Variational proof of the existence of periodic orbits in the anisotropic Kepler problem

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The anisotropic Kepler problem is a model of the motion of free electrons on an n type semiconductor, and is known to be a non-integrable Hamiltonian system. In this paper, we first show that the action functional of the anisotropic Kepler problem has a minimizer under a fixed region condition with boundary conditions on a vertical half-line. Next, we identify the smallest collision trajectory that satisfies the same boundary conditions. By constructing an orbit with an action functional smaller than this collision orbit via local deformation, we show that the collision solution does not become the minimizer. Reversibility allows the periodic orbit to be constructed from the minimizer obtained via the action functional.

1 Introduction

The n-body problem, which has been studied since Newton discovered universal gravitation, is concerned with the motion when n mass points are gravitationally attracted to each other. Poincare showed that the three-body problem cannot be solved under various settings. Therefore, it is not possible to find a general solution of the n-body problem, but it may be possible to find a special solution, in particular, a periodic solution. One way to find such a solution is the variational method. For example, Chenciner and Montgomery famously proved the existence of the figure-eight solution to the 3-body problem with equal masses[6]. In this paper, we apply this method to the anisotropic Kepler problem(AKP).

Equations of motion of a two-dimensional potential system are defined by:

$$\ddot{x} = -\frac{\partial V}{\partial x}, \quad \ddot{y} = -\frac{\partial V}{\partial y} \tag{1.1}$$

where $V(x, y)$ is a potential function. The potential of the AKP is represented by:

$$V(x,y) = -\frac{1}{\sqrt{x^2+\mu y^2}} \quad (\mu > 0)$$

and the Lagrangian by:

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - V(x,y). \tag{1.2}$$

Gutzwiller introduced this problem [9] as an equation modeling free electrons in n type semiconductors. The AKP is non-integrable for $\mu \neq 1$ [13] and has a horseshoe in the range of $\mu > 9/8$ or $\mu < 8/9$ [7].

A number of approximate periodic solutions of the AKP have been obtained via numerical calculation, but none have been mathematically proved to exist. Variational approach has been done by [1, 2, 3, 4]. In this paper, we use the variational method to prove the existence of simple periodic orbits with certain properties in the AKP. Our main result is the following.

Main result. For any $T > 0$ and any μ, there exists a periodic orbit $\boldsymbol{q}(t) = (x(t), y(t))$ that has period $4T$ and satisfies the following properties in the AKP:

- $\dot{x}(0) = \dot{y}(T) = 0$
- $x(-t) = x(t), y(-t) = y(t), x(t+T) = -x(-t+T), y(t+T) = -y(t+T)$
- $\boldsymbol{q}(t)$ is orthogonal to the x- and y- axis at $t = 0$ and $t = T$, respectively;
- For all $t \in (0, T)$, $x(t)$ decreases monotonically and $y(t)$ increases monotonically.

We organize this paper as follows. Section 2 introduces preliminaries for our proof, including reversibility and the conditions for the action functional to have the minimizer. In section 3, we formulate the AKP by the variational method. Section 4 shows that the minimizer has no collision. In section 5, we show the result of numerical calculation using AUTO.

2 Preliminaries

2.1 Reversibility

Consider the following ordinary differential equation:

$$\dot{\boldsymbol{q}} = F(\boldsymbol{q}), \quad \boldsymbol{q} \in \mathbb{R}^d \tag{2.1}$$

Definition 2.1 (*Reversibility*)**.** Let R be an involutory linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$, i.e. $R^2 = E_n$. If (2.1) satisfies:

$$F(R\boldsymbol{x}) + R(F\boldsymbol{x}) = 0$$

then (2.1) is said to be reversible with respect to R.

With a simple calculation,we obtain the following lemma.

Lemma 2.2. Assume that (2.1) satisfies reversibility. Then if $\boldsymbol{q}(t)$ is a solution of (2.1), then so is to $R\boldsymbol{q}(-t)$.

We define

$$\mathrm{Fix}(R) := \{\boldsymbol{q} \in \mathbb{R}^d \mid R\boldsymbol{q} = \boldsymbol{q}\}$$

Lemma 2.3. For a solution $\boldsymbol{q}(t)$ of (2.1), $\boldsymbol{q}(T) \in \mathrm{Fix}(R)$ is satisfied if and only if $\boldsymbol{q}(T+t) = R\boldsymbol{q}(T-t)$.

Proof. If $\boldsymbol{q}(T+t) = R\boldsymbol{q}(T-t)$, then $\boldsymbol{q}(T) \in \mathrm{Fix}(R)$ by substituting $t = 0$. Conversely, we assume that $\boldsymbol{q}(T) \in \mathrm{Fix}(R)$. From Lemma 2.2, if $\boldsymbol{q}(T+t)$ is a solution, then so is to $R\boldsymbol{q}(T-t)$. Since the initial values match from $\boldsymbol{q}(T) = R\boldsymbol{q}(T)$, the lemma follows owing to the uniqueness of the solution. $\square$

2.2 Minimizers of the action functional

We describe the known results of the variational problem of the Lagrangian system. Let $I = [0, T]$. Let $\mathcal{D}$ be a configuration space in $\mathbb{R}^d$. We define $A, B \subset \mathcal{D}$ as nonempty affine spaces. Let TA and TB be linear spaces created by translating A and$B$ so that they pass through the origin. We set:

$$\mathcal{C}_{A,B} = \{\boldsymbol{q} \in C^2(I, \mathcal{D}) | \boldsymbol{q}(0) \in A, \boldsymbol{q}(T) \in B\}. \tag{2.2}$$

The Lagrangian is $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and the action functional is:

$$\mathcal{A}(\boldsymbol{q}) = \int_0^T L(\boldsymbol{q}, \dot{\boldsymbol{q}})\, dt.$$

A path $\boldsymbol{q}$ that satisfies $\mathcal{A}'(\boldsymbol{q}) = 0$ is called the critical point of $\mathcal{A}(\boldsymbol{q})$. From the first variational formula, We consider a Sobolev space;

$$H^1(I, \mathcal{D}) = \left\{\boldsymbol{q} : I \to \mathcal{D} | \boldsymbol{q} \in L^2(I, \mathcal{D}), \frac{d\boldsymbol{q}}{dt} \in L^2(I, \mathcal{D})\right\}$$

and set the norm as:

$$||\boldsymbol{q}||_{H^1} := \sqrt{\int_0^T |\boldsymbol{q}|^2 + |\dot{\boldsymbol{q}}|^2\, dt}.$$

Definition 2.4 (*coercive*)**.** Let $\Omega \in H^1(I, \mathcal{D})$. The action functional $\mathcal{A}|_\Omega$ is said to be coercive when it satisfies

$$\mathcal{A}(\boldsymbol{q}) \to \infty \quad as \quad ||\boldsymbol{q}||_{H^1} \to \infty \quad (\boldsymbol{q} \in \Omega)$$

We set

$$\Omega(A, B) = \{\boldsymbol{q} \in H^1(I, \mathcal{D}) | \boldsymbol{q}(0) \in A, \boldsymbol{q}(T) \in B\}$$

Lemma 2.5 (Proposition 2.1 [5])**.** Let $A, B \subset \mathcal{D}$. If there exists $C_0 < 1$ that satisfies:

$$\boldsymbol{a} \cdot \boldsymbol{b} \leq C_0 |\boldsymbol{a}||\boldsymbol{b}| \tag{2.3}$$

for any $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in B$, then $\mathcal{A}|_\Omega$ is coercive.

Lemma 2.6 ([12])**.** Suppose that $\mathcal{A}|_\Omega$ is coercive. Then there exists a minimizer $\boldsymbol{q}^* \in \overline{\Omega}$. Moreover, if the minimizer $\boldsymbol{q}^*$ has no collision, then it satisfies $\boldsymbol{q}^* \in \mathcal{C}_{A,B}$, i.e. $\boldsymbol{q}^*$ is smooth.

3 Formulation of the AKP by the variational method

We set $\mathcal{D} = \mathbb{R}^2 - \{(0,0)\}$ and consider the boundary condition:

$$A = \{(x, 0) \in \mathcal{D} | x > 0\}, \quad B = \{(0, y) \in \mathcal{D} | y > 0\} \tag{3.1}$$

It is clear that A and B satisfy (2.3). Therefore, from Lemma 2.5 and 2.6, there exists minimizer in $\overline{\Omega}(A, B)$. Moreover, from the first variational formula, $\dot{\boldsymbol{q}}^*(0)$ is orthogonal to A and $\dot{\boldsymbol{q}}^*(T)$ is orthogonal to B. The minimizer $\boldsymbol{q^*}$ is a solution of AKP unless $\boldsymbol{q^*}$ is a collision orbit (Figure 1). The monotonicity will be shown later.

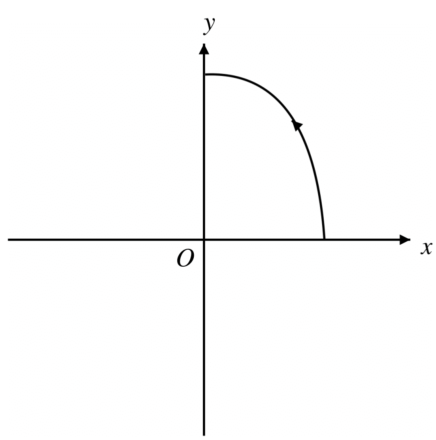

Figure 1: The minimizer we aim to find.

4 Estimation of collision orbit

Without loss of generality, we can limit the parameter μ in the AKP to $0 < \mu < 1$ by substituting $1/\mu$ instead of μ and applying appropriate change of variables.

Now we identify the collision orbit that minimizes the action functional. Using the polar coordinates $\boldsymbol{q} = (r\cos\theta, r\sin\theta)$, the action functional in the AKP is expressed by:

$$\mathcal{A}(\boldsymbol{q}) = \int \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{r\sqrt{\cos^2\theta + \mu\sin^2\theta}}.$$

From this, we can see the following about the collision orbit that minimizes the action functional:

- $\dot{\theta} = 0$ from the minimun of the kinetic energy term. In other words, the collision satisfying $\dot{\theta} \neq 0$ is not a minimizer because the trajectory transformed into $\dot{\theta} = 0$ minimize action functional.
- $\theta = 0$ from the minimum of the potential energy term. In other words, the collision satisfying $\theta \neq 0$ is not minimizer because the trajectory transformed into $\theta = 0$ minimize action functional.

Therefore, it suffices to show that the collision orbit that starts from positive part of the x-axis and moves to the origin along x-axis is not a minimizer. In the following, we discuss the orbit that starts from the origin and moves the to positive part of the x-axis along the x-axis because both have the same action functional and the latter is easier to handle.

4.1 Local transformation

We use the following for local evaluation of the collision orbit.

Lemma 4.1 (Sundman's estimate, [10]). We assume that collition trajectories $\boldsymbol{q}_1, \cdots, \boldsymbol{q}_l$ in a system with a Kepler-type potential collide with c at $t = t_0$. Then:

$$\boldsymbol{q}_k = \boldsymbol{c} + (t - t_0)^{2/3}\boldsymbol{a}_k + O(t - t_0) \quad (k = 1, \ldots, l).$$

From Lemma 4.1, the collision orbit $\boldsymbol{q}_{\mathrm{col}}$ that starts from the origin and moves along the positive part of the x-axis is represented by:

$$\boldsymbol{q}_{\mathrm{col}} = (at^{2/3} + O(t), 0), \quad a = \sqrt[3]{\frac{9}{2}}.$$

The main term of the action functional $\mathcal{A}(\boldsymbol{q}_{\mathrm{col}})$ at $t \in [0, \epsilon]$ is :

$$\left(\frac{2}{3}a^2 + \frac{3}{a}\right)\epsilon^{1/3}$$

We define a local tramsformation $\boldsymbol{\delta}_\epsilon$ at $t \in [0, \epsilon]$ as follows, where $n, m \in \mathbb{N}$ and free for the moment.

$$\boldsymbol{q}_{\mathrm{col}} + \boldsymbol{\delta}_\epsilon(t) = \begin{cases} (a\epsilon^{-1/3}t, c\epsilon^m) & (0 \leq t \leq \epsilon^n) \\ \left(a\epsilon^{-1/3}t, c\epsilon^m\frac{\epsilon - t}{\epsilon - \epsilon^n}\right) & (\epsilon^n \leq t \leq \epsilon) \\ \boldsymbol{q}_{\mathrm{col}}(t) & (t > \epsilon) \end{cases}$$

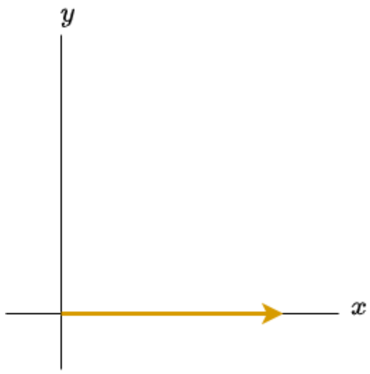

Figure 2: Collision orbit that minimizes the action functional.

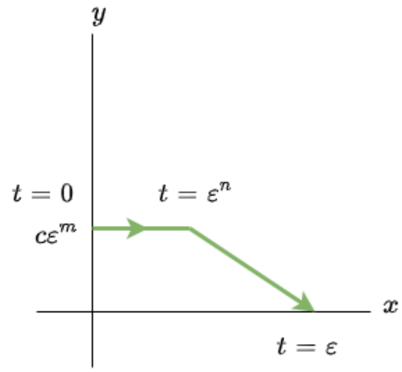

Figure 3: Local transformation.

We can estimate the action functional at $t \in [0, \epsilon]$ after transformation by:

$$\mathcal{A}(\boldsymbol{q}_{\mathrm{col}} + \boldsymbol{\delta}_\epsilon) < \left(\frac{1}{2}a^2 + \frac{\sqrt{2}}{a}\right)\epsilon^{1/3} + \frac{1}{\sqrt{\mu}c}\epsilon^{n-m} + \frac{c^2}{2}\epsilon^{2m-1}$$

Therefore, by choosing n, m so that:

$$m \geq 2/3, \quad n \geq m + 1/3,$$

we obtain $\mathcal{A}(\boldsymbol{q}_{\mathrm{col}}) > \mathcal{A}(\boldsymbol{q}_{\mathrm{col}} + \boldsymbol{\delta}_\epsilon)$. The above remarks demonstrate that the collision orbit is not a minimizer.

4.2 Constructing a periodic solution

We discuss how to construct a periodic solution from the minimizer in the main theorem.

Lemma 4.2. We assume that $\boldsymbol{q} = (x(t), y(t))$ is a minimizer of AKP under the boundary conditon $\Omega(A, B)$. Then $x(t)$ and $y(t)$ are monotone.

Proof. From (3.1), $x(0) > x(T) = 0$ and it is monotonically dicreasing if it is monotone. Assume that $x(t)$ is not monotonically dicreasing. Then there exist $a, b \in [0, T]$ such that $a < b$ and $x(a) < x(b)$. Then, by the mean value theorem, there exist $c \in [a, b]$ and $d \in [b, T]$ such that $\dot{x}(c) > 0$ and $\dot{x}(d) < 0$. From the intermediate value theorem, there exists $e \in (c, d)$ such that $\dot{x}(e) = 0$. Therefore we find that $x(t)$ takes a local maximam value at $t = e$. We set

$$x^*(t) = \begin{cases} \max(x(t), x(e)) & (t \in [0, e]) \\ x(t) & (t \in (e, T]) \end{cases} .$$

Since $x(t) \leq x^*(t)$, we see that $|x'(t)| \geq |x^{*\prime}(t)|$ and $\mathcal{A}(x, y) > \mathcal{A}(x^*, y)$. This is contrary to the fact that $x(t)$ is a minimizer. Similar arguments apply for the case of y. $\square$

The AKP can be represented by:

$$\dot{x} = p_x, \quad \dot{y} = p_y, \tag{4.1}$$

$$\dot{p_x} = -\frac{\partial V}{\partial x}, \quad \dot{p_y} = -\frac{\partial V}{\partial y} \tag{4.2}$$

and (4.2) has reversibility with respect to:

$$R_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We assume that $\boldsymbol{q}(t) = (x(t), y(t))$ is a solution at $t \in I$. Then, from Proposition 2.2, $\boldsymbol{q}_1(t) = (-x(-t), y(-t)), \boldsymbol{q}_2(t) = (x(-t), -y(-t))$, $\boldsymbol{q}_3(t) = (-x(t), -y(t))$ are also solutions, as illustrated in Figure 4.

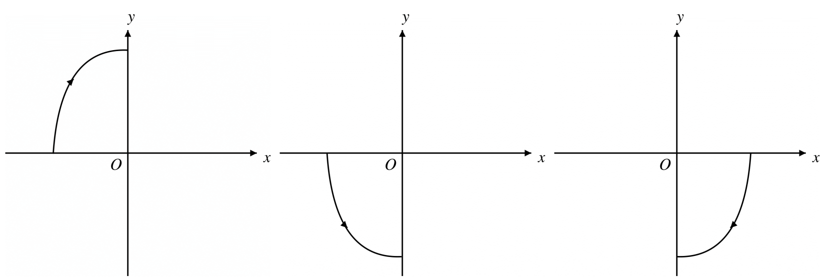

Figure 4: Reversible solutions.

The following lemma shows that these solutions are connected smoothly:

Lemma 4.3. $\dot{x}(0) = 0$ and $\dot{y}(T) = 0$

Proof. From [8], the variation in $t \in [0, T]$ is

$$\delta\mathcal{A} = \int_0^T \left(L_x - \frac{d}{dt}L_{\dot{x}}\right) h_x(t)dt + \int_0^T \left(L_y - \frac{d}{dt}L_{\dot{y}}\right) h_y(t)dt + (L_{\dot{x}}\delta x + L_{\dot{y}}\delta y)\,|_{t=0}^{t=T} \tag{4.3}$$

where $h_x(t)$ and $h_y(t)$ are increments and $\delta x(0), \delta x(T), \delta y(0), \delta y(T)$ are the boundary coordinate increments. Since $\boldsymbol{q}$ is a minimizer, $\delta J = 0$. From the boundary conditions, $\delta x(T) = 0$ and $\delta y(0) = 0$. Therefore, by substituting them in (4.3), we obtain:

$$\dot{x}(0)\delta x(0) = 0, \quad \dot{y}(T)\delta y(T) = 0$$

for any increment. $\square$

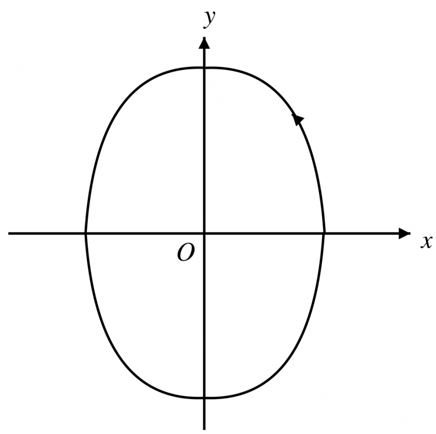

Figure 5: Periodic solutions.

From the above lemma, we obtain a smooth periodic solution like Figure 5.

5 Numerical calculation

Figure 6 shows the results of numerical calculation of the periodic solution of the AKP using AUTO.The initial solution is the solution of the Kepler problem with $\mu = 1$:

$$x(t) = \left(\frac{\pi}{2}\right)^{-\frac{2}{3}} \cos\frac{\pi t}{2}, y(t) = \left(\frac{\pi}{2}\right)^{-\frac{2}{3}} \sin\frac{\pi t}{2}, 0 \leq t \leq 1.$$

We continuate by reducing μ from 1 to 0. The boundary conditions are:

$$x(1) = y(0) = \dot{x}(0) = \dot{y}(0) = 0.$$

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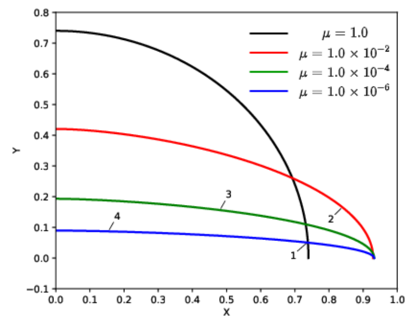

Figure 6: Calculation results.

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