DESIGNS ON TAUTOLOGICAL BUNDLE

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1. Introduction

In the area of combinatorics, many researchers have studied point arrangements called “designs”. Roughly speaking, “designs” are “good” point arrangements with approximate given space. Here, we introduce a difinition of spherical t-design.

Definition 1.1. (Delsart, Gothals, Seidel(1977) [4]) Let

$$S^{n-1} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}. \tag{1.1}$$

A finite subset X of S^{n-1} is called a spherical t-design if

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} f d\mu = \frac{1}{|X|} \sum_{u \in X} f(u) \tag{1.2}$$

for any polynomial $f(x_1, \cdots x_n)$ of degree t or less.

For spherical designs, refer to [2]. Also designs on following spaces have been studied:

- Unitary groups [5]
- Grassmannian spaces [1]
- Compact symmetric spaces [3]

In this paper, we see a new definition of designs and its examples.

2. Definition of τ-design

Definition 2.1. Let

- Ω :a set,
- $W, \mathcal{H}$: $\mathbb{C}$-vector spaces,
- $\mathcal{H}_0$:a subset of $\mathcal{H}$,
- $\{V_p\}_{p\in\Omega}$:a family of vector spaces,
- $\{e_p : \mathcal{H} \to V_p\}_{p\in\Omega}$:a family of linear maps,
- $\tau : \mathcal{H}_0 \to W$:a linear map.

For a finite subset $X \subset \Omega$ and linear functions $\lambda_x : V_x \to W$, a pair $(X, \{\lambda_x\}_{x\in X})$ is a τ-design if for any $s \in \mathcal{H}_0$ the following equation holds:

$$\tau(s) = \sum_{x\in X} (\lambda_x \circ e_x)(s) \tag{2.1}$$

Example 2.2. Now, we rewrite the spherical design with the definition of τ-design. Let

- $\Omega = S^{n-1}$,
- $V = \mathbb{R}, \mathcal{H} = C^\infty(\mathbb{R}^n), \mathcal{H}_0 = \mathrm{Pol}_{\le t}(\mathbb{R}^n)|_{S^{n-1}}$,
- $V_p = \mathbb{R}\ (p \in \Omega), e_p : \mathcal{H} \to V_p, f \mapsto f(p)$,

- $$\tau : \mathcal{H}_0 \to W, f \mapsto \frac{1}{|S^{n-1}|}\int_{S^{n-1}} f d\mu \tag{2.2}$$

  where μ is a radon measure of S^{n-1},
- For $x \in X \subset \Omega$

  $$\lambda_x : V_x \to W, z \mapsto \frac{1}{|X|}z. \tag{2.3}$$

Then, it is easy to see that

$$X \text{ is a spherical } t\text{-design} \Leftrightarrow (X, \lambda) \text{ is a } \tau\text{-design}.$$

3. Designs on tautological bundle

3.1. Tautological bundle

We define an action $\mathrm{SU}(2) \curvearrowright \mathbb{C}P^1$ as follows:

$$g \cdot V = \{gv \mid v \in V\} \tag{3.1}$$

and let $v_0 = \{(x, 0) \mid x \in \mathbb{C}\}$.

Proposition 3.1.

$$\varpi : \mathrm{SU}(2) \to \mathbb{C}P^1, g \mapsto g \cdot v_0 \tag{3.2}$$

is a principal bundle. Moreover, an isotropy subgroup

$$\mathrm{Iso}_{v_0}(\mathrm{SU}(2)) := \{g \in \mathrm{SU}(2) \mid g \cdot v_0 = v_0\} = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) \tag{3.3}$$

leads the following isomorphism:

$$\mathbb{C}P^1 \cong \mathrm{SU}(2)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)), \tag{3.4}$$

that is (3.2) is a principal $\mathrm{S}(\mathrm{U}(1) \times U(1))$-bundle.

Now, we define the following action $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) \curvearrowright \mathbb{C}$:

$$g \cdot x = g_1 x \quad \left(g = \left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2 \end{array}\right) \in \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)), x \in \mathbb{C}\right). \tag{3.5}$$

Then,

$$\pi : T_{2,1} \to \mathbb{C}P^1 \tag{3.6}$$

denotes an associated bundle to (3.2) with fiber $\mathbb{C}$.

Lemma 3.2.

$$T_{2,1} = \{(V, v) \in \mathbb{C}P^1 \times \mathbb{C}^2 \mid v \in V\}. \tag{3.7}$$

The right-hand side of (3.7) is called a tautological bundle.

3.2. Section of tautological bundle

In this section, we see a decomposition of $\Gamma(T_{2,1})$, which is a set of smooth section of $T_{2,1}$.

Proposition 3.3. $\Gamma(T_{2,1})$ has the following irreducible decomposition of SU(2) representation:

$$\Gamma(T_{2,1}) = \bigoplus_{k\geq 1} \Gamma^{2k}(T_{2,1}), \tag{3.8}$$

where $\dim_{\mathbb{C}} \Gamma^{2k}(T_{2,1}) = 2k$.

From the general theory of representations in SU(2), we can see $\Gamma^{2k}(T_{2,1})$ as a symmetric tensor.

Lemma 3.4. Let $S^n(\mathbb{C}^2)$ be the set of all symmetric tensors of order n defined on $\mathbb{C}^2$ and we define

$$s : S^{2k-1}(\mathbb{C}^2) \to \Gamma^{2k}(T_{2,1}) \tag{3.9}$$

as

$$s_\alpha(l) = \left(l, \alpha^l_k \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \quad (\alpha \in S^{2k-1}(\mathbb{C}^2), l \in \mathbb{C}P^1) \tag{3.10}$$

where α^l_k is a coefficient of $e_1^k e_2^{n-k}$ of α when the basis of $\mathbb{C}^2$ are $e_1 = (x, y), e_2 = (-\bar{y}, \bar{x})$ for $l = [x, y]((x, y) \in S^3)$. Then, s gives a homeomorphism between $S^{2k-1}(\mathbb{C}^2)$ and $\Gamma^{2k}(T_{2,1})$.

Then, let $H = \bigoplus_{k=1}^5 \Gamma^{2k}(T_{2,1})$ and

$$\tau : H \to \Gamma^2(T_{2,1}) \tag{3.11}$$

be the projection.

3.3. τ-design

Recalling that the associated bundle to (3.2) is

$$\pi : T^1_{2,1} \to \mathbb{C}P^1, \tag{3.12}$$

$\Omega = \mathbb{C}P^1, \mathcal{H} = \Gamma(T_{2,1})$ and $\mathcal{H}_0 = H(= \bigoplus_{k=1}^5 \Gamma^{2k}(T_{2,1}))$. Also, let $\{V_l\}_{l\in\mathbb{C}P^1} = \{l \in \mathbb{C}P^1\}$ and

$$e_l : \Gamma(T^1_{2,1}) \to l, e_l(s) = s^{(2)}(l). \tag{3.13}$$

where $s^{(2)}(l)$ is the second component of $s(l)$.

Now we define a G-invariant polynomial ring.

Definition 3.5. For a subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$, we define

$$\mathbb{C}[x_1, \cdots, x_n]^G = \{f \in \mathbb{C}[x_1, \cdots, x_n] \mid f(Ax_1, \cdots, Ax_n) = f(x_1, \cdots, x_n)\} \tag{3.14}$$

and $\mathbb{C}[x_1, \cdots, x_n]^G_d$ denotes a set of all elements of $\mathbb{C}[x_1, \cdots, x_n]^G$ whose degree is d. The following series is called Hilbert series:

$$P^G_n(t) = \sum_{d=0}^{\infty} (\dim_{\mathbb{C}} \mathbb{C}[x_1, \cdots, x_n]^G_d) t^d. \tag{3.15}$$

Then, let

$$S = \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, T = \frac{1}{\varepsilon^2 - \varepsilon^3} \begin{pmatrix} \varepsilon + \varepsilon^4 & 1 \\ 1 & -\varepsilon - \varepsilon^4 \end{pmatrix}, U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.16}$$

where ε is an primitive 5th root of 1.

Proposition 3.6.

$$G_{icosa} = \langle S, T, U \rangle \tag{3.17}$$

is a subgroup of SU(2) and $\#G_{icosa} = 120$.

$$P_2^{G_{icosa}}(t) = \frac{1+t^{30}}{(1-t^{12})(1-t^{20})} = 1 + t^{12} + t^{20} + t^{24} + \cdots \tag{3.18}$$

For a subset $X \subset \mathbb{C}P^1$, linear functions $\lambda_x : V_x \to \Gamma^2(T_{2,1})$ and $g \in G \subset \mathrm{SU}(2)$, we define

$$\Psi_{(X,\lambda)} : H \to \Gamma^2(T_{2,1}), \Psi_{(X,\lambda)}(s) = \sum_{x \in X} \lambda_x \circ e_x(s), \tag{3.19}$$

and

$$\Psi_{g\cdot(X,\lambda)} : H \to \Gamma^2(T_{2,1}), \Psi_{g\cdot(X,\lambda)}(s) = \sum_{x \in X} (g \cdot \lambda_x) \circ e_{gx}(s) \tag{3.20}$$

where

$$g \cdot \lambda_x : V_{gx} \to \Gamma^2(T_{2,1}), (g \cdot \lambda_x)(z) = g \cdot \lambda_x(g^{-1}z). \tag{3.21}$$

For subsets $X, Y \subset \mathbb{C}P^1$, we define the sum of (X, λ_X) and (Y, λ_Y) as follows:

$$(X, \lambda_X) + (Y, \lambda_Y) = (X \cup Y, \lambda_{(X \cup Y)}) \tag{3.22}$$

where

$$\lambda_{(X\cup Y)}(z) = \begin{cases} \lambda_X(z) & (z \in X \backslash Y) \\ \lambda_X(z) + \lambda_Y(z) & (z \in X \cap Y) \\ \lambda_Y(z) & (z \in Y \backslash X) \end{cases} . \tag{3.23}$$

Then, we define

$$G \cdot (X, \lambda) = \sum_{g \in G} (g \cdot X, g \cdot \lambda_x) . \tag{3.24}$$

Lemma 3.7. Let G be a finite subset of SU(2) and $(Y, \lambda_Y) = \frac{1}{\#G} G \cdot (X, \lambda_X)$. Then, $\Psi_{(Y,\lambda_Y)}$ is a G-intertwining operator.

Theorem 3.8. For a subset $X \subset \mathbb{C}P^1$, if $\mathrm{tr}_{\mathbb{C}}(\Psi_{(X,\lambda)}) = 2$, $(Y, \lambda_Y) = \frac{1}{\#G_{icosa}} G_{icosa} \cdot (X, \lambda)$ is a τ-design.

This theorem insists that

$$\Psi_{(Y,\lambda)}|_{\Gamma^{2k}(T_{2,1})} = \begin{cases} \mathrm{id} & (k = 1) \\ 0 & (k = 2, 3, 4, 5) \end{cases} . \tag{3.25}$$

Finally, we compose a τ-design. Let $x_0 = \{(x, 0) \mid x \in \mathbb{C}\}$ and $\lambda_0 : x_0 \to \Gamma^2(T^1_{2,1})$ be

$$\lambda_0(x, 0)(l) = (l, 2\mathrm{pr}_l(x, 0)). \tag{3.26}$$

Then, $\mathrm{tr}_{\mathbb{C}}(\Psi_{(x_0,\lambda_0)}) = 2$. Therefore, from Theorem3.8,

$$\frac{1}{\#G_{icosa}} G_{icosa} \cdot (x_0, \lambda_0) \tag{3.27}$$

is a τ-design. That is

$$\tau = \frac{1}{120} \sum_{g \in G_{icosa}} (g \cdot \lambda_0) \circ e_{gx_0} \tag{3.28}$$

Furthermore, since the following lemma, we can calculate the equation (3.28) as follows:

$$\tau = \frac{1}{12}\sum_{i=1}^{12}(g_i \cdot \lambda_0) \circ e_{g_i x_0} \tag{3.29}$$

where $\{g_i\}$ are the representatives of $G_{icosa}/\mathrm{Iso}_{x_0}(G_{icosa})$.

Lemma 3.9.

$$\#\mathrm{Iso}_{x_0}(G_{icosa}) = 10 \tag{3.30}$$

and

$$\forall k \in \#\mathrm{Iso}_{x_0}(G_{icosa}), k \cdot \lambda_0 = \lambda_0, \tag{3.31}$$

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