

CRYSTAL STRUCTURE ON LOCALIZED QUANTUM COORDINATE RINGS

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1. INTRODUCTION

The article is a summary of [12]. Let R be a quiver Hecke algebra associated with a simple Lie algebra \mathfrak{g} and $R\text{-gmod}$ the category of finite-dimensional graded R -modules. We set $\mathcal{K}(R\text{-gmod})$ to be the Grothendieck ring of $R\text{-gmod}$. It is well-known that the unipotent quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ is categorified by $\mathcal{K}(R\text{-gmod})$. The basic theory of localization for the monoidal category $\widetilde{R}\text{-gmod}$ of $R\text{-gmod}$ is initiated by [5] and its Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ defines the localized (unipotent) quantum coordinate ring $\widetilde{\mathcal{A}}_q(\mathfrak{n})$. In [11], Lauda-Vazirani defined certain crystal structure on the family of simple modules of $R\text{-gmod}$ and they have shown that this crystal is isomorphic to the crystal $B(\infty)$ of the nilpotent half of $U_q(\mathfrak{g})$. In this article, considering the family of self-dual simple module $\mathbb{B}(\widetilde{R}\text{-gmod})$ of the localized category $\widetilde{R}\text{-gmod}$, we define a crystal structure of $\widetilde{\mathcal{A}}_q(\mathfrak{n})$ and show that it is isomorphic to the cellular crystal \mathbb{B}_i , which is defined to a reduced word for the longest Weyl group element w_0 . This result can be seen as a localized version of the result by Lauda-Vazirani.

2. PRELIMINARIES

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_- = \langle e_i, h_i, f_i \rangle_{i \in I := \{1, 2, \dots, n\}}$ be a simple Lie algebra associated with a Cartan matrix $A = (a_{ij})_{i, j \in I}$ where $\{e_i, f_i, h_i\}_{i \in I}$ are the standard Chevalley generators and $\mathfrak{n} = \langle e_i \rangle_{i \in I}$ (resp. $\mathfrak{t} = \langle h_i \rangle_{i \in I}$, $\mathfrak{n}_- = \langle f_i \rangle_{i \in I}$) is the positive nilpotent subalgebra (resp. the Cartan subalgebra, the negative nilpotent subalgebra).

Let $\{\alpha_i\}_{i \in I}$ be the set of simple roots of \mathfrak{g} and $\langle \ , \ \rangle$ a pairing on $\mathfrak{t} \times \mathfrak{t}^*$ satisfying $a_{ij} = \langle h_i, \alpha_j \rangle_{i, j \in I}$. We also define a symmetric bilinear form $(\ , \)$ on \mathfrak{t}^* such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ and $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $\lambda \in \mathfrak{t}^*$.

Let $P := \{\lambda \in \mathfrak{t}^* \mid \langle h_i, \lambda \rangle \in \mathbb{Z} \text{ for any } i \in I\}$ be the weight lattice and $P_+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for any } i \in I\}$ the set of dominant weights. Set $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ (resp. $Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$), which is called the root lattice (resp. positive root lattice). For an element $\beta = \sum_i m_i \alpha_i \in Q_+$ define $|\beta| = \sum_i m_i$, which is called the height of β . Let $W = \langle s_i \mid s_i \rangle_{i \in I}$ be the Weyl group associated with P , where s_i is the simple reflection defined by $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$ ($\lambda \in P$).

We denote the dual weight lattice of P by $P^* := \{h \in \mathfrak{t} \mid \langle h, P \rangle \subset \mathbb{Z}\}$. Let $U_q(\mathfrak{g}) := \langle e_i, f_i, q^h \rangle_{i \in I, h \in P^*}$ be the quantum algebra associated with \mathfrak{g} with the defining relations (see e.g., [1, 2]) and $U_q^-(\mathfrak{g}) := \langle f_i \rangle_{i \in I}$ (resp. $U_q^+(\mathfrak{g}) := \langle e_i \rangle_{i \in I}$) the negative (resp. positive) nilpotent subalgebras of $U_q(\mathfrak{g})$. We also define the \mathbb{Z} -form $U_{\mathbb{Z}[q, q^{-1}]}$ of $U_q^-(\mathfrak{g})$ as in [5]. Set $q_i := q^{(\alpha_i, \alpha_i)/2}$, $[n]_i = (q_i^n - q_i^{-1}) / (q_i - q_i^{-1})$, $[n]_i! := \prod_{0 \leq k \leq n} [k]_i$ and $X_i^{(n)} := X_i^n / [n]_i!$ for $X_i = f_i, e_i$ for $i \in I$, $n \in \mathbb{Z}_{\geq 0}$.

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Now, let us define the (*unipotent*) *quantum coordinate ring* $\mathcal{A}_q(\mathfrak{n})$ by

$$\mathcal{A}_q(\mathfrak{n}) = \bigoplus_{\beta \in Q_-} \mathcal{A}_q(\mathfrak{n})_\beta \quad \mathcal{A}_q(\mathfrak{n})_\beta := \text{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q))$$

Note that $U_q^-(\mathfrak{g}) \cong \mathcal{A}_q(\mathfrak{n})$ as a $\mathbb{Q}(q)$ -algebra. The \mathbb{Z} -form $\mathcal{A}(\mathfrak{n})_{\mathbb{Z}[q, q^{-1}]}$ is defined as in [5].

3. CRYSTAL BASES AND CRYSTALS

3.1. Crystal Base of $U_q^-(\mathfrak{g}) \cong \mathcal{A}_q(\mathfrak{n})$. Let us define the crystal base $(L(\infty), B(\infty))$ of $U_q^-(\mathfrak{g})([1])$. For $i \in I$ the operator $e'_i \in \text{End}(U_q^-(\mathfrak{g}))$ is defined by the formula

$$e'_i(PQ) = e'_i(P)Q + q_i^{\langle h_i, \beta \rangle} P e'_i(Q), \quad e'_i(f_j) = \delta_{i,j}, \quad e'_i(1) = 0,$$

for any $P \in U_q(\mathfrak{g})_\beta$, $Q \in U_q(\mathfrak{g})$, $i, j \in I$. By the fact that for $P \in U_q(\mathfrak{g})_\beta$, there exists the following unique decomposition

$$(3.1) \quad P = \sum_{k \geq 0} f_i^{(k)} P_n,$$

where $P_n \in \text{Ker}(e'_i) \cap U_q^-(\mathfrak{g})_{\beta+k\alpha_i}$. And define the operators $\tilde{e}_i, \tilde{f}_i \in \text{End}(U_q^-(\mathfrak{g}))$ on $P \in U_q^-(\mathfrak{g})_\beta$ by using the decomposition (3.1)

$$\tilde{e}_i P = \sum_{k > 0} f_i^{(k-1)} P_n, \quad \tilde{f}_i P = \sum_{k \geq 0} f_i^{(k+1)} P_n,$$

which are called *Kashiwara operators*. Now, set

$$L(\infty) := \sum_{k \geq 0, i_1, \dots, i_k \in I} \mathbb{A} \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\infty, \quad B(\infty) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\infty \bmod qL(\infty) \mid k \geq 0, i_1, \dots, i_k \in I \} \setminus \{0\},$$

$$\varepsilon_i(b) = \max\{k : \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle,$$

where $u_\infty = 1 \in U_q(\mathfrak{g})$ and $\mathbb{A} \subset \mathbb{Q}(q)$ is the local subring at $q = 0$.

Theorem 3.1 ([1]). A pair $(L(\infty), B(\infty))$ is a crystal base of $U_q^-(\mathfrak{g})$. Indeed, we obtain

$$\begin{aligned} \tilde{e}_i L(\infty) &\subset L(\infty), & \tilde{f}_i L(\infty) &\subset L(\infty), \\ \tilde{e}_i B(\infty) &\subset B(\infty) \sqcup \{0\}, & \tilde{f}_i B(\infty) &\subset B(\infty) \sqcup \{0\}, \\ \text{wt}(\tilde{e}_i b) &= \text{wt}(b) + \alpha_i \text{ for } b, \tilde{e}_i b \in B(\infty), & \text{wt}(\tilde{f}_i b) &= \text{wt}(b) - \alpha_i \text{ for } b, \tilde{f}_i b \in B(\infty), \\ \varepsilon_i(\tilde{e}_i b) &= \varepsilon_i(b) - 1 & \varphi_i(\tilde{e}_i b) &= \varepsilon_i(b) + 1, \text{ for } b, \tilde{e}_i b \in B(\infty), \\ \varepsilon_i(\tilde{f}_i b) &= \varphi_i(b) + 1 & \varphi_i(\tilde{f}_i b) &= \varphi_i(b) - 1, \text{ for } b, \tilde{f}_i b \in B(\infty), \\ \tilde{f}_i b = b' &\iff \tilde{e}_i b' = b, & \text{for } b, b' &\in B(\infty) \end{aligned}$$

3.2. Crystals. We shall introduce the notion *crystal* following [2], which is a combinatorial object obtained by abstracting the properties of crystal bases in Theorem 3.1.

Definition 3.2 ([2]). A 6-tuple $(B, \text{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\tilde{e}_i\}, \{\tilde{f}_i\})_{i \in I}$ is a *crystal* if B is a set and there exists a certain special element 0 outside of B and maps:

$$(3.2) \quad \text{wt} : B \rightarrow P, \quad \varepsilon_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\} \quad (i \in I),$$

$$(3.3) \quad \tilde{e}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\} \quad (i \in I),$$

satisfying :

- (1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$.
- (2) If $b, \tilde{e}_i b \in B$, then $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.
- (3) If $b, \tilde{f}_i b \in B$, then $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.

- (4) For $b, b' \in B$ and $i \in I$, one has $\tilde{f}_i b = b'$ iff $b = \tilde{e}_i b'$.
 (5) If $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = \tilde{f}_i b = 0$ and $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$.

Here, a *ccrystal graph* of crystal B is a I -colored oriented graph defined by $b \xrightarrow{i} b' \Leftrightarrow \tilde{f}_i(b) = b'$ for $b, b' \in B$.

Definition 3.3 ([2]). For crystals B_1 and B_2 , Ψ is a *strict embedding* (resp. *isomorphism*) from B_1 to B_2 if $\Psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ is an injective (resp. bijective) map satisfying that $\Psi(0) = 0$, $\text{wt}(\Psi(b)) = \text{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for any $b \in B_1$ and Ψ commutes with all \tilde{e}_i 's and \tilde{f}_i 's.

We obtain the tensor structure of crystals as follows([1, 2]):

Proposition 3.4. For crystals B_1 and B_2 , set

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 := (b_1, b_2) \mid b_1 \in B_1, b_2 \in B_2\} (= B_1 \times B_2).$$

Then, $B_1 \otimes B_2$ becomes a crystal by defining:

$$(3.4) \quad \text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),$$

$$(3.5) \quad \varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle),$$

$$(3.6) \quad \varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle),$$

$$(3.7) \quad \tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$(3.8) \quad \tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Example 3.5. For $i \in I$, set $B_i := \{(n)_i \mid n \in \mathbb{Z}\}$ and

$$\text{wt}((n)_i) = n\alpha_i, \quad \varepsilon_i((n)_i) = -n, \quad \varphi_i((n)_i) = n,$$

$$\varepsilon_j((n)_i) = \varphi_j((n)_i) = -\infty \quad (i \neq j),$$

$$\tilde{e}_i((n)_i) = (n+1)_i, \quad \tilde{f}_i((n)_i) = (n-1)_i,$$

$$\tilde{e}_j((n)_i) = \tilde{f}_j((n)_i) = 0 \quad (i \neq j).$$

Then B_i ($i \in I$) possesses a crystal structure. Note that as a set the crystal B_i can be identified with the set of integers \mathbb{Z} .

3.3. Explicit structure of the crystal $B_{i_1} \otimes \cdots \otimes B_{i_m}$. Here we shall describe an explicit structure of tensor product of B_i 's. Fix a sequence of indices $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ and write

$$(x_1, \dots, x_m) := \tilde{f}_{i_1}^{x_1}(0)_{i_1} \otimes \cdots \otimes \tilde{f}_{i_m}^{x_m}(0)_{i_m} = (-x_1)_{i_1} \otimes \cdots \otimes (-x_m)_{i_m},$$

where if $n < 0$, then $\tilde{f}_i^n(0)_i$ means $\tilde{e}_i^{-n}(0)_i$. Note that here we do not necessarily assume that \mathbf{i} is a reduced word though later we will take \mathbf{i} to be a reduced longest word. By the tensor structure of crystals in Proposition 3.4, for the sequence \mathbf{i} as above, we can describe the explicit crystal structure on $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_m}$ as follows: For $x = (x_1, \dots, x_m) \in \mathbb{B}_{\mathbf{i}}$, define

$$\sigma_k(x) := x_k + \sum_{j < k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j$$

and for $i \in I$ define

$$\tilde{\sigma}^{(i)}(x) := \max\{\sigma_k(x) \mid 1 \leq k \leq m \text{ and } i_k = i\},$$

$$\tilde{M}^{(i)} = \tilde{M}^{(i)}(x) := \{k \mid 1 \leq k \leq m, i_k = i, \sigma_k(x) = \tilde{\sigma}^{(i)}(x)\},$$

$$\tilde{m}_f^{(i)} = \tilde{m}_f^{(i)}(x) := \max \tilde{M}^{(i)}(x), \quad \tilde{m}_e^{(i)} = \tilde{m}_e^{(i)}(x) := \min \tilde{M}^{(i)}(x).$$

Now, the actions of the Kashiwara operators \tilde{e}_i, \tilde{f}_i and the functions ε_i, φ_i and wt are written explicitly:

$$(3.9) \quad \tilde{f}_i(x)_k := x_k + \delta_{k, \tilde{m}_f^{(i)}}, \quad \tilde{e}_i(x)_k := x_k - \delta_{k, \tilde{m}_e^{(i)}},$$

$$(3.10) \quad \text{wt}(x) := -\sum_{k=1}^m x_k \alpha_{i_k}, \quad \varepsilon_i(x) := \tilde{\sigma}^{(i)}(x), \quad \varphi_i(x) := \langle h_i, \text{wt}(x) \rangle + \varepsilon_i(x).$$

Define the function $\beta_k^{(i)}$ on \mathbb{B}_i by :

$$(3.11) \quad \beta_k^{(i)}(x) := \sigma_{k^+}(x) - \sigma_k(x) = x_k + \sum_{k < j < k^+} \langle h_i, \alpha_{i_j} \rangle x_j + x_{k^+},$$

for $x = (x_1, \dots, x_m) \in \mathbb{B}_i$, where for $k \in [1, N]$, k^+ (resp. k^-) is the minimum (resp. maximum) number $j \in [1, N]$ such that $k < j$ (resp. $l < k$) and $i_k = i_j$ if it exists, otherwise $N+1$ (resp. 0). Here one knows that $\tilde{m}_f^{(i)}(x)$ and $\tilde{m}_e^{(i)}(x)$ are determined by $\{\beta_k^{(i)}(x) \mid 1 \leq k \leq N, i_k = i\}$.

3.4. Braid-type isomorphism. We shall introduce some isomorphism of crystals, called "braid-type isomorphism".

Set $c_{ij} := \langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle$, $c_1 := -\langle h_i, \alpha_j \rangle$ and $c_2 := -\langle h_j, \alpha_i \rangle$. In the sequel, for $x \in \mathbb{Z}$, put

$$x_+ := \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Proposition 3.6 ([13]). There exist the following isomorphisms of crystals $\phi_{ij}^{(k)}$ ($k = 0, 1, 2, 3$)

(1) If $c_{ij} = 0$,

$$(3.12) \quad \phi_{ij}^{(0)} : B_i \otimes B_i \xrightarrow{\sim} B_j \otimes B_i,$$

where $\phi_{ij}^{(0)}((x)_i \otimes (y)_j) = (y)_j \otimes (x)_i$.

(2) If $c_{ij} = 1$,

$$(3.13) \quad \phi_{ij}^{(1)} : B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_j \otimes B_i \otimes B_j,$$

where

$$\phi_{ij}^{(1)}((x)_i \otimes (y)_j \otimes (z)_i) = (z + (-x + y - z)_+)_j \otimes (x + z)_i \otimes (y - z - (-x + y - z)_+)_j.$$

(3) If $c_{ij} = 2$,

$$(3.14) \quad \phi_{ij}^{(2)} : B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i,$$

where $\phi_{ij}^{(2)}$ is given by the following: for $(x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j$ we set $(X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i := \phi_{ij}^{(2)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j)$.

$$(3.15) \quad X = w + (-c_2 x + y - w + c_2(x - c_1 y + z)_+)_+,$$

$$(3.16) \quad Y = x + c_1 w + (-x + z - c_1 w + (x - c_1 y + z)_+)_+,$$

$$(3.17) \quad Z = y - (-c_2 x + y - w + c_2(x - c_1 y + z)_+)_+,$$

$$(3.18) \quad W = z - c_1 w - (-x + z - c_1 w + (x - c_1 y + z)_+)_+.$$

(4) If $c_{ij} = 3$, the map

$$(3.19) \quad \phi_{ij}^{(3)} : B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i,$$

is defined by the following: for $(x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j$ we set $A := -x + c_1 y - z$, $B := -y + c_2 z - u$, $C := -z + c_1 u - v$ and $D := -u + c_2 v - w$. Then $(X)_j \otimes (Y)_i \otimes (Z)_j \otimes (U)_i \otimes (V)_j \otimes (W)_i := \phi_{ij}^{(3)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (u)_j \otimes (v)_i \otimes (w)_j)$ is given by

$$\begin{aligned} X &= w + (D + (c_2 C + (2B + A_+)_+)_+)_+, \\ Y &= x + c_1 w + (c_1 D + (3C + (2c_1 B + 2A_+)_+)_+)_+, \\ Z &= y + u + w - X - V, \\ U &= x + z + v - Y - W, \\ V &= u - w - (2D + (2c_2 C + (3B + c_2 A_+)_+)_+)_+, \\ W &= v - c_1 w - (c_1 D + (2C + (c_1 B + A_+)_+)_+)_+. \end{aligned}$$

They also satisfy $\phi_{ij}^{(k)} \circ \phi_{ji}^{(k)} = \text{id}$.

We call such isomorphisms of crystals *braid-type isomorphisms*.

We also define a *braid-move* on the set of reduced words of $w \in W$ to be a composition of the following transformations induced from braid relations:

$$\begin{aligned} \cdots i j \cdots &\rightarrow \cdots j i \cdots (c_{ij} = 0), & \cdots i j i \cdots &\rightarrow \cdots j i j \cdots (c_{ij} = 1), \\ \cdots i j i \cdots &\rightarrow \cdots j i j i \cdots (c_{ij} = 2), & \cdots i j i j i \cdots &\rightarrow \cdots j i j i j i \cdots (c_{ij} = 3), \end{aligned}$$

which are called by 2-move, 3-move, 4-move, 6-move respectively.

3.5. Cellular Crystal $\mathbb{B}_{\mathbf{i}} = \mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes \cdots \otimes B_{i_k}$. For a reduced word $\mathbf{i} = i_1 i_2 \cdots i_k$ of some Weyl group element, we call the crystal $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_k}$ a *cellular crystal* associated with a reduced word \mathbf{i} . Indeed, it is obtained by applying the tropicalization functor to the geometric crystal on the Langlands-dual Schubert cell ${}^L X_w$, where $w = s_{i_1} \cdots s_{i_k}$ is an element of the Weyl group W ([14]). It is immediate from the braid-type isomorphisms that for any $w \in W$ and its reduced words $i_1 \cdots i_l$ and $j_1 \cdots j_l$, we get the following isomorphism of crystals:

$$(3.20) \quad B_{i_1} \otimes \cdots \otimes B_{i_l} \cong B_{j_1} \otimes \cdots \otimes B_{j_l}.$$

3.6. Half potential and the crystal $B(\infty)$. For a Laurent polynomial $\phi(x_1, \cdots, x_n)$ with positive coefficients, the tropicalization of ϕ is denoted by $\tilde{\phi} := \text{Trop}(\phi)$, which is given by the rule: $\text{Trop}(ax + by) = \min(x, y)$ with $a, b > 0$, $\text{Trop}(xy) = x + y$ and $\text{Trop}(x/y) = x - y$ and $\text{Trop}(c) = 0$ for $c > 0$. In [10], the crystal $B(\infty)$ has been realized as a certain subset of $\mathbb{B}_{\mathbf{i}}$ defined as follows:

Theorem 3.7 ([10, Theorem 5.11]). Define the subset of $\mathbb{B}_{\mathbf{i}}$:

$$(\tilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)}, \Theta_{\mathbf{i}}} = \{x = (x_1, \cdots, x_N) \in \mathbb{B}_{\mathbf{i}} \mid \tilde{\Phi}^{(+)}(x) \geq 0\},$$

where $\mathbb{B}_{w_0}^-$ is a certain geometric crystal, $\tilde{\Phi}^{(+)}$ is a tropicalization of the half potential $\Phi^{(+)}$ which is a Laurent polynomial with positive coefficients in N variables and $\Theta_{\mathbf{i}}$ is a certain positive structure on the geometric crystal $\mathbb{B}_{w_0}^-$. Then, $(\tilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)}, \Theta_{\mathbf{i}}} \cong B(\infty)$.

Remark 3.8. To define the crystal structure on $(\tilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)}, \Theta_{\mathbf{i}}}$, it is supposed that if $\tilde{e}_i x \notin (\tilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)}, \Theta_{\mathbf{i}}}$, then $\tilde{e}_i x = 0$. Thus, in this sense, the embedding $B(\infty) \cong (\tilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+)}, \Theta_{\mathbf{i}}} \hookrightarrow \mathbb{B}_{\mathbf{i}}$ is not a strict embedding. In [15, 14], it has been given the strict embedding of $B(\infty) \hookrightarrow \mathbb{B}_{\mathbf{i}}$, which is called "Kashiwara embedding" and the method to describe the image of this embedding, called "polyhedral realization".

3.7. **Subspace \mathcal{H}_i .** The object \mathcal{H}_i will play a significant role for this article.

Fix a reduced longest word $\mathbf{i} = i_1 \cdots i_N$ and take the function $\beta_k^{(\mathbf{i})}(x) = x_k + \sum_{k < j < k^+} \langle h_{i_k} \alpha_{i_j} \rangle x_j + x_{k^+}$ ($1 \leq k \leq N$) as in (3.11). In what follows, let us identify the \mathbb{Z} -lattice \mathbb{Z}^N with B_i and then we define the summation of elements $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ by $x + y = (x_1 + y_1, \dots, x_N + y_N)$ as a standard one in \mathbb{Z}^N . Here, we define the subspace $\mathcal{H}_i \subset \mathbb{Z}^N$ by

$$(3.21) \quad \mathcal{H}_i := \{x \in \mathbb{Z}^N (= \mathbb{B}_i) \mid \beta_k^{(\mathbf{i})}(x) = 0 \text{ for any } k \text{ such that } k^+ \leq N\} \subset \mathbb{B}_i.$$

The following result was presented in [10]:

Proposition 3.9 ([10]). For $\mathbf{i} = i_1 i_2 \cdots i_N$, $k = 1, 2, \dots, N$ and a fundamental weight Λ_i , set

$$(3.22) \quad h_i^{(k)} := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda_i \rangle \quad \text{and} \quad \mathbf{h}_i := (h_i^{(1)}, h_i^{(2)}, \dots, h_i^{(N)}) \in \mathbb{B}_i$$

Then, we obtain that $\{\mathbf{h}_1, \dots, \mathbf{h}_N\}$ is a \mathbb{Z} -basis of \mathcal{H}_i , namely,

$$(3.23) \quad \mathcal{H}_i = \mathbb{Z}\mathbf{h}_1 \oplus \mathbb{Z}\mathbf{h}_2 \oplus \cdots \oplus \mathbb{Z}\mathbf{h}_n.$$

Example 3.10. In $\mathfrak{g} = G_2$ -case. Set $a_{12} = -1$ and $a_{21} = -3$. Taking a reduced longest word $\mathbf{i} = 121212$, one has

$$\beta_1^{(\mathbf{i})}(x) = x_1 - x_2 + x_3, \quad \beta_2^{(\mathbf{i})}(x) = x_2 - 3x_3 + x_4, \quad \beta_3^{(\mathbf{i})}(x) = x_3 - x_4 + x_5, \quad \beta_4^{(\mathbf{i})}(x) = x_4 - 3x_5 + x_6.$$

By the formula (3.22), one gets

$$\mathbf{h}_1 = (1, 3, 2, 3, 1, 0), \quad \mathbf{h}_2 = (0, 1, 1, 2, 1, 1).$$

Then the solution space \mathcal{H}_i of $\beta_1^{(\mathbf{i})}(x) = \beta_2^{(\mathbf{i})}(x) = \beta_3^{(\mathbf{i})}(x) = \beta_4^{(\mathbf{i})}(x) = 0$ is given by

$$\mathcal{H}_i = \{c_1 \mathbf{h}_1 + c_2 \mathbf{h}_2 = (c_1, c_2 + 3c_1, c_2 + 2c_1, 2c_2 + 3c_1, c_2 + c_1, c_2) \mid c_1, c_2 \in \mathbb{Z}\}.$$

Lemma 3.11. The braid-type isomorphisms are well-defined on \mathcal{H}_i , that is, $\phi_{ij}^{(k)}(\mathcal{H}_i) = \mathcal{H}_{i'}$, where i' is the reduced word obtained by applying the corresponding braid-moves. We also obtain the following formula:

- (1) For any $h = (\cdots, x, y, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes \cdots \in \mathcal{H}_i$, assume that $a_{ij} = a_{ji} = 0$. Applying the braid-type isomorphism $\phi_{ij}^{(0)}$ on (x, y) in h , we have

$$(3.24) \quad \phi_{ij}^{(0)}(h) = (\cdots, y, x, \cdots) = \cdots \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{i'}$$

- (2) For any $h = (\cdots, x, y, z, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes \cdots \in \mathcal{H}_i$, assume that $a_{ij} = a_{ji} = -1$. Applying the braid-type isomorphism $\phi_{ij}^{(1)}$ on (x, y, z) in h , we have

$$(3.25) \quad \phi_{ij}^{(1)}(h) = (\cdots, z, y, x, \cdots) = \cdots \otimes (-z)_j \otimes (-y)_i \otimes (-x)_j \otimes \cdots \in \mathcal{H}_{i'}$$

- (3) For $h = (\cdots, x, y, z, w, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes (-w)_j \cdots \in \mathcal{H}_i$, assume that $a_{ij} \cdot a_{ji} = 2$. Applying the braid-type isomorphism $\phi_{ij}^{(2)}$ on (x, y, z, w) in h , we have

$$(3.26) \quad \phi_{ij}^{(2)}(h) = (\cdots, w, z, y, x, \cdots) = \cdots \otimes (-w)_j \otimes (-z)_i \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{i'}$$

- (4) For $h = (\cdots, x, y, z, u, v, w, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes (-u)_j \otimes (-v)_i \otimes (-w)_j \cdots \in \mathcal{H}_i$, assume that $a_{ij} \cdot a_{ji} = 3$. Applying the braid-type isomorphism $\phi_{ij}^{(3)}$ on (x, y, z, u, v, w) in h , we have

$$(3.27) \quad \phi_{ij}^{(3)}(h) = (\cdots, w, v, u, z, y, x, \cdots) = \cdots \otimes (-w)_j \otimes (-z)_i \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{i'}$$

In [10, Sect.8], we have shown the following statements under the condition " \mathbf{H}_i ", where we omit the explicit form of \mathbf{H}_i since we do not need it here. But, we succeed in showing the following proposition without the condition \mathbf{H}_i since in [10] we have shown that there exists a specific reduced longest word \mathbf{i}_0 satisfying the condition $\mathbf{H}_{\mathbf{i}_0}$ for each simple Lie algebra \mathfrak{g} and we got Lemma 3.11.

Proposition 3.12. Let $\mathbf{i} = i_1 i_2 \cdots i_N$ be an arbitrary reduced longest word. Here if the crystal $B(\infty)$ is realized in $\mathbb{B}_{\mathbf{i}}$ as in 3.6, we shall denote it by $B(\infty)_{\mathbf{i}}$ to emphasize the word \mathbf{i} . For $h \in \mathcal{H}_{\mathbf{i}}$, define

$$B^h(\infty)_{\mathbf{i}} := \{x + h \in \mathbb{Z}^N (= \mathbb{B}_{\mathbf{i}}) \mid x \in B(\infty)_{\mathbf{i}}\} \subset \mathbb{B}_{\mathbf{i}}.$$

(1) For any $x + h \in B^h(\infty)_{\mathbf{i}}$ and $i \in I$, we obtain

$$(3.28) \quad \tilde{e}_i(x + h) = \tilde{e}_i(x) + h, \quad \tilde{f}_i(x + h) = \tilde{f}_i(x) + h.$$

(2) For any $h \in \mathcal{H}_{\mathbf{i}}$, we have $B(\infty)_{\mathbf{i}} \cap B^h(\infty)_{\mathbf{i}} \neq \emptyset$.

(3)

$$\mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^h(\infty)_{\mathbf{i}}$$

Remark 3.13. In the setting of the half-potential method in [10], as mentioned in Remark 3.8, the crystal $B(\infty)$ is realized as a subset of $\mathbb{B}_{\mathbf{i}}$ and it is supposed that $\tilde{e}_i x = 0$ if $\tilde{e}_i x \notin (\widetilde{\mathbb{B}}_{w_0}^-)_{\Phi^{(+), \Theta_i}} \cong B(\infty)$. At the statement (2), since $x \in B(\infty)_{\mathbf{i}}$ is considered as an element of $\mathbb{B}_{\mathbf{i}}$, $\tilde{e}_i x$ is also considered as an element in $\mathbb{B}_{\mathbf{i}}$. That is, even if $\tilde{e}_i x \notin B(\infty)$, we consider that $\tilde{e}_i x \in \mathbb{B}_{\mathbf{i}}$ and then it never vanishes.

It is immediate from this proposition that one has the following theorem:

Theorem 3.14 ([10]). For any simple Lie algebra \mathfrak{g} and any reduced word $i_1 i_2 \cdots i_k$, the cellular crystal $\mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_k}$ is connected as a crystal graph.

4. QUIVER HECKE ALGEBRA AND ITS MODULES

In this section, we shall introduce the quiver Hecke algebra and its basic properties (see [4, 5, 7, 16]).

4.1. Definition of Quiver Hecke Algebra. For a finite index set I and a field \mathbf{k} , let $(\mathcal{Q}_{i,j}(u, v))_{i,j \in I} \in \mathbf{k}[u, v]$ be polynomials satisfying:

- (1) $\mathcal{Q}_{i,j}(u, v) = \mathcal{Q}_{j,i}(v, u)$ for any $i, j \in I$.
- (2) $\mathcal{Q}_{i,j}(u, v)$ is in the form:

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} \sum_{a(\alpha_i, \alpha_i) + b(\alpha_j, \alpha_j) = -2(\alpha_i, \alpha_j)} t_{i,j,a,b} u^a v^b & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j,-a_{ij},0} \in \mathbf{k}^\times$.

For $\beta = \sum_i m_i \alpha_i \in Q_+$ with $|\beta| := \sum_i m_i = m$, set $I^\beta := \{v = (v_1, \dots, v_m) \in I^m \mid \sum_{k=1}^m \alpha_{v_k} = \beta\}$.

Definition 4.1. For $\beta \in Q_+$, the *quiver Hecke algebra* $R(\beta)$ associated with a Cartan matrix A and polynomials $\mathcal{Q}_{i,j}(u, v)$ is the \mathbf{k} -algebra generated by

$$\{e(v) \mid v \in I^\beta\}, \quad \{x_k \mid 1 \leq k \leq n\}, \quad \{\tau_i \mid 1 \leq i \leq n-1\}$$

with the following relations:

$$\begin{aligned} e(\nu)e(\nu') &= \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1, \quad e(\nu)x_k = x_k e(\nu), \quad x_k x_l = x_l x_k, \\ \tau_l e(\nu) &= e(s_l(\nu))\tau_l, \quad \tau_k \tau_l = \tau_l \tau_k \quad \text{if } |k-l| > 1, \\ \tau_k^2 e(\nu) &= \mathcal{Q}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1})e(\nu), \\ (\tau_k x_l - x_{s_k(l)} \tau_k) e(\nu) &= \begin{cases} -e(\nu) & \text{if } l = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\ (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) &= \begin{cases} \overline{\mathcal{Q}}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\overline{\mathcal{Q}}_{i,j}(u, v, w) = \frac{\mathcal{Q}_{i,j}(u,v) - \mathcal{Q}_{i,j}(w,v)}{u-w} \in \mathbf{k}[u, v, w]$.

- (1) The relations above are homogeneous if we define

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

Thus, $R(\beta)$ becomes a \mathbb{Z} -graded algebra. Here we define the *weight* of $R(\beta)$ -module M as $\text{wt}(M) = -\beta$.

- (2) Let $M = \bigoplus_{k \in \mathbb{Z}} M_k$ be a \mathbb{Z} -graded $R(\beta)$ -module. Define a *grading shift functor* q on the category of graded $R(\beta)$ -modules $R(\beta)\text{-Mod}$ by

$$qM := \bigoplus_{k \in \mathbb{Z}} (qM)_k, \quad \text{where } (qM)_k = M_{k-1}.$$

- (3) For $M, N \in R(\beta)\text{-Mod}$, let $\text{Hom}_{R(\beta)}(M, N)$ be the space of degree preserving morphisms and define $\text{Hom}_{R(\beta)}(M, N) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{R(\beta)}(q^k M, N)$, which is a space of morphisms up to grading shift. We define $\deg(f) = k$ for $f \in \text{Hom}_{R(\beta)}(q^k M, N)$.
- (4) Let ψ be the anti-automorphism of $R(\beta)$ preserving all generators. For $M \in R(\beta)\text{-Mod}$, define $M^* := \text{Hom}_{\mathbf{k}}(M, \mathbf{k})$ with the $R(\beta)$ -module structure by $(r \cdot f)(u) := f(\psi(r)u)$ for $r \in R(\beta)$, $u \in M$ and $f \in M^*$, which is called a *dual module* of M . In particular, if $M \cong M^*$ we call M is *self-dual*.
- (5) For $\beta, \gamma \in Q_+$, set $e(\beta, \gamma) = \sum_{\nu \in I^\beta, \nu' \in I^\gamma} e(\nu, \nu')$. We define an injective homomorphism $\xi_{\beta, \gamma} : R(\beta) \otimes R(\gamma) \rightarrow e(\beta, \gamma)R(\beta + \gamma)e(\beta, \gamma)$ by $\xi(\beta, \gamma)(e(\nu) \otimes e(\nu')) = e(\nu, \nu')$, $\xi(\beta, \gamma)(x_k e(\beta) \otimes 1) = x_k e(\beta, \gamma)$, $\xi(\beta, \gamma)(1 \otimes x_k e(\gamma)) = x_{k+|\beta|} e(\beta, \gamma)$, $\xi(\beta, \gamma)(\tau_k e(\beta) \otimes 1) = \tau_k e(\beta, \gamma)$, $\xi(\beta, \gamma)(1 \otimes \tau_k e(\gamma)) = \tau_{k+|\beta|} e(\beta, \gamma)$.
- (6) For $M \in R(\beta)\text{-Mod}$ and $N \in R(\gamma)\text{-Mod}$, define the *convolution product* \circ by

$$M \circ N := R(\beta + \gamma)e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} (M \otimes N)$$

For simple $M \in R(\beta)\text{-Mod}$ and simple $N \in R(\gamma)\text{-Mod}$, we say M and N *strongly commutes* if $M \circ N$ is simple and M is *real* if $M \circ M$ is simple.

- (7) For $M \in R(\beta)\text{-Mod}$ and $N \in R(\gamma)\text{-Mod}$, denote by $M \nabla N := \text{hd}(M \circ N)$ the head of $M \circ N$ and $M \Delta N := \text{soc}(M \circ N)$ the socle of $M \circ N$, where the head of module M is the quotient by its radical and the socle of module M is the summation of all simple submodules.

4.2. Categorification of quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$. Let $R(\beta)\text{-gmod}$ be the full subcategory of $R(\beta)\text{-Mod}$ whose objects are finite-dimensional graded $R(\beta)$ -modules and set $R\text{-gmod} = \bigoplus_{\beta \in Q_+} R(\beta)\text{-gmod}$. Define the functors

$$E_i : R(\beta)\text{-gmod} \rightarrow R(\beta - \alpha_i)\text{-gmod}, \quad F_i : R(\beta)\text{-gmod} \rightarrow R(\beta + \alpha_i)\text{-gmod},$$

by $E_i(M) := e(\alpha_i, \beta - \alpha_i)M$, $F_i(M) = L(i) \circ M$, where $e(\alpha_i, \beta - \alpha_i) := \sum_{v \in \beta, v_1 = i} e(v)$ and $L(i) := R(\alpha_i)/R(\alpha_i)_{x_1}$ is a 1-dimensional simple $R(\alpha_i)$ -module. Let $\mathcal{K}(R\text{-gmod})$ be the Grothendieck ring of R -gmod and then $\mathcal{K}(R\text{-gmod})$ becomes a $\mathbb{Z}[q, q^{-1}]$ -algebra with the multiplication induced by the convolution product and $\mathbb{Z}[q, q^{-1}]$ -action induced by the grading shift functor q . Here, one obtain the following:

Theorem 4.2 ([4, 16]). As a $\mathbb{Z}[q, q^{-1}]$ -algebra there exists an isomorphism

$$\mathcal{K}(R\text{-gmod}) \cong \mathcal{A}_{q(1)}_{\mathbb{Z}[q, q^{-1}]}.$$

4.3. Categorification of the crystal $B(\infty)$ by Lauda and Vazirani [11]. The following lemma is given in [4]:

Lemma 4.3 ([4]). For any simple $R(\beta)$ -module M , $\text{soc}(E_i M)$, $\text{hd}(E_i M)$ and $\text{hd}(F_i M)$ are all simple modules. Here we also have that $\text{soc}(E_i M) \cong \text{hd}(E_i M)$ up to grading shift.

For $M \in R(\beta)\text{-gmod}$, define

$$(4.1) \quad \text{wt}(M) = -\beta, \quad \varepsilon_i(M) = \max\{n \in \mathbb{Z} \mid E_i^n M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \text{wt}(M) \rangle,$$

$$(4.2) \quad \widetilde{E}_i M := q_i^{1-\varepsilon_i(M)} \text{soc}(E_i M) \cong q_i^{\varepsilon_i(M)-1} \text{hd}(E_i M), \quad \widetilde{F}_i M := q_i^{\varepsilon_i(M)} \text{hd}(F_i M).$$

Set $\mathbb{B}(R\text{-gmod}) := \{S \mid S \text{ is a self-dual simple module in } R\text{-gmod}\}$. Then, it follows from Lemma 4.3 that \widetilde{E}_i and \widetilde{F}_i are well-defined on $\mathbb{B}(R\text{-gmod})$.

Theorem 4.4 ([11]). The 6-tuple $(\mathbb{B}(R\text{-gmod}), \{\widetilde{E}_i\}, \{\widetilde{F}_i\}, \text{wt}, \{\varepsilon_i\}, \{\varphi_i\}_{i \in I})$ holds a crystal structure and there exists the following isomorphism of crystals:

$$\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty).$$

Remark 4.5. Note that Lauda and Vasirani showed this theorem under more general setting that \mathfrak{g} is arbitrary symmetrizable Kac-Moody Lie algebra. Here we assume that \mathfrak{g} is a simple Lie algebra. The definition of \widetilde{E}_i and \widetilde{F}_i in (4.2) differs from the one in [11], which follows the one in [7].

5. LOCALIZATION OF MONOIDAL CATEGORY

Here we shall review the theory of localization for monoidal category following [5].

5.1. Braiders and Real Commuting Family. Let Λ be \mathbb{Z} -lattice and $\mathcal{T} = \bigoplus_{\lambda \in \Lambda} \mathcal{T}_\lambda$ be a \mathbf{k} -linear Λ -graded monoidal category with a data consisting of a bifunctor $\otimes : \mathcal{T}_\lambda \times \mathcal{T}_\mu \rightarrow \mathcal{T}_{\lambda+\mu}$, an isomorphism $a(X, Y, Z) : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ satisfying $a(X, Y, Z \otimes W) \circ a(X \otimes Y, Z, W) = \text{id}_X \otimes a(Y, Z, W) \circ a(X, Y \otimes Z, W) \circ a(X, Y, Z) \otimes \text{id}_W$ and an object $\mathbf{1} \in \mathcal{T}_0$ endowed with an isomorphism $\epsilon : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ such that the functor $X \mapsto X \otimes \mathbf{1}$ and $X \mapsto \mathbf{1} \otimes X$ are fully-faithful.

Definition 5.1 ([5]). Let q be the grading shift functor on \mathcal{T} . A *graded braider* is a triple (C, R_C, ϕ) , where $C \in \mathcal{T}$, \mathbb{Z} -linear map $\phi : \Lambda \rightarrow \mathbb{Z}$ and a morphism:

$$R_C : C \otimes X \rightarrow q^{\phi(\lambda)} X \otimes C \quad (X \in \mathcal{T}_\lambda),$$

satisfying the following commutative diagram:

$$\begin{array}{ccc} C \otimes X \otimes Y & \xrightarrow{R_C(X \otimes Y)} & q^{\phi(\lambda)} X \otimes C \otimes Y & (X \in \mathcal{T}_\lambda, Y \in \mathcal{T}_\mu) \\ & \searrow R_C(X \otimes Y) & \downarrow X \otimes R_C(Y) & \\ & & q^{\phi(\lambda+\mu)}(X \otimes Y) \otimes C & \end{array}$$

and being functorial, that is, for any $X, Y \in \mathcal{T}$ and $f \in \text{Hom}_{\mathcal{T}}(X, Y)$ it satisfies the following commutative diagram:

$$\begin{array}{ccc} C \otimes X & \xrightarrow{\text{id} \otimes f} & C \otimes Y \\ R_C(X) \downarrow & & \downarrow R_C(Y) \\ X \otimes C & \xrightarrow{f \otimes \text{id}} & Y \otimes C \end{array}$$

Definition 5.2 ([5]). Let I be an index set and $(C_i, R_{C_i}, \phi_i)_{i \in I}$ a family of graded braiders in \mathcal{T} . We say that $(C_i, R_{C_i}, \phi_i)_{i \in I}$ is a *real commuting family of graded braiders* in \mathcal{T} if

- (1) $C_i \in \mathcal{T}_{\lambda_i}$ for some $\lambda_i \in \Lambda$, and $\phi_i(\lambda_i) = 0$, $\phi_i(\lambda_j) + \phi_j(\lambda_i) = 0$ for any $i, j \in I$.
- (2) $R_{C_i}(C_i) \in \mathbf{k}^{\times} \text{id}_{C_i \otimes C_i}$ for any $i \in I$.
- (3) $R_{C_i}(C_j) \otimes R_{C_j}(C_i) \in \mathbf{k}^{\times} \text{id}_{C_i \otimes C_j}$ for any $i, j \in I$.

Note that R_{C_i} 's satisfy so-called "Yang-Baxter equation", such as,

$$R_{C_i}(C_j) \circ R_{C_i}(C_k) \circ R_{C_j}(C_k) = R_{C_j}(C_k) \circ R_{C_i}(C_k) \circ R_{C_i}(C_j) \quad \text{on } C_i \circ C_j \circ C_k.$$

For a finite index set I , set $\Gamma := \bigoplus_{i \in I} \mathbb{Z} e_i$ and $\Gamma_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} e_i$.

Lemma 5.3 ([5]). Suppose that we have a real commuting family of graded braiders $(C_i, R_{C_i}, \phi_i)_{i \in I}$. We can choose a bilinear map $H : \Gamma \times \Gamma \rightarrow \mathbb{Z}$ such that $\phi_i(\lambda_j) = H(e_i, e_j) - H(e_j, e_i)$ and there exist

- (1) an object C^α for any $\alpha \in \Gamma_+$.
- (2) an isomorphism $\xi_{\alpha, \beta} : C^\alpha \otimes C^\beta \xrightarrow{\sim} q^{H(\alpha, \beta)} C^{\alpha + \beta}$ for any $\alpha, \beta \in \Gamma_+$

such that $C^0 = 1$ and $C^{e_i} = C_i$.

5.2. Localization. Let \mathcal{T} and $(C_i, R_{C_i}, \phi_i)_{i \in I}$ be as above and $\{C^\alpha\}_{\alpha \in \Gamma_+}$ objects as in the previous lemma. We define a partial order \leq on Γ by

$$\alpha \leq \beta \iff \beta - \alpha \in \Gamma_+$$

For $\alpha_1, \alpha_2, \dots \in \Gamma$, define

$$\mathcal{D}_{\alpha_1, \alpha_2, \dots} := \{\delta \in \Gamma \mid \alpha_j + \delta \in \Gamma_+ \text{ for any } j = 1, 2, \dots\}.$$

For $X \in \mathcal{T}_\lambda, Y \in \mathcal{T}_\mu$ and $\delta \in \mathcal{D}_{\alpha, \beta}$, set

$$H_\delta((X, \alpha), (Y, \beta)) := \text{Hom}_{\mathcal{T}}(C^{\delta + \alpha} \otimes X, q^{P(\alpha, \beta, \delta, \mu)} Y \otimes C^{\beta + \delta}),$$

where a \mathbb{Z} -valued function $P(\alpha, \beta, \delta, \mu) := H(\delta, \beta - \alpha) + \phi(\delta + \beta, \mu)$ and the map $\phi : \Gamma \times \Lambda \rightarrow \mathbb{Z}$ is defined by $\phi(\alpha, L(\beta)) = H(\alpha, \beta) - H(\beta, \alpha)$ and $L : \Gamma \rightarrow \Lambda$ is defined by $L(e_i) = \lambda_i$ ([5]).

Lemma 5.4 ([5]). For $\delta \leq \delta'$ there exists the map

$$\zeta_{\delta, \delta'} : H_\delta((X, \alpha), (Y, \beta)) \rightarrow H_{\delta'}((X, \alpha), (Y, \beta))$$

satisfying

$$\zeta_{\delta, \delta'} \circ \zeta_{\delta', \delta''} = \zeta_{\delta, \delta''} \text{ for } \delta \leq \delta' \leq \delta''.$$

Therefore, we find that $\{H_\delta((X, \alpha), (Y, \beta))\}_{\delta \in \mathcal{D}_{\alpha, \beta}}$ becomes an inductive system.

Definition 5.5 (Localization [5]). We define the category $\widetilde{\mathcal{T}}$ by

$$\begin{aligned} \text{Ob}(\widetilde{\mathcal{T}}) &:= \text{Ob}(\mathcal{T}) \times \Gamma, \\ \text{Hom}_{\widetilde{\mathcal{T}}}((X, \alpha), (Y, \beta)) &:= \varinjlim_{\substack{\delta \in \mathcal{D}_{\alpha, \beta}, \\ \lambda + L(\alpha) = \mu + L(\beta)}} H_\delta((X, \alpha), (Y, \beta)), \end{aligned}$$

where $X \in \mathcal{T}_\lambda, Y \in \mathcal{T}_\mu$ and the function $L : \Gamma \rightarrow \Lambda$ ($e_i \mapsto \lambda_i$) is as above. We call this $\widetilde{\mathcal{T}}$ a *localization* of \mathcal{T} by $(C_i, R_{C_i}, \phi_i)_{i \in I}$ and denote it by $\mathcal{T}[C_i^{\otimes -1} \mid i \in I]$ when we emphasize $\{C_i \mid i \in I\}$.

Theorem 5.6 ([5]). $\widetilde{\mathcal{T}}$ becomes a monoidal category. Moreover, there exists a monoidal functor $\Upsilon : \mathcal{T} \rightarrow \widetilde{\mathcal{T}}$ such that

- (1) $\Upsilon(C_i)$ is invertible in $\widetilde{\mathcal{T}}$ for any $i \in I$, namely, the functors $X \mapsto X \otimes \Upsilon(C_i)$ and $X \mapsto \Upsilon(C_i) \otimes X$ are equivalence of categories.
- (2) For any $i \in I$ and $X \in \mathcal{T}$, $\Upsilon(R_{C_i}(X)) : \Upsilon(C_i \otimes X) \rightarrow \Upsilon(X \otimes C_i)$ is an isomorphism.
- (3) The functor Υ holds the following universality: If there exists another monoidal category \mathcal{T}' and a monoidal functor $\Upsilon' : \mathcal{T} \rightarrow \mathcal{T}'$ satisfying the above statements (1) and (2), then there exists a monoidal functor $F : \widetilde{\mathcal{T}} \rightarrow \mathcal{T}'$ (unique up to iso.) such that $\Upsilon' = F \circ \Upsilon$.

Proposition 5.7 ([5]). Under the setting above, we obtain

- (1) $(X, \alpha + \beta) \cong q^{-H(\beta, \alpha)}(C^\alpha \otimes X, \beta)$, $(1, \beta) \otimes (1, -\beta) \cong q^{-H(\beta, \beta)}(1, 0)$ for $\alpha \in \Gamma_+$, $\beta \in \Gamma$ and $X \in \widetilde{\mathcal{T}}$.
- (2) If \mathcal{T} is an abelian category, then so is $\widetilde{\mathcal{T}}$.
- (3) The functors $\Upsilon : \mathcal{T} \rightarrow \widetilde{\mathcal{T}}$ is exact.
- (4) If the functor $-\otimes Y$ and $Y \otimes -$ are exact for any Y in \mathcal{T} , then the functors $\widetilde{\mathcal{T}} \rightarrow \widetilde{\mathcal{T}}$ ($X \mapsto X \otimes Y$ (resp. $X \rightarrow Y \otimes X$)) are exact for any Y in $\widetilde{\mathcal{T}}$.

6. LOCALIZATION OF THE CATEGORY R -gmod

In this section, we shall apply the method of localization to the category R -gmod.

6.1. Determinantal Modules. Here we just go back to the setting as in Sect.4. Let $L(i^n) := q_i^{\frac{n(n-1)}{2}} L(i)^{\circ n}$ be a simple $R(n\alpha_i)$ -module satisfying $\text{qdim}(L(i^n)) = [n]_i! := \prod_{k=1}^n \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}$ ($q_i := q^{\frac{\langle \alpha_i, \alpha_i \rangle}{2}}$).

Definition 6.1 ([5, 7]). For $M \in R$ -gmod, define

$$\widetilde{F}_i^n(M) := L(i^n) \nabla M.$$

For a Weyl group element w , let $s_{i_1} \cdots s_{i_l}$ be its reduced expression. For a dominant weight $\Lambda \in P_+$, set

$$m_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_l} \Lambda \rangle \quad (k = 1, \dots, l).$$

We define the *determinantal module* associated with w and Λ by

$$\mathbf{M}(w\Lambda, \Lambda) := \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_l}^{m_l} \mathbf{1},$$

where $\mathbf{1}$ is a trivial $R(0)$ -module.

Note that in general, one can define determinantal modules $\mathbf{M}(w\Lambda, u\Lambda)$ ($w, u \in W$) which corresponds to the generalized minor $\Delta_{w\Lambda, u\Lambda}$.

Now, let us see some similarity between the family of determinantal modules $\{\mathbf{M}(w_0\Lambda, \Lambda)\}_{\Lambda \in P_+}$ and the subspace $\mathcal{H}_{\mathbf{i}}$. As has seen above that for a reduced longest word $\mathbf{i} = i_1 \cdots i_N$, the subspace $\mathcal{H}_{\mathbf{i}} \subset \mathbb{B}_{\mathbf{i}}$ is presented by

$$\mathcal{H}_{\mathbf{i}} = \bigoplus_{i \in I} \mathbb{Z} \mathbf{h}_i, \quad \mathbf{h}_i = ((h_i^{(k)}) := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda_i \rangle)_{k=1, \dots, N}.$$

Furthermore, we also get

Proposition 6.2. For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$ and $\Lambda \in P_+$, set

$$m_k := \langle h_{i_k}, s_{i_{k+1}} s_{i_{k+2}} \cdots s_{i_N} \Lambda \rangle \quad (k = 1, 2, \dots, N) \quad \text{and} \quad \mathbf{h}_\Lambda := (m_1, \dots, m_N).$$

Then we obtain

$$\mathbf{h}_\Lambda = \widetilde{f}_{i_1}^{m_1} \widetilde{f}_{i_2}^{m_2} \cdots \widetilde{f}_{i_N}^{m_N} ((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) = \widetilde{f}_{i_1}^{m_1}(0)_{i_1} \otimes \widetilde{f}_{i_2}^{m_2}(0)_{i_2} \otimes \cdots \otimes \widetilde{f}_{i_N}^{m_N}(0)_{i_N} \in \mathcal{H}_{\mathbf{i}},$$

where note that for $\Lambda = \sum_i a_i \Lambda_i$, one has $\mathbf{h}_\Lambda = \sum_i a_i \mathbf{h}_{\Lambda_i}$.

By this proposition, one observes that there would exist a certain correspondence

$$(6.1) \quad \mathbf{M}(w_0\Lambda, \Lambda) = \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_l}^{m_l} \mathbf{1} \longleftrightarrow \mathbf{h}_\Lambda = \widetilde{f}_{i_1}^{m_1} \cdots \widetilde{f}_{i_N}^{m_N} ((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}).$$

Definition 6.3 ([5]). For $\beta \in Q_+$, define a central element in $R(\beta)$ by

$\mathfrak{p}_i := \sum_{v \in I^\beta} \left(\prod_{a \in \{1, 2, \dots, \text{ht}(\beta)\}, v_a = i} x_a \right) e(v) \in R(\beta)$. For a simple $M \in R(\beta)\text{-gmod}$, define an *affinization* \widehat{M} of M with degree d :

- (1) There is an endomorphism $z : \widehat{M} \rightarrow \widehat{M}$ of degree $d > 0$ such that \widehat{M} is finitely generated free module of $\mathbf{k}[z]$ and $\widehat{M}/z\widehat{M} \cong M$.
- (2) $\mathfrak{p}_i \widehat{M} \neq 0$ for any $i \in I$.

Theorem 6.4 ([5, Theorem 3.26]). For any $\Lambda \in P_+$ and $w \in W$, the determinantal module $\mathbf{M}(w\Lambda, \Lambda)$ is a real simple module and admits an affinization $\widehat{\mathbf{M}}(w\Lambda, \Lambda)$.

Note that indeed, if \mathfrak{g} is simply-laced, then the affinization \widehat{M} always exists for any simple $M \in R(\beta)\text{-gmod}$ as ([3]),

$$\widehat{M} = \mathbf{k}[z] \otimes_{\mathbf{k}} M.$$

6.2. Localization.

Definition 6.5 ([5]). Let M be a simple R -module. A graded braider (M, R_M, ϕ) is *non-degenerate* if $R_M(L(i)) : M \circ L(i) \rightarrow L(i) \circ M$ is a non-zero homomorphism.

For $R\text{-gmod}$, there exists a non-degenerate real commuting family of graded braiders $(C_i, R_{C_i}, \phi_i)_{i \in I}$ ([5]). Set $C_\Lambda := \mathbf{M}(w_0\Lambda, \Lambda)$ and denote C_{Λ_i} by C_i .

Proposition 6.6 ([8]). For $\Lambda = \sum_i m_i \Lambda_i \in P_+$, we obtain the following isomorphism up to grading shift:

$$(6.2) \quad C_\Lambda := \mathbf{M}(w_0\Lambda, \Lambda) \cong C_1^{o m_1} \circ \cdots \circ C_n^{o m_n}.$$

Theorem 6.7 ([5, Proposition 5.1]). Define the function $\phi_i : Q \rightarrow \mathbb{Z}$ by

$$\phi_i(\beta) := -(\beta, w_0\Lambda_i + \Lambda_i).$$

Then there exists $\{(C_i, R_{C_i}, \phi_i)\}_{i \in I}$ a non-degenerate real commuting family of graded braiders of the monoidal category $R\text{-gmod}$.

Now, we take $\Gamma = P = \bigoplus_i \mathbb{Z}\Lambda_i$ and $\Gamma_+ = P_+ = \bigoplus_i \mathbb{Z}_{\geq 0}\Lambda_i$. Here, we obtain the localization $R\text{-gmod}[C_i^{o-1} \mid i \in I]$ by $\{(C_i, R_{C_i}, \phi_i)\}_{i \in I}$, which will be denoted by $\widetilde{R}\text{-gmod}$.

By the above Proposition, it holds the following properties:

Proposition 6.8 ([5]). Let $\Phi : R\text{-gmod} \rightarrow \widetilde{R}\text{-gmod}$ be the canonical functor. Then,

- (1) $\widetilde{R}\text{-gmod}$ is an abelian category and the functor Φ is exact.
- (2) For any simple object $S \in R\text{-gmod}$, $\Phi(S)$ is simple in $\widetilde{R}\text{-gmod}$.
- (3) $\widetilde{C}_i := \Phi(C_i)$ ($i \in I$) is invertible central graded braider in $\widetilde{R}\text{-gmod}$.

For $\mu \in P$, define \widetilde{C}_μ such that $\widetilde{C}_\mu := \Phi(C_\mu)$ for $\mu \in P_+$, $\widetilde{C}_{-\Lambda_i} = C_i^{o-1}$ and $\widetilde{C}_{\lambda+\mu} = \widetilde{C}_\lambda \circ \widetilde{C}_\mu$ for $\lambda, \mu \in P$ up to grading shift.

- (4) Any simple object in $\widetilde{R}\text{-gmod}$ is isomorphic to $\widetilde{C}_\Lambda \circ \Phi(S)$ for some simple module $S \in R\text{-gmod}$ and $\Lambda \in P$.

Note that in (4) $\Lambda \in P$ and $S \in R\text{-gmod}$ are not necessarily unique.

Remark 6.9. In [5], the localization is applied to more general category \mathcal{C}_w , which is the full subcategory of $R\text{-gmod}$ associated with a Weyl group element w . The category $R\text{-gmod}$ here coincides with \mathcal{C}_{w_0} associated with the longest element w_0 in W .

Definition 6.10. The category $\widetilde{R}\text{-gmod}$ is abelian and monoidal. Therefore, its Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ holds a natural $\mathbb{Z}[q, q^{-1}]$ -algebra structure, which defines a *localized quantum coordinate ring* $\widetilde{\mathcal{A}}_q(\mathfrak{n}) := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathcal{K}(\widetilde{R}\text{-gmod})$.

Indeed, the Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ is described as follows:

Proposition 6.11 ([5, Corollary 5.4]). The Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ is isomorphic to the left ring of quotients of the ring $\mathcal{K}(R\text{-gmod})$ with respect to the multiplicative set

$$\mathcal{S} := \{q^k \prod_{i \in I} [C_i]^{a_i} \mid k \in \mathbb{Z}, (a_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I\},$$

that is, $\mathcal{K}(\widetilde{R}\text{-gmod}) \cong \mathcal{S}^{-1} \mathcal{K}(R\text{-gmod})$.

7. CRYSTAL STRUCTURE ON LOCALIZED QUANTUM COORDINATE RINGS

We shall mention the main theorem, crystal structure on localized quantum coordinate ring $\widetilde{\mathcal{A}}_q(\mathfrak{n})$. More precisely, we shall define a crystal structure on a family of self-dual simple objects in the category $\widetilde{R}\text{-gmod}$ (Theorem 7.4) and mention that it is isomorphic to the cellular crystal $\mathbb{B}_{\mathbf{i}}$ (Theorem 7.5), where \mathbf{i} is a reduced word for the longest Weyl group element w_0 .

Lemma 7.1 ([4, Proposition 2.18]). For any $i \in I, \beta, \gamma \in \mathcal{Q}_+$, any modules $M \in R(\beta)\text{-gmod}$ and $N \in R(\gamma)\text{-gmod}$, one has the following exact sequence in $R(\beta + \gamma - \alpha_i)\text{-gmod}$:

$$(7.1) \quad 0 \longrightarrow E_i M \circ N \longrightarrow E_i(M \circ N) \longrightarrow q^{-(\alpha, \beta)} M \circ E_i N \longrightarrow 0.$$

For $i \in I$, let $i^* \in I$ be a unique index satisfying $\Lambda_{i^*} = -w_0 \Lambda_i$.

Lemma 7.2. (1) For $S \in R\text{-gmod}$ and $i \in I$, if $E_i S = 0$, then the module $E_i C_{\Lambda_{i^*}} \circ S$ is a simple module.

(2) If $E_i S = 0$ for $S \in R\text{-gmod}$, then we get for $\Lambda \in P_+$ with $\langle h_{i^*}, \Lambda \rangle > 0$,

$$(7.2) \quad \text{soc}(E_i(C_\Lambda \circ S)) \cong C_{\Lambda - \Lambda_{i^*}} \circ (E_i C_{i^*} \circ S),$$

up to grading shift.

We set

$$\mathbb{B}(\widetilde{R}\text{-gmod}) := \{L \mid L \text{ is a self-dual simple module in } \widetilde{R}\text{-gmod}\}.$$

Lemma 7.3 ([5]). For any simple $L \in \widetilde{R}\text{-gmod}$, there exists a unique $n \in \mathbb{Z}$ such that $q^n L$ is self-dual simple. For a simple module $L \in \widetilde{R}\text{-gmod}$ we define $\delta(L)$ to be this integer n .

Then by this lemma, we find that $\mathbb{B}(\widetilde{R}\text{-gmod})$ includes all simple modules in $\widetilde{R}\text{-gmod}$ up to grading shift. For a simple object $\widetilde{C}_\Lambda \circ \Phi(S) \in \widetilde{R}\text{-gmod}$ we write simply $C_\Lambda \circ S$ if there is no confusion.

Now let us define the Kashiwara operators \widetilde{F}_i and \widetilde{E}_i ($i \in I$) on $\mathbb{B}(\widetilde{R}\text{-gmod})$ by

$$(7.3) \quad \widetilde{F}_i(C_\Lambda \circ S) = q^{\delta(C_\Lambda \circ \widetilde{F}_i S)} C_\Lambda \circ \widetilde{F}_i S,$$

$$(7.4) \quad \widetilde{E}_i(C_\Lambda \circ S) = \begin{cases} q^{\delta(C_\Lambda \circ \widetilde{E}_i S)} C_\Lambda \circ \widetilde{E}_i S & \text{if } E_i S \neq 0, \\ q^{\delta(C_{\Lambda - \Lambda_{i^*}} \circ (\widetilde{E}_i C_{\Lambda_{i^*}} \circ S))} C_{\Lambda - \Lambda_{i^*}} \circ (\widetilde{E}_i C_{\Lambda_{i^*}} \circ S) & \text{if } E_i S = 0, \end{cases}$$

where $C_\Lambda \circ S$ is a self-dual simple module in $\widetilde{R}\text{-gmod}$, the actions $\widetilde{E}_i S$ and $\widetilde{F}_i S$ are given in (4.2), which is defined on the family of all self-dual simple modules in $R\text{-gmod}$ and in (7.4) the module $\widetilde{E}_i C_{\Lambda_{i^*}} \circ S$ is simple by Lemma 7.2. Note that for any $m > 0$, $\widetilde{E}_i^m(C_\Lambda \circ S) \neq 0$, $\widetilde{F}_i^m(C_\Lambda \circ S) \neq 0$.

Let $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ be as in Theorem 4.4. For $C_\Lambda \circ S \in \mathbb{B}(\widetilde{R}\text{-gmod})$, we also define

$$(7.5) \quad \begin{aligned} \varepsilon_i(C_\Lambda \circ S) &= \varepsilon_i(\Psi(S)) - \langle h_i, w_0 \Lambda \rangle, & \text{wt}(C_\Lambda \circ S) &= \text{wt}(\Psi(S)) + w_0 \Lambda - \Lambda, \\ \varphi_i(C_\Lambda \circ S) &= \varepsilon_i(\Psi(C_\Lambda \circ S)) + \langle h_i, \text{wt}(C_\Lambda \circ S) \rangle. \end{aligned}$$

Theorem 7.4. The 6-tuple $(\mathbb{B}(\widetilde{R}\text{-gmod}), \text{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\widetilde{E}_i\}, \{\widetilde{F}_i\}_{i \in I})$ is a crystal.

Here, by Proposition 6.2 we observe that there seems to exist a certain correspondence:

$$\begin{aligned} \{C_\Lambda \mid \Lambda \in P_+\} \subset R\text{-gmod} &\longleftrightarrow \mathcal{H}_i \\ C_\Lambda = \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_N}^{m_N} \mathbf{1} &\longleftrightarrow \mathbf{h}_\Lambda = \widetilde{f}_{i_1}^{m_1} \widetilde{f}_{i_2}^{m_2} \cdots \widetilde{f}_{i_N}^{m_N} ((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) \end{aligned}$$

Together with the result of Proposition 3.12, we obtain the following:

Theorem 7.5. For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$, there exists an isomorphism of crystals:

$$\begin{aligned} \widetilde{\Psi} : \mathbb{B}(\widetilde{R}\text{-gmod}) &\xrightarrow{\sim} \mathbb{B}_i = \bigcup_{h \in \mathcal{H}_i} B^h(\infty) \\ C_\Lambda \circ S &\longmapsto \mathbf{h}_\Lambda + \Psi(S) \in B^{\mathbf{h}_\Lambda}(\infty), \end{aligned}$$

where $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ is the isomorphism of crystals given in Theorem 4.4, S is simple in $\mathbb{B}(R\text{-gmod})$ and for $\Lambda = \sum_i a_i \Lambda_i$ set $\mathbf{h}_\Lambda = \sum_i a_i \mathbf{h}_i$.

8. APPLICATION AND FURTHER PROBLEMS

8.1. Operator $\tilde{\alpha}$. Define the $\mathbb{Q}(q)$ -linear anti-automorphism \star of $U_q(\mathfrak{g})$ by

$$(q^h)^\star = q^{-h}, \quad e_i^\star = e_i, \quad f_i^\star = f_i.$$

Theorem 8.1 ([2]). Set $L^\star(\infty) := \{u^\star \mid u \in L(\infty)\}$, $B^\star(\infty) := \{b^\star \mid b \in B(\infty)\}$. Then we have

$$L^\star(\infty) = L(\infty), \quad B^\star(\infty) = B(\infty).$$

From the proof of Theorem 5.13 in [5] we get

Proposition 8.2 ([5]). For $\nu = (\nu_1, \nu_2, \dots, \nu_{m-1}, \nu_m) \in I^\beta$ ($m := |\beta|$) set $\bar{\nu} = (\nu_m, \nu_{m-1}, \dots, \nu_2, \nu_1)$. Define the automorphism α on $R(\beta)$ by

$$\alpha(e(\nu)) = e(\bar{\nu}), \quad \alpha(x_i e(\nu)) = x_{m-i+1} e(\bar{\nu}), \quad \alpha(\tau_j e(\nu)) = -\tau_{m-j} e(\bar{\nu}).$$

Then, there exists the functor $\alpha : R\text{-gmod} \rightarrow R\text{-gmod}$ such that $\alpha(C_i) = C_{i^*}$ ($\forall i \in I$), $\alpha^2 \cong \text{id}$ and $\alpha(X \circ Y) \cong \alpha(Y) \circ \alpha(X)$ for $X, Y \in R\text{-gmod}$. Furthermore, it is extended to the functor $\tilde{\alpha} : \widetilde{R}\text{-gmod} \rightarrow \widetilde{R}\text{-gmod}$ which satisfies

$$(8.1) \quad \tilde{\alpha}^2 \cong \text{id}, \quad \text{and} \quad \tilde{\alpha}(X \circ Y) \cong \tilde{\alpha}(Y) \circ \tilde{\alpha}(X) \quad \text{for } X, Y \in \widetilde{R}\text{-gmod}.$$

Note that α (resp. $\tilde{\alpha}$) induces the operation \star on $\mathcal{A}_q(\mathfrak{n})$ (resp. $\widetilde{\mathcal{A}}_q(\mathfrak{n})$) since $\alpha(L(i)) = L(i)$ and then one has $\alpha(f_i) = f_i$ (resp. $\tilde{\alpha}(f_i) = f_i$) on $\mathcal{A}_q(\mathfrak{n})$ (resp. $\widetilde{\mathcal{A}}_q(\mathfrak{n})$). Now, we obtain the following:

Proposition 8.3. Let $\tilde{\alpha} : \widetilde{R}\text{-gmod} \rightarrow \widetilde{R}\text{-gmod}$ be the functor as above. It yields

$$(8.2) \quad \tilde{\alpha}(\mathbb{B}(\widetilde{R}\text{-gmod})) = \mathbb{B}(\widetilde{R}\text{-gmod}).$$

Here note that Proposition 8.3 can be seen as a generalization of Theorem 8.1.

Since as crystals $\mathbb{B}(\widetilde{R}\text{-gmod}) \cong \mathbb{B}_i$ for any reduced longest word \mathbf{i} , the proposition above gives rise to the following problem.

Problem 1. Can we describe $\tilde{\alpha}$ -operation on $\mathbb{B}_i = B_{i_1} \otimes \cdots \otimes B_{i_N}$ explicitly?

Of course, this problem is non-trivial since even for the case $B(\infty)$ the explicit description has not yet been done before in \mathbb{B}_i .

8.2. **Category $\widetilde{\mathcal{C}}_w$.** In [5], it has been shown that for an arbitrary symmetrizable Kac-Moody Lie algebra and any Weyl group element $w \in W$, there exists a subcategory $\mathcal{C}_w \subset R\text{-gmod}$ and it admits a localization

$$\widetilde{\mathcal{C}}_w = \mathcal{C}_w[C_i^{\circ-1} \mid i \in I], \quad (C_i = M(w\Lambda_i, \Lambda_i))$$

Indeed, note that for finite type Lie algebra setting, $\mathcal{C}_{w_0} = R\text{-gmod}$.

Problem 2. We conjecture that the localization $\widetilde{\mathcal{C}}_w$ possess a crystal $\mathbb{B}(\widetilde{\mathcal{C}}_w)$. If so, we also conjecture that there is an isomorphism of crystals

$$\mathbb{B}(\widetilde{\mathcal{C}}_w) \xrightarrow{\sim} B_{i_1} \otimes \cdots \otimes B_{i_m},$$

where $i_1 \cdots i_m$ is a reduced word of w .

8.3. Rigidity.

Definition 8.4. Let X, Y be objects in a monoidal category \mathcal{T} , and $\varepsilon : X \otimes Y \rightarrow 1$ and $\eta : 1 \rightarrow Y \otimes X$ morphisms in \mathcal{T} . We say that a pair (X, Y) is *dual pair* or X is a *left dual* to Y or Y is a *right dual* to X if the following compositions are identities:

$$X \simeq X \otimes 1 \xrightarrow{\text{id} \otimes \eta} X \otimes Y \otimes X \xrightarrow{\varepsilon \otimes \text{id}} 1 \otimes X \simeq X, \quad Y \simeq 1 \otimes Y \xrightarrow{\eta \otimes \text{id}} Y \otimes X \otimes Y \xrightarrow{\text{id} \otimes \varepsilon} Y \otimes 1 \simeq Y$$

We denote a right dual to X by $\mathcal{D}(X)$ and a left dual to X by $\mathcal{D}^{-1}(X)$.

Theorem 8.5 ([5]). For any finite type R , $\widetilde{R}\text{-gmod}$ is rigid, i.e., every object in $\widetilde{R}\text{-gmod}$ has left and right duals.

Note that in [6], it is shown that for any symmetrizable Kac-Moody setting the localized category $\widetilde{\mathcal{C}}_w$ is rigid.

Problem 3. For a simple object $C_\Lambda \circ S \in \mathbb{B}(\widetilde{R}\text{-gmod})$, describe the right and left duals explicitly:

$$\widetilde{\Psi}(\mathcal{D}(C_\Lambda \circ S)), \quad \widetilde{\Psi}(\mathcal{D}^{-1}(C_\Lambda \circ S)) \in \mathbb{B}_i.$$

REFERENCES

- [1] Kashiwara M. On crystal bases of the q -analogue of universal enveloping algebras, *Duke Math. J.*, 63 (2), (1991), 465–516.
- [2] Kashiwara M. Crystal base and Littelmann’s refined Demazure character formula. *Duke Math. J.* **1993**, 71 (3), 839–858.
- [3] Kashiwara M. and Park E., Affinizations and R-matrices for quiver Hecke algebras, *J. Eur. Math.Soc.* 20, (2018), 1161–1193.
- [4] Khovanov M. and Lauda A., A diagrammatic approach to categorification of quantum groups I, *Represent. Theory* 13 (2009), 309–347.
- [5] Kashiwara M., Kim M., Oh S-J. and Park E., Localization for quiver Hecke algebras, *Pure Appl.Math. Q.* 17(2021), no.4, 1465–1548.
- [6] Kashiwara M., Kim M., Oh S-J. and Park E., Localization for quiver Hecke algebras II, arXiv:2208.01255.
- [7] Kang S-J., Kashiwara M., Kim M. and Oh S-J., Monoidal categorification of cluster algebras, *J.Amer.Math.Soc.*, **31**, No.2, (2017), 349–426.
- [8] Kashiwara M, Kim M., Oh S.-j., and Park E., Monoidal categories associated with strata of flag manifolds, *Adv. Math.* 328 (2018), 959–1009.
- [9] Kang S-J, Kashiwara M., Kim M., and Oh S-J, Simplicity of heads and socles of tensor products, *Compos. Math.* 151 (2015), no. 2, 377–396.
- [10] Kanakubo Y. and Nakashima T., Half potential on geometric crystals and connectedness of cellular crystals, arXiv:1910.06182.
- [11] Lauda A.D. and Vazirani M., Crystals from categorified quantum groups, *Adv.Math.*, **228**, (2011), 803–861.
- [12] Nakashima T., Categorified crystal structure on localized quantum coordinate rings, arXiv:2208.08396.
- [13] Nakashima T., Polyhedral Realizations of Crystal Bases and Braid-type Isomorphisms, *Contemporary Mathematics* **248**, (1999), pp419–435.
- [14] Nakashima T., Geometric Crystals on Schubert Varieties, *J.Geometry and Physics*, **53** (2), 197–225, (2005).
- [15] Nakashima T. and Zelevinsky A., Polyhedral Realizations of Crystal Bases for Quantized Kac-Moody Algebras, *Adv. Math.* **131**, No.1, (1997), 253–278.
- [16] Rouquier R., 2-Kac-Moody algebras, arXiv:0812.5023v1.