4$, there exist $\delta > 0$ and $C > 0$ (independent of u) such that if $\|u(0)\|_{L^2} \leq \delta$ then we have*

$$\|u\|_{E^s(T)}^2 \leq C(\|u(0)\|_{H^s}^2 + \|u\|_{F^s(T)}^6).$$

In fact, this is the main part of the proof of Theorem 1.1. Recall that the $E^s(T)$ norm takes L_t^∞ before the ℓ^2 summation over dyadic frequency blocks, and so it is fairly stronger than the $L_t^\infty H_x^s$ norm.

4.1 A reduction

First of all, we reduce Proposition 4.1 to the following estimate on a “modified energy”.

Proposition 4.2 *Let $0 < T \leq 1$ and $u \in C([0, T]; H^\infty)$ be a solution to (1.1) on the time interval $[0, T]$. Let $s > 1/4$, and assume that a smooth symbol $a \in C^\infty(\mathbb{R})$ has the following properties:*

$$\begin{cases} a \text{ is positive, even, nondecreasing in } [0, \infty), \text{ constant on } [-1, 1], \\ a(2\xi) \lesssim a(\xi) \text{ for any } \xi > 0, \\ \frac{a(N_1)}{a(N_2)} \gtrsim (\frac{N_1}{N_2})^{1/2} \text{ for any } N_1 > N_2 \geq 1, \\ |\partial_\xi^j a(\xi)| \lesssim \langle \xi \rangle^{-j} a(\xi) \text{ for any } \xi \in \mathbb{R} \text{ and } 1 \leq j \leq 5. \end{cases} \tag{4.1}$$

Then there exist $\delta > 0$ and $C > 0$ depending on s and the implicit constants in (4.1) (but not on u) such that if $\|u(0)\|_{L^2} \leq \delta$ then we have

$$\begin{aligned} E_0^a(u(t)) &:= \|\sqrt{a(D)}u(t)\|_{L^2}^2 \\ &\leq C \left(E_0^a(u(0)) + \|u\|_{F^s(T)}^4 \sum_{N \geq 1} a(N) \|P_N u\|_{F_N(T)}^2 \right), \quad t \in [0, T]. \end{aligned}$$

Remark 4.3 (i) The Sobolev weight $a(\xi) = \langle \xi \rangle^{2s}$ ($s \geq 1/4$) satisfies conditions (4.1) (after modifying on $[-1, 1]$). With this choice of a , we can obtain from Proposition 4.2 an $L^\infty([0, T]; H^s)$ *a priori* estimate. This is, however, weaker than what we want to prove in Proposition 4.1.

(ii) To obtain an $E^s(T)$ bound, one may consider estimating localized H^s norms $N^{2s} \|\psi_N(D)u(t)\|_{L^2}^2$ for dyadic numbers $N \geq 1$ and summing them up. This is indeed the approach taken in [3]. On the other hand, we will improve the bound by adding a correction term to the energy functional. For this purpose, it will be convenient to introduce a modified energy $\|\sqrt{a(D)}u(t)\|_{L^2}^2$ and estimate it instead of the localized H^s norms, where a symbol $a(\xi)$ is chosen so that it is positive everywhere and its derivatives are controlled by itself as $|\partial_\xi^j a(\xi)| \lesssim \langle \xi \rangle^{-j} a(\xi)$. Such a modified energy has been used for the cubic NLS in [6] and for the modified Benjamin–Ono (and also the DNLS) equation in [4, 5].

(iii) Our choice of $a(\xi)$ is slightly simpler than that in [4–6] (see the proof of Proposition 4.1). Indeed, the modified energies in these papers are defined from a sequence of positive numbers $\{\beta_N\}$ depending on the initial data, but we do not use such a sequence.

Proof of Proposition 4.1 from Proposition 4.2 Let $s > 1/4$, $\varepsilon := s - 1/4 > 0$. For each dyadic integer \bar{N} , we define the positive sequence $\{a_N^{\bar{N}}\}_{N \geq 1}$ by

$$a_N^{\bar{N}} := N^{2s} \left(\frac{N}{\bar{N}} \wedge \frac{\bar{N}}{N} \right)^\varepsilon = \begin{cases} \bar{N}^{-\varepsilon} N^{2s+\varepsilon} & (N \leq \bar{N}), \\ \bar{N}^\varepsilon N^{2s-\varepsilon} & (N \geq \bar{N}). \end{cases}$$

It is clear that $\{a_N^{\bar{N}}\}$ is increasing in N . In fact, the growth of $\{a_N^{\bar{N}}\}$ is controlled as

$$2^{2s-\varepsilon} \leq \frac{a_{2N}^{\bar{N}}}{a_N^{\bar{N}}} \leq 2^{2s+\varepsilon} \quad (N \geq 1).$$

Now, define the smooth symbol $a^{\bar{N}} \in C^\infty(\mathbb{R})$ by

$$a^{\bar{N}}(\xi) := \sum_{N \geq 1} a_N^{\bar{N}} \psi_N(\xi).$$

It is easy to see that $a^{\bar{N}}$ satisfies all the properties in (4.1) with implicit constants independent of \bar{N} . Applying Proposition 4.2 and restricting the left-hand side of the resulting estimate to the target frequency $\{\langle \xi \rangle \sim \bar{N}\}$, we have

$$\sup_{t \in [0, T]} \bar{N}^{2s} \|P_{\bar{N}} u(t)\|_{L^2}^2 \lesssim \sum_{N \geq 1} \left(\frac{N}{\bar{N}} \wedge \frac{\bar{N}}{N} \right)^\varepsilon N^{2s} [\|P_N u(0)\|_{L^2}^2 + \|u\|_{F^s(T)}^4 \|P_N u\|_{F_N(T)}^2]$$

for any smooth solution $u \in C([0, T]; H^\infty)$ to (1.1) with $\|u(0)\|_{L^2}$ sufficiently small. Summing up in \bar{N} , we obtain the claimed estimate. \square

4.2 Construction of the modified energies

Now, we start proving Proposition 4.2. The argument is very similar to the estimate of the modified energy with correction terms in the I -method, where an important role is played by various cancelations after symmetrization of the energy functionals (see, e.g., Colliander, Keel, Staffilani, Takaoka, and Tao [17]). However, there are fewer symmetries compared to the DNLS case, and more delicate analysis is required. In particular, some of the highest order terms cannot be canceled out, and we need to recognize these terms to be nonpositive by making use of the dissipative nature of the equation.

Let $a \in C^\infty(\mathbb{R})$ be a symbol satisfying (4.1). Hereafter, the notation $\xi_{ij\dots} = \xi_i + \xi_j + \dots$ will be frequently used. Our proof will be designed for the periodic problem; however, in view of Remark 2.9, the same argument can be applied in the nonperiodic setting. For a smooth solution u of (1.1), we have

$$\begin{aligned} \partial_t \hat{u}(\xi) &= -i\xi^2 \hat{u}(\xi) + i\alpha \xi \sum_{\xi=\xi_{123}} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) + \beta \xi \sum_{\xi=\xi_{123}} \operatorname{sgn}(\xi_{12}) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3), \\ \partial_t \hat{u}(\xi) &= i\xi^2 \hat{u}(\xi) + i\alpha \xi \sum_{\xi=\xi_{123}} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) + \beta \xi \sum_{\xi=\xi_{123}} \operatorname{sgn}(\xi_{23}) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3). \end{aligned}$$

The derivative of $E_0^a(u(t)) = \sum_{\xi_{12}=0} a(\xi_1) \hat{u}(\xi_1) \hat{u}(\xi_2)$ is computed as

$$\begin{aligned} \frac{d}{dt} E_0^a(u(t)) &= i\alpha \sum_{\xi_{1234}=0} [a(\xi_{123}) \xi_{123} + a(\xi_1) \xi_{234}] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) \\ &\quad + \beta \sum_{\xi_{1234}=0} [a(\xi_{123}) \xi_{123} \operatorname{sgn}(\xi_{12}) + a(\xi_1) \xi_{234} \operatorname{sgn}(\xi_{34})] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4). \end{aligned}$$

The first term is the same as that appearing in the DNLS case, and it is symmetrized as follows:

$$\frac{\alpha}{2i} \sum_{\xi_{1234}=0} [\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4).$$

We observe that the multiplier part $\xi_1 a(\xi_1) + \dots + \xi_4 a(\xi_4)$ vanishes for the resonant frequencies:

$$\xi_{1234} = 0, \quad \xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 = -2\xi_{12}\xi_{23} = 0.$$

Then this quadrilinear term can be canceled with the leading term of the derivative of the quadrilinear functional

$$-\frac{\alpha}{2i} \sum_{\substack{\xi_{1234}=0 \\ \xi_{12}\xi_{23} \neq 0}} \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{-i(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2)} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4),$$

which can be used as an appropriate correction term to $E_0^a(u)$.

In the following, we assume $\alpha = 0$ for simplicity and consider the term

$$\beta \sum_{\xi_{1234}=0} [a(\xi_{123})\xi_{123}\text{sgn}(\xi_{12}) + a(\xi_1)\xi_{234}\text{sgn}(\xi_{34})]\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4).$$

This term has less symmetry due to the sign functions. In fact, this is symmetrized as

$$\frac{\beta}{2} \sum_{\xi_{1234}=0} [(\xi_1 a(\xi_1) + \xi_2 a(\xi_2))\text{sgn}(\xi_{12}) + (\xi_3 a(\xi_3) + \xi_4 a(\xi_4))\text{sgn}(\xi_{34})]\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4),$$

and the multiplier part does not vanish when $\xi_{23} = 0 \neq \xi_{12}$ (in this case $\text{sgn}(\xi_{34}) = -\text{sgn}(\xi_{12}) \neq 0$). Now, we observe that the function $\xi \mapsto \xi a(\xi)$ is odd and strictly increasing on \mathbb{R} , and hence

$$(\xi_1 a(\xi_1) + \xi_2 a(\xi_2))\text{sgn}(\xi_{12}) = |\xi_1 a(\xi_1) + \xi_2 a(\xi_2)| > 0$$

for any $\xi_1, \xi_2 \in \mathbb{R}$ with $\xi_{12} \neq 0$. Then we decompose this term as

$$\begin{aligned} &\beta \sum_{\xi_{1234}=0} \sqrt{|\xi_1 a(\xi_1) + \xi_2 a(\xi_2)|} \sqrt{|\xi_3 a(\xi_3) + \xi_4 a(\xi_4)|} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4) \\ &\quad + \frac{\beta}{2} \sum_{\xi_{1234}=0} [\sqrt{|\xi_1 a(\xi_1) + \xi_2 a(\xi_2)|} - \sqrt{|\xi_3 a(\xi_3) + \xi_4 a(\xi_4)|}]^2 \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4) \\ &=: \mathcal{Q}_1 + \mathcal{Q}_2. \end{aligned}$$

On the one hand, for $\beta < 0$, we have

$$\mathcal{Q}_1 = \beta \sum_{\xi} \left| \sum_{\xi_{12}=\xi} \sqrt{|\xi_1 a(\xi_1) + \xi_2 a(\xi_2)|} \hat{u}(\xi_1)\hat{u}(\xi_2) \right|^2 \leq 0.$$

On the other hand, it will turn out that the multiplier part of \mathcal{Q}_2 vanishes when $\xi_{12} = 0$ and also when $\xi_{23} = 0$. \mathcal{Q}_2 is then canceled out by adding the correction term

$$E_1^a(u) := \sum_{\substack{\xi_{1234}=0 \\ \xi_{12}\xi_{23} \neq 0}} b_4^a(\xi_1, \xi_2, \xi_3, \xi_4)\text{sgn}(\xi_{12})\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4)$$

to the modified energy $E_0^a(u)$, where

$$b_4^a(\xi_1, \xi_2, \xi_3, \xi_4) := -\frac{\beta}{2} \frac{[\sqrt{|\xi_1 a(\xi_1) + \xi_2 a(\xi_2)|} - \sqrt{|\xi_3 a(\xi_3) + \xi_4 a(\xi_4)|}]^2}{2i\xi_{12}\xi_{23}\text{sgn}(\xi_{12})}.$$

We can show that b_4^a is extended to a smooth function on Γ_4 , where

$$\Gamma_m = \{(\xi_j)_{1 \leq j \leq m} : \xi_{12\dots m} = 0\}$$

(we put $\text{sgn}(\xi_{12})$ outside to make b_4^a smooth). Moreover, $\xi_{12}\xi_{23} = 0$ implies $b_4^a(\xi_1, \xi_2, \xi_3, \xi_4) = 0$ (and hence the restriction $\xi_{12}\xi_{23} \neq 0$ for the sum in $E_1^a(u)$ can be disregarded). In fact,

noticing that

$$q(\xi_1, \xi_2) := \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 + \xi_2} > 0 \quad (\xi_1, \xi_2 \in \mathbb{R}, \xi_{12} \neq 0),$$

on $\Gamma_4 \cap \{\xi_{12}\xi_{23} \neq 0\}$, we compute it as

$$\begin{aligned} b_4^a(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{i\beta}{4\xi_{23}} \left[\sqrt{\frac{|\xi_1 a(\xi_1) + \xi_2 a(\xi_2)|}{|\xi_{12}|}} - \sqrt{\frac{|\xi_3 a(\xi_3) + \xi_4 a(\xi_4)|}{|\xi_{34}|}} \right]^2 \\ &= \frac{i\beta}{4\xi_{23}} \left[\sqrt{q(\xi_1, \xi_2)} - \sqrt{q(\xi_3, \xi_4)} \right]^2 \\ &= \frac{i\beta}{4\xi_{23}} \left[\frac{q(\xi_1, \xi_2) - q(\xi_1 + \xi_{23}, \xi_2 - \xi_{23})}{\sqrt{q(\xi_1, \xi_2)} + \sqrt{q(\xi_3, \xi_4)}} \right]^2 \\ &= \frac{i\beta\xi_{23}}{4} \frac{[\int_0^1 (\partial_1 q - \partial_2 q)(\xi_1 + \xi_{23}t, \xi_2 - \xi_{23}t) dt]^2}{[\sqrt{q(\xi_1, \xi_2)} + \sqrt{q(\xi_3, \xi_4)}]^2}, \end{aligned}$$

while $q(\xi_1, \xi_2)$ is actually positive and smooth on \mathbb{R}^2 , as we see in the following lemma.

Lemma 4.4 $q(\xi_1, \xi_2)$ can be extended to a smooth positive function on \mathbb{R}^2 . Moreover, the following hold (with $\xi_{\max} := |\xi_1| \vee |\xi_2|$):

- (i) $q(\xi_1, \xi_2) \sim a(\xi_{\max})$, $|\partial_1^{\gamma_1} \partial_2^{\gamma_2} q|(\xi_1, \xi_2) \lesssim \langle \xi_1 \rangle^{-\gamma_1} \langle \xi_2 \rangle^{-\gamma_2} a(\xi_{\max})$ ($1 \leq |\gamma| \leq 3$);
- (ii) $|\partial_1^{\gamma_1} \partial_2^{\gamma_2} (\partial_1 - \partial_2)q|(\xi_1, \xi_2) \lesssim \langle \xi_1 \rangle^{-\gamma_1} \langle \xi_2 \rangle^{-\gamma_2} \langle \xi_{\max} \rangle^{-1} a(\xi_{\max})$ ($0 \leq |\gamma| \leq 3$).

Proof For $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{\xi_{12} = 0\}$, we have

$$q(\xi_1, \xi_2) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 + \xi_2} \tag{4.2}$$

$$= \frac{\xi_1 a(\xi_1) - (-\xi_2) a(-\xi_2)}{\xi_1 - (-\xi_2)} = \int_0^1 (\xi a(\xi))'(-\xi_2 + \xi_{12}t) dt. \tag{4.3}$$

Since a is smooth and $(\xi a(\xi))' = a(\xi) + \xi a'(\xi) \geq a(\xi) \geq a(0) > 0$ ($\xi \in \mathbb{R}$), (4.3) defines a positive smooth function on \mathbb{R}^2 .

To show $q(\xi_1, \xi_2) \sim a(\xi_{\max})$, it suffices to consider the following three cases. If $\xi_{\max} \leq 1$, then $q(\xi_1, \xi_2) = a(0) = a(\xi_{\max})$. If $\xi_{\max} > 1$ and $|\xi_{12}| \sim \xi_{\max}$, then the claim follows from expression (4.2) (and some more arguments). If $\xi_{\max} > 1$ and $|\xi_{12}| \ll \xi_{\max}$, then we have $|\xi_1 + \xi_{12}t| \sim |\xi_1| \sim \xi_{\max}$ for $t \in [0, 1]$, and the claim follows from (4.3) since $(\xi a(\xi))' = a(\xi) + \xi a'(\xi) \sim a(\xi)$.

For the derivatives of q , we may focus on the case $\xi_{\max} > 1$. In the case $|\xi_{12}| \sim \xi_{\max}$, we first observe that

$$|\partial_1^{\gamma_1} \partial_2^{\gamma_2} [\xi_{12}^{-1}]| \lesssim \langle \xi_{\max} \rangle^{-1-|\gamma|}.$$

On the other hand, using the property $|\partial_\xi^j a(\xi)| \lesssim \langle \xi \rangle^{-j} a(\xi)$ for $j \leq 5$, we have

$$|\partial_\xi^j (\xi a(\xi))| \lesssim \langle \xi \rangle^{1-j} a(\xi),$$

which implies that

$$|\partial_1^{\gamma_1} \partial_2^{\gamma_2} (\xi_1 a(\xi_1) + \xi_2 a(\xi_2))| \begin{cases} \lesssim \langle \xi_1 \rangle^{1-j} a(\xi_{\max}) & (\gamma = (j, 0)), \\ \lesssim \langle \xi_2 \rangle^{1-j} a(\xi_{\max}) & (\gamma = (0, j)), \\ = 0 & (\gamma_1 \geq 1 \text{ and } \gamma_2 \geq 1). \end{cases}$$

The claimed estimate follows from these estimates and expression (4.2). When $|\xi_{12}| \ll \xi_{\max}$, we deduce from expression (4.3) that

$$[\partial_1^{\gamma_1} \partial_2^{\gamma_2} q](\xi_1, \xi_2) = \int_0^1 t^{\gamma_1} (t-1)^{\gamma_2} [\partial_\xi^{1+|\gamma|} (\xi a(\xi))](-\xi_2 + \xi_{12}t) dt.$$

Recalling that $|-\xi_2 + \xi_{12}t| \sim \xi_{\max}$ for $t \in [0, 1]$, we have

$$|[\partial_1^{\gamma_1} \partial_2^{\gamma_2} q](\xi_1, \xi_2)| \lesssim \langle \xi_{\max} \rangle^{-|\gamma|} a(\xi_{\max}) \lesssim \langle \xi_1 \rangle^{-\gamma_1} \langle \xi_2 \rangle^{-\gamma_2} a(\xi_{\max}).$$

This proves (i). For (ii), we compute

$$\begin{aligned} (\partial_1 q - \partial_2 q)(\xi_1, \xi_2) &= \frac{a(\xi_1) - a(\xi_2)}{\xi_{12}} + \frac{\xi_1 a'(\xi_1) - \xi_2 a'(\xi_2)}{\xi_{12}} \\ &= \int_0^1 (\xi a(\xi))'(-\xi_2 + \xi_{12}t) dt. \end{aligned}$$

Using these expressions instead of (4.2)–(4.3), the desired estimate is verified by an argument similar to (i). □

We have the following estimate on $E_1^a(u(t))$ for each t .

Lemma 4.5 *We have*

$$|E_1^a(f)| \lesssim \|f\|_{L^2_x}^2 E_0^a(f)$$

for any $f \in L^2_x$ such that $E_0^a(f) < \infty$.

Proof Let us begin with the dyadic decomposition:

$$|E_1^a(f)| \lesssim \sum_{N_1, \dots, N_4 \geq 1} \sum_{\Gamma_4} |b_4^a(\xi_1, \xi_2, \xi_3, \xi_4)| \cdot \psi_{N_1}(\xi_1) |\hat{f}(\xi_1)| \cdots \psi_{N_4}(\xi_4) |\hat{f}(\xi_4)|.$$

We can show that

$$|b_4^a(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim \frac{a(N_1^*)}{N_1^*} \quad \text{on } (I_{N_1} \times \cdots \times I_{N_4}) \cap \Gamma_4,$$

where we renumber N_1, \dots, N_4 as N_1^*, \dots, N_4^* such that $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. (We will actually prove a stronger result including estimates on derivatives of b_4^a in the proof of Lemma 4.7.) Then, by Hölder we have

$$|E_1^a(f)| \lesssim \sum_{N_1^* \sim N_2^* \geq N_3^* \geq N_4^* \geq 1} \frac{a(N_1^*)}{N_1^*} (N_3^* N_4^*)^{1/2} \prod_{j=1}^4 \|P_{N_j^*} f\|_{L^2}$$

$$\begin{aligned} &\lesssim \|f\|_{L^2}^2 \sum_{N_1^* \sim N_2^*} a(N_1^*) \|P_{N_1^*} f\|_{L^2} \|P_{N_2^*} f\|_{L^2} \\ &\lesssim \|f\|_{L^2}^2 \sum_N a(N) \|P_N f\|_{L^2}^2 \\ &\lesssim \|f\|_{L^2}^2 E_0^a(f), \end{aligned}$$

as desired. □

By differentiating $E_1^a(u(t))$ in t and substituting the equation, we obtain

$$\begin{aligned} \frac{d}{dt} (E_0^a(u(t)) + E_1^a(u(t))) &= Q_1 + \beta \sum_{\Gamma_6} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) \hat{u}(\xi_5) \hat{u}(\xi_6) \\ &\quad \times \left[-b_4^a(\xi_{123}, \xi_4, \xi_5, \xi_6) \operatorname{sgn}(\xi_{56}) \xi_{123} \operatorname{sgn}(\xi_{12}) \right. \\ &\quad \left. - b_4^a(\xi_1, \xi_{234}, \xi_5, \xi_6) \operatorname{sgn}(\xi_{56}) \xi_{234} \operatorname{sgn}(\xi_{34}) \right. \\ &\quad \left. + b_4^a(\xi_1, \xi_2, \xi_{345}, \xi_6) \operatorname{sgn}(\xi_{12}) \xi_{345} \operatorname{sgn}(\xi_{34}) \right. \\ &\quad \left. + b_4^a(\xi_1, \xi_2, \xi_3, \xi_{456}) \operatorname{sgn}(\xi_{12}) \xi_{456} \operatorname{sgn}(\xi_{56}) \right] \\ &=: Q_1 + \beta(\mathcal{R}_1 + \dots + \mathcal{R}_4). \end{aligned}$$

Here, it turns out that $\mathcal{R}_1 = \mathcal{R}_3 = \overline{\mathcal{R}_2} = \overline{\mathcal{R}_4}$. To see this, we start with \mathcal{R}_1 and first change variables as $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \mapsto (\xi_3, \xi_4, \xi_5, \xi_6, \xi_1, \xi_2)$ to obtain \mathcal{R}_3 . We then see $\overline{\mathcal{R}_1} = \mathcal{R}_4$ and $\overline{\mathcal{R}_2} = \mathcal{R}_3$ by taking the complex conjugate, using $\overline{\hat{u}(\xi)} = \hat{u}(-\xi)$ and changing variables as $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \mapsto (-\xi_6, -\xi_5, -\xi_4, -\xi_3, -\xi_2, -\xi_1)$. Therefore, it suffices to consider

$$\begin{aligned} \mathcal{R}(u) &:= -\mathcal{R}_1 \\ &= \sum_{\Gamma_6} b_4^a(\xi_{123}, \xi_4, \xi_5, \xi_6) \xi_{123} \operatorname{sgn}(\xi_{12}) \operatorname{sgn}(\xi_{56}) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4) \hat{u}(\xi_5) \hat{u}(\xi_6), \end{aligned}$$

which satisfies

$$\frac{d}{dt} (E_0^a(u(t)) + E_1^a(u(t))) = Q_1(u(t)) - 4\beta \Im \mathcal{R}(u(t)).$$

We need to prove the following estimate on the remainder term $\mathcal{R}(u)$, which is the hardest part in the overall proof of the main theorem.

Proposition 4.6 *Let $s > 1/4$. For $0 < T \leq 1$, we have*

$$\left| \int_0^T \mathcal{R}(u(t)) dt \right| \lesssim \|u\|_{F^s(T)}^4 \sum_{N \geq 1} a(N) \|P_N u\|_{F_N(T)}^2.$$

Let us postpone the proof of this proposition and verify Proposition 4.2.

Proof of Proposition 4.2 By Lemma 4.5, Proposition 4.6, and the fact that any smooth solution of (1.1) reduces its L^2 norm, we have

$$\sup_{t \in [0, T]} E_0^a(u(t))$$

$$\begin{aligned} &\leq E_0^a(u_0) + \sup_{t \in [0, T]} |E_1^a(u(t))| + C \|u\|_{F^s(T)}^4 \sum_{N \geq 1} a(N) \|P_N u\|_{F_N(T)}^2 \\ &\leq E_0^a(u_0) + C \|u(0)\|_{L^2}^2 \sup_{t \in [0, T]} E_0^a(u(t)) + C \|u\|_{F^s(T)}^4 \sum_{N \geq 1} a(N) \|P_N u\|_{F_N(T)}^2. \end{aligned}$$

Assuming $\|u(0)\|_{L^2} \ll 1$, this yields the claimed estimate. □

4.3 Estimate on the remainder term

It remains to prove Proposition 4.6. A difficulty here is that we cannot directly apply point-wise bounds on the multipliers (as we did in the proof of Lemma 4.5), because $u \in F^s(T)$ does not in general imply $\mathcal{F}_\xi^{-1}[|\hat{u}(t, \xi)|] \in F^s(T)$. Indeed, linear solutions $u = e^{it\partial_x^2} u_0$ can be considered as counterexamples.

We prepare the following lemma, which allows us to separate variables in the multiplier b_4^a . This idea has also been used in [5, 6, 18].

Lemma 4.7 *Let N_1, \dots, N_4 be dyadic integers such that $N_1^* \sim N_2^*$, where N_1^*, \dots, N_4^* denote the numbers N_1, \dots, N_4 rearranged in decreasing order. Let $\tilde{\psi}_1, \dots, \tilde{\psi}_4 \in C_0^\infty(\mathbb{R})$ be bump functions such that $\text{supp}(\tilde{\psi}_j(\cdot/N_j)) \subset \mathcal{I}_{N_j}$, with \mathcal{I}_{N_j} defined as in Definition 2.2 (i.e., $\tilde{\psi}_j$ is supported in $[-2, 2]$ if $N_j = 1$ and in $[-2, 2] \setminus (-\frac{1}{2}, \frac{1}{2})$ if $N_j > 1$).*

Then there is a sequence $\hat{c} \in \ell^1(\mathbb{Z}^4)$ such that

$$\begin{aligned} &b_4^a(\xi_1, \xi_2, \xi_3, \xi_4) \xi_1 \cdot \tilde{\psi}_1\left(\frac{\xi_1}{N_1}\right) \tilde{\psi}_2\left(\frac{\xi_2}{N_2}\right) \tilde{\psi}_3\left(\frac{\xi_3}{N_3}\right) \tilde{\psi}_4\left(\frac{\xi_4}{N_4}\right) \\ &= \sum_{k_1, \dots, k_4 \in \mathbb{Z}} \hat{c}(k_1, k_2, k_3, k_4) e^{i\left(\frac{k_1}{N_1}\xi_1 + \frac{k_2}{N_2}\xi_2 + \frac{k_3}{N_3}\xi_3 + \frac{k_4}{N_4}\xi_4\right)}, \quad (\xi_1, \dots, \xi_4) \in \Gamma_4 \end{aligned}$$

and

$$\sum_{k_1, \dots, k_4 \in \mathbb{Z}} |\hat{c}(k_1, k_2, k_3, k_4)| \lesssim N_1 \frac{a(N_1^*)}{N_1^*}.$$

Proof Following the argument in [18], we first construct a smooth function $\tilde{b}_4^a(\xi_1, \dots, \xi_4)$ on $\mathcal{I}_{N_1} \times \dots \times \mathcal{I}_{N_4}$, which extends $b_4^a(\xi_1, \dots, \xi_4)$ (defined on Γ_4) and satisfies

$$|\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \partial_4^{\gamma_4} \tilde{b}_4^a(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim \frac{1}{N_1^{\gamma_1} N_2^{\gamma_2} N_3^{\gamma_3} N_4^{\gamma_4}} \frac{a(N_1^*)}{N_1^*} \quad (0 \leq |\gamma| \leq 3). \tag{4.4}$$

We use the following extensions of b_4^a :

$$\begin{aligned} \tilde{b}_1(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{i\beta}{4\xi_{23}} \frac{[q(\xi_1, \xi_2) - q(\xi_3, \xi_4)]^2}{[\sqrt{q(\xi_1, \xi_2)} + \sqrt{q(\xi_3, \xi_4)}]^2} \quad (\text{on } \mathbb{R}^4 \setminus \{\xi_{23} = 0\}), \\ \tilde{b}_2(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{i\beta\xi_{23}}{4\xi_{12}^2} \frac{[q(\xi_2, \xi_3) - q(\xi_1, \xi_4)]^2}{[\sqrt{q(\xi_1, \xi_2)} + \sqrt{q(\xi_3, \xi_4)}]^2} \quad (\text{on } \mathbb{R}^4 \setminus \{\xi_{12} = 0\}), \\ \tilde{b}_3(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{i\beta\xi_{23}}{4} \frac{[\int_0^1 (\partial_1 q - \partial_2 q)(\xi_1 + \xi_{23}t, \xi_2 - \xi_{23}t) dt]^2}{[\sqrt{q(\xi_1, \xi_2)} + \sqrt{q(\xi_3, \xi_4)}]^2} \quad (\text{on } \mathbb{R}^4). \end{aligned}$$

From Lemma 4.4, we can show that

- $\left| \partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \partial_4^{\gamma_4} \left(\frac{[q(\xi_1, \xi_2) - q(\xi_3, \xi_4)]^2}{[q(\xi_2, \xi_3) - q(\xi_1, \xi_4)]^2} \right) \right| \lesssim \frac{a(N_1^*)^2}{N_1^{\gamma_1} N_2^{\gamma_2} N_3^{\gamma_3} N_4^{\gamma_4}};$
 - $|\partial_1^{\gamma_1} \partial_2^{\gamma_2} \sqrt{q(\xi_1, \xi_2)}| \lesssim \langle \xi_1 \rangle^{-\gamma_1} \langle \xi_2 \rangle^{-\gamma_2} \sqrt{a(|\xi_1| \vee |\xi_2|)} (\xi_1, \xi_2 \in \mathbb{R});$
 - $|\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \partial_4^{\gamma_4} [\sqrt{q(\xi_1, \xi_2)} + \sqrt{q(\xi_3, \xi_4)}]^{-2}| \lesssim \frac{1}{N_1^{\gamma_1} N_2^{\gamma_2} N_3^{\gamma_3} N_4^{\gamma_4} a(N_1^*)};$
 - If $N_1 \sim N_2 \sim N_1^*$ and $|\xi_{23}| \ll N_1^*$, then
- $$\left| \partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \partial_4^{\gamma_4} \left[\int_0^1 (\partial_1 q - \partial_2 q)(\xi_1 + \xi_{23}t, \xi_2 - \xi_{23}t) dt \right]^2 \right| \lesssim \frac{a(N_1^*)^2}{(N_1^*)^{2+|\gamma|}}.$$

Using these bounds, we see that the desired estimates (4.4) hold:

- for \tilde{b}_1 if $|\xi_{23}| \sim N_1^*$;
- for \tilde{b}_2 if $|\xi_{12}| \sim N_1^*$;
- for \tilde{b}_3 if $N_1 \sim N_2 \sim N_1^*$ and $|\xi_{23}| \ll N_1^*$.

Note that $|\xi_{12}| \vee |\xi_{23}| \ll N_1^*$ implies $N_1 \sim N_2 \sim N_3 \sim N_1^*$ under the hypothesis $N_1^* \sim N_2^*$. Therefore, we can define \tilde{b}_4^a by

$$\tilde{b}_4^a := \left[1 - \eta \left(\frac{\xi_{23}}{\frac{1}{100} N_1^*} \right) \right] \tilde{b}_1 + \left[1 - \eta \left(\frac{\xi_{12}}{\frac{1}{100} N_1^*} \right) \right] \eta \left(\frac{\xi_{23}}{\frac{1}{100} N_1^*} \right) \tilde{b}_2 + \eta \left(\frac{\xi_{12}}{\frac{1}{100} N_1^*} \right) \eta \left(\frac{\xi_{23}}{\frac{1}{100} N_1^*} \right) \tilde{b}_3,$$

for instance, where η is defined as in Definition 2.2. It is clear that the above defined \tilde{b}_4^a coincides with b_4^a on Γ_4 and satisfies (4.4).

Now, we define

$$c(\eta_1, \eta_2, \eta_3, \eta_4) := \tilde{b}_4^a(N_1 \eta_1, N_2 \eta_2, N_3 \eta_3, N_4 \eta_4) N_1 \eta_1 \cdot \tilde{\psi}_1(\eta_1) \tilde{\psi}_2(\eta_2) \tilde{\psi}_3(\eta_3) \tilde{\psi}_4(\eta_4),$$

which is a smooth function supported in $[-2, 2]^4$ and thus can be extended to a 2π -periodic smooth function on \mathbb{R}^4 . Let $\hat{c}(k_1, k_2, k_3, k_4)$ be the Fourier coefficients of c , then the claimed identity follows from the Fourier series expansion and the restriction onto Γ_4 . Moreover, we deduce from (4.4) that

$$\|c\|_{C^3([- \pi, \pi]^4)} := \max_{\eta \in [- \pi, \pi]^4, |\gamma| \leq 3} |\partial_1^{\gamma_1} \partial_2^{\gamma_2} \partial_3^{\gamma_3} \partial_4^{\gamma_4} c(\eta_1, \eta_2, \eta_3, \eta_4)| \lesssim N_1 \frac{a(N_1^*)}{N_1^*},$$

which then implies that

$$\|\hat{c}\|_{\ell^1(\mathbb{Z}^4)} \lesssim N_1 \frac{a(N_1^*)}{N_1^*}.$$

This completes the proof. □

We are now in a position to prove Proposition 4.6.

Proof of Proposition 4.6 As usual, we decompose the sum into dyadic pieces:

$$\int_0^T \mathcal{R}(u(t)) dt$$

$$\begin{aligned}
 &= \sum_{N_1, \dots, N_6, N \geq 1} \int_0^T \sum_{\Gamma_6} b_4^\alpha(\xi_{123}, \xi_4, \xi_5, \xi_6) \xi_{123} \psi_N(\xi_{123}) \tilde{\psi}_{N_4}(\xi_4) \tilde{\psi}_{N_5}(\xi_5) \tilde{\psi}_{N_6}(\xi_6) \\
 &\quad \times \operatorname{sgn}(\xi_{12}) \operatorname{sgn}(\xi_{56}) (\psi_{N_1} \hat{u})(t, \xi_1) (\psi_{N_2} \hat{u})(t, \xi_2) \cdots (\psi_{N_6} \hat{u})(t, \xi_6) dt,
 \end{aligned}$$

where for $j = 4, 5, 6$, $\tilde{\psi}_{N_j}(\xi_j) := \tilde{\psi}_j(\xi_j/N_j)$ and $\tilde{\psi}_j \in C_0^\infty(\mathbb{R})$ is chosen so that $\tilde{\psi}_j(\cdot/N_j) \equiv 1$ on $\operatorname{supp}(\psi_{N_j})$ and $\operatorname{supp}(\tilde{\psi}_j(\cdot/N_j)) \subset \mathcal{I}_{N_j}$, with $\psi_{N_j}, \mathcal{I}_{N_j}$ defined as in Definition 2.2. In the following, we write N_1^*, \dots, N_6^* to denote the numbers N_1, \dots, N_6 rearranged in decreasing order. Note that the range of N_1, \dots, N_6, N can be restricted to

$$N_1^* \sim N_2^* \quad \text{and} \quad N \lesssim \min\{\max\{N_1, N_2, N_3\}, \max\{N_4, N_5, N_6\}\}. \tag{4.5}$$

Applying Lemma 4.7 for each (N, N_4, N_5, N_6) , we have

$$\begin{aligned}
 &\int_0^T \mathcal{R}(u(t)) dt \\
 &= \sum_{N_1, \dots, N_6, N \geq 1} \sum_{k, k_4, k_5, k_6 \in \mathbb{Z}} \hat{c}_{N, N_4, N_5, N_6}(k, k_4, k_5, k_6) \int_0^T \sum_{\Gamma_6} \operatorname{sgn}(\xi_{12}) \operatorname{sgn}(\xi_{56}) \\
 &\quad \times e^{i(\frac{k}{N} \xi_1 + \frac{k}{N} \xi_2 + \frac{k}{N} \xi_3 + \frac{k_4}{N_4} \xi_4 + \frac{k_5}{N_5} \xi_5 + \frac{k_6}{N_6} \xi_6)} \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \cdots \hat{u}_6(t, \xi_6) dt,
 \end{aligned}$$

where we write $u_j = P_{N_j} u$ for brevity, $j = 1, \dots, 6$, and

$$\sum_{k, k_4, k_5, k_6 \in \mathbb{Z}} |\hat{c}_{N, N_4, N_5, N_6}(k, k_4, k_5, k_6)| \lesssim \frac{Na(\max\{N, N_4, N_5, N_6\})}{\max\{N, N_4, N_5, N_6\}}. \tag{4.6}$$

Since multiplication by $e^{i\theta\xi}$ on the Fourier side does not change the $F_{N_j}(T)$ norm of u_j , the proof is reduced to estimating

$$\begin{aligned}
 &\left| \int_0^T \sum_{\Gamma_6} \operatorname{sgn}(\xi_{12}) \operatorname{sgn}(\xi_{56}) \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \cdots \hat{u}_6(t, \xi_6) dt \right| \\
 &= \left| \int_0^T \int_{\mathbb{T}} H(u_1 \bar{u}_2) u_3 \bar{u}_4 H(u_5 \bar{u}_6) dx dt \right|.
 \end{aligned}$$

To obtain a bound with the short-time norms, we have to divide the time interval into small subintervals of length $\leq (N_1^*)^{-1}$ (denoted by I), which gives a factor of $O(N_1^*)$. The strategy in the previous results on DNLS [4, 5] is to use two bilinear Strichartz and two L^∞ embeddings if $N_1^* \sim N_2^* \gg N_3^*$ or $N_1^* \sim N_3^* \gg N_4^*$; one bilinear Strichartz, one L^∞ embedding, and three L^6 Strichartz if $N_1^* \sim N_4^* \gg N_5^*$ or $N_1^* \sim N_5^* \gg N_6^*$; and six L^6 Strichartz if $N_1^* \sim N_6^*$. For KDNLS, there are some cases where the same argument does not work due to the presence of the Hilbert transformations. For instance, we cannot use the standard bilinear Strichartz estimate (Lemma 2.4) with only one of u_1 and u_2 involved. We can indeed use the modified bilinear Strichartz estimates (Lemmas 2.7, 2.8) instead, but the argument will be even more complicated.

The goal is to prove

$$\begin{aligned} & \text{R.H.S. of (4.6)} \times N_1^* \left| \int_I \int_{\mathbb{T}} H(u_1 \bar{u}_2) u_3 \bar{u}_4 H(u_5 \bar{u}_6) dx dt \right| \\ & \lesssim a(N_1^*) (N_3^* N_4^* N_5^* N_6^*)^{\frac{1}{4}+} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})} \end{aligned} \tag{4.7}$$

for each $N_1, \dots, N_6, N \geq 1$ satisfying (4.5) and each interval $I = [a, b] \subset [0, T]$ with $|I| \leq (N_1^*)^{-1}$. In fact, this is enough to carry out the summations in N_1, \dots, N_6 when $s > 1/4$. For summability in N , notice that either $N \sim N_1^*$ or $N \lesssim N_3^*$ holds.

In the rest of the proof, we shall establish (4.7), dividing into the following five cases:

- (I) $N_1^* \sim N_6^*$; (II) $N_1^* \sim N_5^* \gg N_6^*$; (III) $N_1^* \sim N_4^* \gg N_5^*$;
- (IV) $N_1^* \sim N_3^* \gg N_4^*$; (V) $N_1^* \sim N_2^* \gg N_3^*$.

We will estimate the right-hand side of (4.6) roughly by $a(N_1^*)$, except for a subcase (Vd) of Case (V). For simplicity, we will see in detail only Case (III) and Case (V).¹ Case (I) is the easiest, and it is treated by the L^6 Strichartz estimate (Lemma 2.3). Case (II) is a little more involved, and we need the bilinear Strichartz estimate (Lemma 2.4). In Case (IV) we also use the first modified bilinear Strichartz estimate (Lemma 2.7), but the argument is similar to that in Case (III).

Case (III) $N_1^ \sim N_4^* \gg N_5^*$.* We consider the following three subcases, according to which two frequencies are smaller than N_1^* .

(IIIa) Each of $\{N_1, N_2\}$, $\{N_3, N_4\}$, $\{N_5, N_6\}$ contains at most one frequency $\ll N_1^*$. Consider the case $N_5^* = N_1$ and $N_6^* = N_3$ for instance, but the other cases can be treated similarly. Then we apply Lemma 2.4 to $H(u_1 \bar{u}_2)$, Lemma 2.3 to u_4, u_5, u_6 , and L^∞ embedding to u_3 , which yields

$$\begin{aligned} \mathcal{I} & := \left| \int_I \int_{\mathbb{T}} H(u_1 \bar{u}_2) u_3 \bar{u}_4 H(u_5 \bar{u}_6) dx dt \right| \\ & \lesssim \frac{1}{N_2^{1/2}} N_3^{1/2} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})} \lesssim \frac{(N_3^* \dots N_6^*)^{1/4}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})}. \end{aligned}$$

(IIIb) $\{N_5^*, N_6^*\} = \{N_1, N_2\}$ or $\{N_5, N_6\}$. Consider the former case, for instance. We apply the first modified bilinear Strichartz estimate (Lemma 2.7) for $H(u_1 \bar{u}_2) u_3$ and Lemma 2.3 for the other three u_j s to obtain

$$\mathcal{I} \lesssim \frac{(N_1 \wedge N_2)^{1/2}}{N_3^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})} \lesssim \frac{(N_3^* \dots N_6^*)^{1/4}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})}.$$

(IIIc) $\{N_5^*, N_6^*\} = \{N_3, N_4\}$. Without loss of generality, we assume $N_5^* = N_3$ and $N_6^* = N_4$.

¹An extended version of this article can be found in <https://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1967.pdf>, where we keep a complete proof for the reader's convenience.

(IIIc-i) $N_3 \sim N_4$. In this case, we divide $H(u_1\bar{u}_2)$ into two parts as follows:

$$H(u_1\bar{u}_2) = HP_{\gg N_3}(u_1\bar{u}_2) + HP_{\lesssim N_3}(u_1\bar{u}_2).$$

For the first term, we can eliminate the Hilbert transformations by the fact that the frequency for $H(u_5\bar{u}_6)$ must be much bigger than that of $u_3\bar{u}_4$:

$$\begin{aligned} & \int_I \int_{\mathbb{T}} HP_{\gg N_3}(u_1\bar{u}_2)u_3\bar{u}_4H(u_5\bar{u}_6) dx dt \\ &= - \int_I \int_{\mathbb{T}} P_{\gg N_3}(u_1\bar{u}_2)H[u_3\bar{u}_4H(u_5\bar{u}_6)] dx dt \\ &= - \int_I \int_{\mathbb{T}} P_{\gg N_3}(u_1\bar{u}_2)u_3\bar{u}_4H^2(u_5\bar{u}_6) dx dt \\ &= \int_I \int_{\mathbb{T}} P_{\gg N_3}(u_1\bar{u}_2)u_3\bar{u}_4u_5\bar{u}_6 dx dt. \end{aligned}$$

Now, the desired bound can be obtained by applying Lemma 2.4 to u_3u_5 , L^∞ embedding to u_4 , and Lemma 2.3 to the others, for instance. To estimate the contribution from the second term, we use the second modified bilinear Strichartz estimate (Lemma 2.8) for $HP_{\lesssim N_3}(u_1\bar{u}_2)u_3$ and Lemma 2.3 for u_4, u_5, u_6 . In each case, we obtain the factor $a(N_1^*)(N_1^*N_6^*)^{1/2} \sim a(N_1^*)(N_3^* \dots N_6^*)^{1/4}$.

(IIIc-ii) $N_3 \gg N_4$. In this case, we make a finer decomposition:

$$\begin{aligned} & H(u_1\bar{u}_2)u_3\bar{u}_4H(u_5\bar{u}_6) \\ &= HP_{\ll N_3}(u_1\bar{u}_2)u_3\bar{u}_4H(u_5\bar{u}_6) + H(u_1\bar{u}_2)u_3\bar{u}_4HP_{\ll N_3}(u_5\bar{u}_6) \\ &\quad - HP_{\ll N_3}(u_1\bar{u}_2)u_3\bar{u}_4HP_{\ll N_3}(u_5\bar{u}_6) \\ &\quad + HP_{\gg N_3}(u_1\bar{u}_2)u_3\bar{u}_4HP_{\gtrsim N_3}(u_5\bar{u}_6) + HP_{\sim N_3}(u_1\bar{u}_2)u_3\bar{u}_4HP_{\gtrsim N_3}(u_5\bar{u}_6). \end{aligned}$$

There is no contribution from the third term, while the estimate for the fourth term is exactly the same as the first term in Case (IIIc-i) since in the integral we can replace $P_{\gtrsim N_3}(u_5\bar{u}_6)$ by $u_5\bar{u}_6$. For the first two terms, we can separate two functions of high frequency from the Hilbert transformation; for instance,

$$\begin{aligned} & \int_I \int_{\mathbb{T}} HP_{\ll N_3}(u_1\bar{u}_2)u_3\bar{u}_4H(u_5\bar{u}_6) dx dt \\ &= - \int_I \int_{\mathbb{T}} H[HP_{\ll N_3}(u_1\bar{u}_2)u_3\bar{u}_4]u_5\bar{u}_6 dx dt \\ &= - \int_I \int_{\mathbb{T}} HP_{\ll N_3}(u_1\bar{u}_2)(Hu_3)\bar{u}_4u_5\bar{u}_6 dx dt. \end{aligned}$$

This can again be treated similarly to the first term in Case (IIIc-i). To estimate the contribution from the last term, we first notice that the frequency for $u_5\bar{u}_6$ must stay $\sim N_3$ in the integral; hence let us write it as $\tilde{P}_{\sim N_3}(u_5\bar{u}_6)$. We estimate the integral in two ways. First, we apply Lemma 2.4 to $P_{\sim N_3}(u_1\bar{u}_2)$ and $\tilde{P}_{\sim N_3}(u_5\bar{u}_6)$ and L^∞ embedding to u_3, u_4 , which yields

$$\left| \int_I \int_{\mathbb{T}} HP_{\sim N_3}(u_1\bar{u}_2)u_3\bar{u}_4H\tilde{P}_{\sim N_3}(u_5\bar{u}_6) dx dt \right| \lesssim \frac{1}{N_3^{1/2}} \frac{1}{N_3^{1/2}} N_3^{1/2} N_4^{1/2} \prod_{j=1}^6 \| \chi_I u_j \|_{U_\Delta^2(\mathbb{R})}.$$

Next, applying Lemma 2.8 to $HP_{\sim N_3}(u_1 \bar{u}_2)u_3$ and $H\tilde{P}_{\sim N_3}(u_5 \bar{u}_6)\bar{u}_4$, we have

$$\left| \int_I \int_{\mathbb{T}} HP_{\sim N_3}(u_1 \bar{u}_2)u_3 \bar{u}_4 H\tilde{P}_{\sim N_3}(u_5 \bar{u}_6) dx dt \right| \lesssim \frac{N_3^{1/2}}{N_1^{1/2}} \frac{N_3^{1/2}}{N_5^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})}.$$

Interpolating these estimates, we obtain

$$\begin{aligned} \left| \int_I \int_{\mathbb{T}} HP_{\sim N_3}(u_1 \bar{u}_2)u_3 \bar{u}_4 H\tilde{P}_{\sim N_3}(u_5 \bar{u}_6) dx dt \right| &\lesssim \frac{N_3^{1/4} N_4^{1/4}}{(N_1^*)^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})} \\ &\sim \frac{(N_3^* \dots N_6^*)^{1/4}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})}, \end{aligned}$$

as desired.

Case (V) $N_1^ \sim N_2^* \gg N_3^*$.* Let us divide into the following four cases.

(Va) One of $\{N_1, N_2\}$ and one of $\{N_5, N_6\}$ are comparable to N_1^* . In this case, we can break the binding by the Hilbert transformation between u_1 and \bar{u}_2 and between u_5 and \bar{u}_6 . Applying Lemma 2.4 twice to the pairs (N_1^*, N_3^*) and (N_2^*, N_4^*) and the L^∞ embedding to the functions corresponding to N_5^*, N_6^* , we obtain

$$\mathcal{I} \lesssim \frac{(N_5^* N_6^*)^{1/2}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})} \lesssim \frac{(N_3^* \dots N_6^*)^{1/4}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})}.$$

(Vb) One of $\{N_1, N_2, N_5, N_6\}$ and one of $\{N_3, N_4\}$ are comparable to N_1^* . Assume, say, $N_1 \sim N_3 \sim N_1^*$. We can deal with u_1 and \bar{u}_2 of $H(u_1 \bar{u}_2)$ separately, which allows us to treat two cases $N_2 \geq N_4$ and $N_2 \leq N_4$ in a parallel manner (let us assume $N_2 \leq N_4$, say). Apply Lemma 2.4 to $u_3 \bar{u}_4$, Lemma 2.7 to $(Hu_1)H(u_5 \bar{u}_6)$, and L^∞ embedding to \bar{u}_2 , then we have

$$\mathcal{I} \lesssim \frac{1}{N_3^{1/2}} \frac{(N_5 \wedge N_6)^{1/2}}{N_1^{1/2}} N_2^{1/2} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})} \lesssim \frac{(N_3^* \dots N_6^*)^{1/4}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})}.$$

(Vc) $N_3 \sim N_4 \sim N_1^*$. It suffices to apply Lemma 2.7 twice to $H(u_1 \bar{u}_2)u_3$ and $\bar{u}_4 H(u_5 \bar{u}_6)$, which gives

$$\mathcal{I} \lesssim \frac{(N_1 \wedge N_2)^{1/2}}{N_3^{1/2}} \frac{(N_5 \wedge N_6)^{1/2}}{N_4^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})} \lesssim \frac{(N_3^* \dots N_6^*)^{1/4}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{L^2_{\Delta}(\mathbb{R})}.$$

(Vd) $N_1 \sim N_2 \sim N_1^*$ or $N_5 \sim N_6 \sim N_1^*$. This is the hardest case, and let us focus on the former situation $N_1 \sim N_2 \gg N_3, \dots, N_6$. First, using conditions (4.1) on a and (4.5) on N , we deduce that

$$\text{R.H.S. of (4.6)} \lesssim a(N_1^*) \left(\frac{N_3^*}{N_1^*} \right)^{1/2}. \tag{4.8}$$

(In fact, it is only in the case $N_1 \sim N_2 \sim N_1^*$ that we need to exploit the growth condition on a —the third line in (4.1).) In the following, we only consider the case $N_3 \geq N_4, N_5 \geq N_6$; the other cases are parallel, though.

(Vd-i) $N_3^* \gg N_4^*$, namely, only one of N_3, \dots, N_6 is much bigger than the others. In this case, we can move the Hilbert transformation on $u_1 \bar{u}_2$ to one of u_3, \dots, u_6 ; if $N_3 = N_3^*$ we have

$$\int_I \int_{\mathbb{T}} H(u_1 \bar{u}_2) u_3 \bar{u}_4 H(u_5 \bar{u}_6) dx dt = - \int_I \int_{\mathbb{T}} u_1 \bar{u}_2 H(u_3) \bar{u}_4 H(u_5 \bar{u}_6) dx dt,$$

and if $N_5 = N_3^*$ we have

$$\begin{aligned} \int_I \int_{\mathbb{T}} H(u_1 \bar{u}_2) u_3 \bar{u}_4 H(u_5 \bar{u}_6) dx dt &= - \int_I \int_{\mathbb{T}} u_1 \bar{u}_2 u_3 \bar{u}_4 H^2(u_5) \bar{u}_6 dx dt \\ &= \int_I \int_{\mathbb{T}} u_1 \bar{u}_2 u_3 \bar{u}_4 u_5 \bar{u}_6 dx dt. \end{aligned}$$

Then we can obtain

$$\mathcal{I} \lesssim \frac{(N_6^* N_4^*)^{1/2}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})} \lesssim \frac{(N_3^* \cdots N_6^*)^{1/4}}{N_1^*} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})}$$

by using Lemmas 2.4 and 2.7 when $N_3 = N_3^*$, or Lemma 2.4 twice when $N_5 = N_3^*$. Note that we do not actually need the improved bound (4.8) in this subcase.

(Vd-ii) $N_3^* \sim N_6^*$. In this case, we use Lemma 2.8 for $H(u_1 \bar{u}_2) u_3$ (noticing that $u_1 \bar{u}_2$ may be replaced by $P_{\lesssim N_3^*}(u_1 \bar{u}_2)$), Lemma 2.3 for the others to obtain

$$\mathcal{I} \lesssim \frac{(N_3^*)^{1/2}}{(N_1^*)^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})}.$$

Combining it with (4.8), we have the desired estimate.

(Vd-iii) $N_3^* \sim N_4^* \gg N_6^*$ and $N_5 \gg N_6$. In this case $H(u_5 \bar{u}_6) = (Hu_5) \bar{u}_6$, so that we can separate u_3, \dots, u_6 .

(Vd-iii-1) If $N_3^* \sim N_4^* \gg N_5^*$, then we use Lemma 2.8 for $H(u_1 \bar{u}_2) u_3^*$, Lemma 2.4 for $u_4^* u_5^*$, and the L^∞ embedding for u_6^* , where u_j^* means the function corresponding to N_j^* . The resulting estimate is

$$\mathcal{I} \lesssim \frac{(N_3^*)^{1/2}}{(N_1^*)^{1/2}} \frac{1}{(N_4^*)^{1/2}} (N_6^*)^{1/2} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})} \lesssim \frac{(N_5^* N_6^*)^{1/4}}{(N_1^*)^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_{\Delta}^2(\mathbb{R})},$$

which is sufficient together with (4.8).

(Vd-iii-2) If $N_3^* \sim N_5^* \gg N_6^*$, by the assumption we have $N_3 \sim N_4 \sim N_5 \gg N_6$. In this case, we make the decomposition

$$u_3 \bar{u}_4 = P_{\ll N_3}(u_3 \bar{u}_4) + P_{\sim N_3}(u_3 \bar{u}_4).$$

For the first term, we can separate u_1 and u_2 as

$$\int_I \int_{\mathbb{T}} H(u_1 \bar{u}_2) P_{\ll N_3}(u_3 \bar{u}_4) H(u_5) \bar{u}_6 dx dt = - \int_I \int_{\mathbb{T}} u_1 \bar{u}_2 H[P_{\ll N_3}(u_3 \bar{u}_4) H(u_5) \bar{u}_6] dx dt$$

$$= \int_I \int_{\mathbb{T}} u_1 \bar{u}_2 P_{\ll N_3}(u_3 \bar{u}_4) u_5 \bar{u}_6 \, dx \, dt.$$

Hence, this is similar to Case (Vd-i) and easily treated by applying Lemmas 2.4 and 2.7. For the second term, we use Lemma 2.8 for $H(u_1 \bar{u}_2)H(u_5)$, Lemma 2.4 for $P_{\sim N_3}(u_3 \bar{u}_4)$, and L^∞ embedding for \bar{u}_6 to obtain

$$\mathcal{I} \lesssim \frac{N_3^{1/2}}{N_1^{1/2}} \frac{1}{N_3^{1/2}} N_6^{1/2} \prod_{j=1}^6 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})} \lesssim \frac{(N_5^* N_6^*)^{1/4}}{(N_1^*)^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})},$$

which is again sufficient.

(Vd-iv) $N_3^* \sim N_4^* \gg N_6^*$ and $N_5 \sim N_6$.

(Vd-iv-1) If $N_3 \gg N_5$, so $N_3 \sim N_4 \gg N_5 \sim N_6$, we use Lemma 2.8 for $H(u_1 \bar{u}_2)u_3$ and Lemma 2.7 for $\bar{u}_4 H(u_5 \bar{u}_6)$. We obtain

$$\mathcal{I} \lesssim \frac{N_3^{1/2}}{N_1^{1/2}} \frac{N_5^{1/2}}{N_4^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})} \lesssim \frac{(N_5^* N_6^*)^{1/4}}{(N_1^*)^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})}.$$

(Vd-iv-2) If $N_5 \gtrsim N_3 \sim N_4$, namely, $N_5 \sim N_6 \gg N_3 \sim N_4$, we make a decomposition similar to that in Case (Vdiii-2):

$$H(u_5 \bar{u}_6) = HP_{\gg N_3}(u_5 \bar{u}_6) + HP_{\lesssim N_3}(u_5 \bar{u}_6).$$

For the first term, we can separate u_1 and u_2 as in Case (Vd-iii-2) since $H[u_3 \bar{u}_4 HP_{\gg N_3}(u_5 \bar{u}_6)] = u_3 \bar{u}_4 H^2 P_{\gg N_3}(u_5 \bar{u}_6)$. The estimate is then easy and similar to Case (Vd-i). For the second term, we can put $P_{\lesssim N_3}$ also on $H(u_1 \bar{u}_2)$. Applying Lemma 2.8 to $HP_{\lesssim N_3}(u_1 \bar{u}_2)u_3$ and Lemma 2.3 to the others, we have

$$\mathcal{I} \lesssim \frac{N_3^{1/2}}{N_1^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})} \lesssim \frac{(N_5^* N_6^*)^{1/4}}{(N_1^*)^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})}.$$

(Vd-iv-3) The only remaining case is $N_5 \gtrsim N_3 \gg N_4$, namely, either $N_5 \sim N_6 \sim N_3 \gg N_4$ or $N_5 \sim N_6 \gg N_3 \gg N_4$. We make a slightly finer decomposition

$$H(u_5 \bar{u}_6) = HP_{\ll N_3}(u_5 \bar{u}_6) + \sum_{N_3 \lesssim K \lesssim N_5} HP_K(u_5 \bar{u}_6).$$

The first term is again easy to treat since we can separate u_1 and u_2 by the identity $H[u_3 \bar{u}_4 HP_{\ll N_3}(u_5 \bar{u}_6)] = (Hu_3) \bar{u}_4 HP_{\ll N_3}(u_5 \bar{u}_6)$. For the second term, for each K , we can put $P_{\lesssim K}$ on $H(u_1 \bar{u}_2)$. Hence, by applying Lemma 2.8 to $HP_{\lesssim K}(u_1 \bar{u}_2)u_3$ and using Lemma 2.4 for $HP_K(u_5 \bar{u}_6)$, L^∞ embedding for \bar{u}_4 , we obtain

$$\mathcal{I} \lesssim \sum_{N_3 \lesssim K \lesssim N_5} \frac{K^{1/2}}{N_1^{1/2}} \frac{1}{K^{1/2}} N_4^{1/2} \prod_{j=1}^6 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})} \lesssim (N_3^*)^{0+} \frac{(N_5^* N_6^*)^{1/4}}{(N_1^*)^{1/2}} \prod_{j=1}^6 \|\chi_I u_j\|_{U_\Delta^2(\mathbb{R})},$$

which together with (4.8) shows the desired estimate (4.7).

We have thus completed the case-by-case analysis for the proof of (4.7). □

5 Conclusion

We have proved *a priori* estimates for solutions in a neighborhood of the origin in H^s , $s > 1/4$. As a result, we have proved the continuous dependence of solutions on initial data at the origin in H^s , $s > 1/4$, which constitutes a part of the well-posedness for the Cauchy problem of (1.1).

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Declarations

Competing interests

The authors declare that they have no competing interests.

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