Computation of weighted Bergman inner products on bounded symmetric domains and Parseval–Plancherel-type formulas for $(Sp(r,\mathbb{R}), Sp(r',\mathbb{R}) \times Sp(r'',\mathbb{R}))$

Ryosuke Nakahama*†
NTT Institute for Fundamental Mathematics

Abstract

Let $(G,G') = (G,(G^\sigma)_0)$ be a symmetric pair of holomorphic type, and we consider a pair of Hermitian symmetric spaces $D' = G'/K' \subset D = G/K$, realized as bounded symmetric domains in complex vector spaces $\mathfrak{p}_1^+ := (\mathfrak{p}^+)^\sigma \subset \mathfrak{p}^+$ respectively. Then the universal covering group $\widetilde{G}$ of G acts unitarily on the weighted Bergman space $\mathcal{H}_\lambda(D) \subset \mathcal{O}(D) = \mathcal{O}_\lambda(D)$ on D for sufficiently large λ. Its restriction to the subgroup $\widetilde{G}'$ decomposes discretely and multiplicity-freely, and its branching law is given explicitly by Hua–Kostant–Schmid–Kobayashi's formula in terms of the $\widetilde{K}'$-decomposition of the space $\mathcal{P}(\mathfrak{p}_2^+)$ of polynomials on $\mathfrak{p}_2^+ := (\mathfrak{p}^+)^{-\sigma} \subset \mathfrak{p}^+$. Our goal is to understand the decomposition of the restriction $\mathcal{H}_\lambda(D)|_{\widetilde{G}'}$ by studying the weighted Bergman inner product on each $\widetilde{K}'$-type in $\mathcal{P}(\mathfrak{p}_2^+) \subset \mathcal{H}_\lambda(D)$. In this article we mainly deal with the symmetric pair $(G,G') = (Sp(r,\mathbb{R}), Sp(r',\mathbb{R}) \times Sp(r'',\mathbb{R}))$.

1 Setting

First we review a family of representations, called *holomorphic discrete series representations*, of a Hermitian Lie group G, in the case $G = Sp(r,\mathbb{R})$. We realize the real symplectic group $G = Sp(r,\mathbb{R})$ as

$$G = Sp(r,\mathbb{R}) := \left\{ g \in GL(2r,\mathbb{C}) \;\middle|\; g\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},\; g\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \overline{g} \right\}.$$

Then this is isomorphic to the usual $Sp(r,\mathbb{R})$ via the Cayley transform. Under this realization, G acts transitively on

$$D_r := \{x \in \mathrm{Sym}(r,\mathbb{C}) \mid I - x\overline{x} \text{ is positive definite}\}$$

by the linear fractional transform

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.x := (ax+b)(cx+d)^{-1},$$

and D_r gives the *bounded symmetric domain* realization (*Harish-Chandra realization*) of the Hermitian symmetric space $Sp(r,\mathbb{R})/U(r)$. Next let $\lambda \in \mathbb{C}$, and let (τ, V) be a finite dimensional representation of $GL(r,\mathbb{C})$, with the $K := U(r)$-invariant inner product $(\cdot,\cdot)_V$. Then

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the universal covering group $\widetilde{G}$ of G acts on the space of V-valued holomorphic functions $\mathcal{O}(D_r,V)=\mathcal{O}_\lambda(D_r,V)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1}\right)f(x):=\det(cx+d)^{-\lambda}\,\tau\bigl({}^t(cx+d)\bigr)f\bigl((ax+b)(cx+d)^{-1}\bigr).$$

We note that $\det(cx+d)^{-\lambda}$ is not well-defined on $G\times D$ unless $\lambda\in\mathbb{Z}$, but is well-defined on the universal covering space $\widetilde{G}\times D$. Let $\mathcal{H}_\lambda(D_r,V)\subset\mathcal{O}_\lambda(D_r,V)$ be the non-zero unitary subrepresentation of $\widetilde{G}$ if it exists. We note that such subrepresentation is unique, since the corresponding reproducing kernel is proportional to $\tau(I-x\overline{y})\det(I-x\overline{y})^{-\lambda}$ by the transitivity of the action of G on D_r. Especially, if $\lambda\in\mathbb{R}$ is sufficiently large, then such unitary subrepresentation exists, and its inner product is given by the explicit converging integral

$$\langle f,g\rangle_{\lambda,V}:=C_{\lambda,V}\int_{D_r}\bigl(\tau(I-x\overline{x})^{-1}f(x),g(x)\bigr)_V\det(I-x\overline{x})^{\lambda-(r+1)}dx.$$

This is called a *weighted Bergman inner product*, and the unitary representation $(\tau_\lambda,\mathcal{H}_\lambda(D_r,V))$ is called a *holomorphic discrete series representation*. Especially when $(\tau,V)=\mathbb{C}$ is trivial, then we write $\mathcal{H}_\lambda(D_r,\mathbb{C})=\mathcal{H}_\lambda(D_r)$, and call it of *scalar type*. In this case $\mathcal{H}_\lambda(D)$ becomes a holomorphic discrete series representation if $\lambda>r$, with the inner product

$$\langle f,g\rangle_\lambda:=C_\lambda\int_{D_r}f(x)\overline{g(x)}\det(I-x\overline{x})^{\lambda-(r+1)}dx. \tag{1.1}$$

Here we determine the constant C_λ such that $\|1\|_\lambda=1$ holds.

Next suppose (G,G') is a symmetric pair of *holomorphic type*, that is, both G/K and G'/K' are Hermitian symmetric spaces and the natural embedding $G'/K'\hookrightarrow G/K$ is holomorphic, and let $\mathcal{H}_\lambda(D)$ be a holomorphic discrete series representation of $\widetilde{G}$ of scalar type. Then it is known that the restriction $\mathcal{H}_\lambda(D)|_{\widetilde{G}'}$ decomposes discretely and multiplicity-freely, and its branching law is explicitly determined (see Kobayashi [13]). In the following, we consider the case $(G,G'):=(Sp(r,\mathbb{R}),Sp(r',\mathbb{R})\times Sp(r'',\mathbb{R}))$ with $r=r'+r''$, $r'\le r''$, and give the description of the branching law of $\mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}$. To do this, let

$$\mathfrak{p}^+:=\mathrm{Sym}(r,\mathbb{C}),\quad \mathfrak{p}^+_{11}:=\mathrm{Sym}(r',\mathbb{C}),\quad \mathfrak{p}^+_{12}:=M(r',r'';\mathbb{C}),\quad \mathfrak{p}^+_{22}:=\mathrm{Sym}(r'',\mathbb{C}),$$

and write the elements $x\in\mathfrak{p}^+$ as

$$\mathfrak{p}^+=\mathfrak{p}^+_{11}\oplus\mathfrak{p}^+_{12}\oplus\mathfrak{p}^+_{22}\ni x=\begin{pmatrix} x_{11} & x_{12}\\ {}^tx_{12} & x_{22}\end{pmatrix}.$$

Also, let

$$\mathbb{Z}^r_{++}:=\{\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}^r\mid k_1\ge\cdots\ge k_r\ge 0\}.$$

Then the space of polynomials $\mathcal{P}(\mathfrak{p}^+_{12})$ on $\mathfrak{p}^+_{12}$ is decomposed under $K':=U(r')\times U(r'')$ as

$$\mathcal{P}(\mathfrak{p}^+_{12})=\mathcal{P}(M(r',r'';\mathbb{C}))=\bigoplus_{\mathbf{k}\in\mathbb{Z}^{r'}_{++}}\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})\simeq\bigoplus_{\mathbf{k}\in\mathbb{Z}^{r'}_{++}}V^{(r')\vee}_{\mathbf{k}}\boxtimes V^{(r'')\vee}_{\mathbf{k}},$$

where $V^{(r')\vee}_{\mathbf{k}}$ is the irreducible representation of $U(r')$ with the lowest weight $-\mathbf{k}$ under a suitable identification of the weight lattice for $U(r')$ and $\mathbb{Z}^{r'}$, and similar for $V^{(r'')\vee}_{\mathbf{k}}$, where we identify $\mathbf{k}$ and $(\mathbf{k},0,\dots,0)$. According to this decomposition, for $\lambda>r$, $\mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}$ is decomposed as

$$\mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}\simeq\sum_{\mathbf{k}\in\mathbb{Z}^{r'}_{++}}^{\oplus}\mathcal{H}_\lambda(D_{r'},V^{(r')\vee}_{\mathbf{k}})\,\hat{\boxtimes}\,\mathcal{H}_\lambda(D_{r''},V^{(r'')\vee}_{\mathbf{k}}) \tag{1.2}$$

(see Kobayashi [13, Theorem 8.3]). Now we want to understand this decomposition concretely by considering the inner product

$$\left\langle f(x_{12}), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} \qquad \left(f(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+),\ x = \begin{pmatrix} x_{11} & x_{12} \\ {}^t x_{12} & x_{22} \end{pmatrix}, z \in \mathfrak{p}^+ \right), \tag{1.3}$$

where the subscript x stands for the variable of integration.

For example, suppose $r' = r''$, and we consider the case $\mathbf{k} = (k, \ldots, k)$. Then we have $\mathcal{P}_{(k,\ldots,k)}(\mathfrak{p}_{12}^+) = \mathcal{P}_{(k,\ldots,k)}(M(r', \mathbb{C})) = \mathbb{C} \det(x_{12})^k$. In this setting, the above inner product is explicitly computable.

Theorem 1.1 ([21, Theorem 6.8 (2)])**.** *Suppose $r' = r''$. Then for $k \in \mathbb{Z}_{\geq 0}$, $\mathrm{Re}\,\lambda > 2r'$, $z = \left(\begin{smallmatrix} z_{11} & z_{12} \\ {}^t z_{12} & z_{22} \end{smallmatrix}\right) \in \mathfrak{p}^+$, we have*

$$\begin{aligned} &\left\langle \det(x_{12})^k, e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} \\ &= \frac{\prod_{i=r'+1}^{2r'} \left(\lambda + \left\lceil \frac{k}{2} \right\rceil - \frac{i}{2}\right)_{\lfloor k/2 \rfloor}}{\prod_{i=1}^{r'} \left(\lambda - \frac{i-1}{2}\right)_k \prod_{i=r'+1}^{2r'} \left(\lambda - \frac{i-1}{2}\right)_{\lfloor k/2 \rfloor}} \det(z_{12})^k {}_2F_1 \left(\begin{matrix} -\frac{k}{2}, -\frac{k-1}{2} \\ -\lambda - k + r' + 1 \end{matrix}; z_{11} {}^t z_{12}^{-1} z_{22} z_{12}^{-1} \right). \end{aligned}$$

Here, $(\lambda)_m := \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+m-1)$. We omit the definition of ${}_2F_1$, but this coincides with a special case of Heckman–Opdam's multivariate hypergeometric function of type $BC_{r'}$ under a suitable change of variables. By using this, we can construct explicitly the $\widetilde{G}'$-intertwining operator (symmetry breaking operator) from $\mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}$ to $\mathcal{H}_{\lambda+k}(D_{r'}) \hat{\boxtimes} \mathcal{H}_{\lambda+k}(D_{r'})$ (see [21, Theorem 8.6]). Also, by the theorem we can immediately determine the top term (i.e., the value at $z_{11} = 0$, $z_{22} = 0$) of (1.3) as

$$\left\langle \det(x_{12})^k, e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} \Big|_{z_{11}=0,\, z_{22}=0} = \frac{\prod_{i=r'+1}^{2r'} \left(\lambda + \left\lceil \frac{k}{2} \right\rceil - \frac{i}{2}\right)_{\lfloor k/2 \rfloor}}{\prod_{i=1}^{r'} \left(\lambda - \frac{i-1}{2}\right)_k \prod_{i=r'+1}^{2r'} \left(\lambda - \frac{i-1}{2}\right)_{\lfloor k/2 \rfloor}} \det(z_{12})^k,$$

and determine the poles of (1.3) with respect to $\lambda \in \mathbb{C}$, that is,

$$\prod_{i=1}^{r'} \left(\lambda - \frac{i-1}{2}\right)_k \prod_{i=r'+1}^{2r'} \left(\lambda - \frac{i-1}{2}\right)_{\lfloor k/2 \rfloor} \left\langle \det(x_{12})^k, e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$. In the following, we consider general partitions $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$. Then we cannot compute explicitly (1.3) so far, but can compute the top term and the poles. This is applied for the determination of the Parseval–Plancherel-type formula for the decomposition of $\mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}$.

2 Main theorems and applications

As before, let $\mathfrak{p}^+ := \mathrm{Sym}(r, \mathbb{C}) \supset \mathfrak{p}_{12}^+ := M(r', r''; \mathbb{C})$ with $r = r' + r''$, $r' \leq r''$, and write $x = \left(\begin{smallmatrix} x_{11} & x_{12} \\ {}^t x_{12} & x_{22} \end{smallmatrix}\right) \in \mathfrak{p}^+$. First we give a result on top terms of (1.3).

Theorem 2.1. *Let $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$ and put $k_{r'+1} := 0$. Then for $\mathrm{Re}\,\lambda > r$, $f(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+)$, we have*

$$\left\langle f(x_{12}), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} \Big|_{z_{11}=0,\, z_{22}=0} = C(\lambda, \mathbf{k}) f(z_{12}),$$

where

$$
\begin{aligned}
C(\lambda,\mathbf{k}) &= \frac{2^{|\mathbf{k}|}\prod_{1\le i<j\le r'}(2\lambda-(i+j))_{k_i+k_j}}{\prod_{1\le i<j\le r'+1}(2\lambda-(i+j-1))_{k_i+k_j}}\frac{\prod_{i=1}^{r'}(\lambda-i)_{k_i}}{\prod_{i=1}^{r'}(\lambda-(i-1))_{k_i}} \\
&= \frac{\prod_{1\le i<j\le r'}\left(\lambda-\frac{i+j-1}{2}\right)_{\left\lfloor\frac{k_i+k_j}{2}\right\rfloor}\prod_{1\le i\le j\le r'}\left(\lambda-\frac{i+j}{2}\right)_{\left\lceil\frac{k_i+k_j}{2}\right\rceil}}{\prod_{1\le i\le j\le r'+1}\left(\lambda-\frac{i+j-2}{2}\right)_{\left\lfloor\frac{k_i+k_j}{2}\right\rfloor}\prod_{1\le i<j\le r'+1}\left(\lambda-\frac{i+j-1}{2}\right)_{\left\lceil\frac{k_i+k_j}{2}\right\rceil}} \qquad (2.1)\\
&= \frac{\prod_{a=3}^{2r'-1}\prod_{i=\max\{1,a-r'\}}^{\lceil a/2\rceil-1}\left(\lambda-\frac{a-1}{2}\right)_{\left\lfloor\frac{k_i+k_{a-i}}{2}\right\rfloor}\prod_{a=2}^{2r'}\prod_{i=\max\{1,a-r'\}}^{\lfloor a/2\rfloor}\left(\lambda-\frac{a}{2}\right)_{\left\lceil\frac{k_i+k_{a-i}}{2}\right\rceil}}{\prod_{a=1}^{2r'}\prod_{i=\max\{1,a-r'\}}^{\lceil a/2\rceil}\left(\lambda-\frac{a-1}{2}\right)_{\left\lfloor\frac{k_i+k_{a+1-i}}{2}\right\rfloor}\prod_{a=2}^{2r'}\prod_{i=\max\{1,a-r'\}}^{\lfloor a/2\rfloor}\left(\lambda-\frac{a}{2}\right)_{\left\lceil\frac{k_i+k_{a+1-i}}{2}\right\rceil}}.
\end{aligned}
$$

Next we give a result on poles of (1.3).

Theorem 2.2. *For $\mathbf{k}\in\mathbb{Z}_{++}^{r'}$, define $\phi(\mathbf{k})\in\mathbb{Z}_{++}^{2r'}$ by*

$$
\phi(\mathbf{k})_a := \min\left\{\left\lfloor\frac{k_i+k_j}{2}\right\rfloor \;\middle|\; 1\le i\le j\le r'+1,\ i+j=a+1\right\}\quad (1\le a\le 2r'), \qquad (2.2)
$$

where $k_{r'+1}:=0$. Then for $f(x_{12})\in\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+)$,

$$
\prod_{a=1}^{2r'}\left(\lambda-\frac{a-1}{2}\right)_{\phi(\mathbf{k})_a}\left\langle f(x_{12}),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda,x}
$$

is holomorphically continued for all $\lambda\in\mathbb{C}$.

Remark 2.3. *We can easily verify the restriction of Theorem 2.2 to $z_{11}=0$, $z_{22}=0$ by using Theorem 2.1, that is,*

$$
\begin{aligned}
&\prod_{a=1}^{2r'}\left(\lambda-\frac{a-1}{2}\right)_{\phi(\mathbf{k})_a}C(\lambda,\mathbf{k}) = \prod_{a=1}^{2r'}\left(\lambda-\frac{a-1}{2}\right)_{\min_i\left\lfloor\frac{k_i+k_{a+1-i}}{2}\right\rfloor} \\
&\times\frac{\prod_{a=3}^{2r'-1}\prod_{i=\max\{1,a-r'\}}^{\lceil a/2\rceil-1}\left(\lambda-\frac{a-1}{2}\right)_{\left\lfloor\frac{k_i+k_{a-i}}{2}\right\rfloor}}{\prod_{a=1}^{2r'}\prod_{i=\max\{1,a-r'\}}^{\lceil a/2\rceil}\left(\lambda-\frac{a-1}{2}\right)_{\left\lfloor\frac{k_i+k_{a+1-i}}{2}\right\rfloor}}\frac{\prod_{a=2}^{2r'}\prod_{i=\max\{1,a-r'\}}^{\lfloor a/2\rfloor}\left(\lambda-\frac{a}{2}\right)_{\left\lceil\frac{k_i+k_{a-i}}{2}\right\rceil}}{\prod_{a=2}^{2r'}\prod_{i=\max\{1,a-r'\}}^{\lfloor a/2\rfloor}\left(\lambda-\frac{a}{2}\right)_{\left\lceil\frac{k_i+k_{a+1-i}}{2}\right\rceil}}
\end{aligned}
$$

is holomorphic for all $\lambda\in\mathbb{C}$.

Next we consider some applications of the theorems. Let $(G,G')=(Sp(r,\mathbb{R}),Sp(r',\mathbb{R})\times Sp(r'',\mathbb{R}))$ as before. Then since $\mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}$ decomposes as in (1.2) for $\lambda>r$, for each $\mathbf{k}\in\mathbb{Z}_{++}^{r'}$ there exists uniquely (up to scalar) a $\widetilde{G}'$-intertwining operator (symmetry breaking operator)

$$
\mathcal{F}_{\lambda,\mathbf{k}}\colon \mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}\longrightarrow \mathcal{H}_\lambda(D_{r'},V_{\mathbf{k}}^{(r')\vee})\hat{\boxtimes}\mathcal{H}_\lambda(D_{r''},V_{\mathbf{k}}^{(r'')\vee}).
$$

We fix the normalization of $\mathcal{F}_{\lambda,\mathbf{k}}$ such that

$$
\|\mathcal{F}_{\lambda,\mathbf{k}}(f(x_{12}))\|^2_{\mathcal{H}_\lambda(D_{r'},V_{\mathbf{k}}^{(r')\vee})\hat{\boxtimes}\mathcal{H}_\lambda(D_{r''},V_{\mathbf{k}}^{(r'')\vee})} = \overline{f}\left(\frac{1}{2}\frac{\partial}{\partial z_{12}}\right)f(z_{12})\bigg|_{z_{12}=0}\qquad (f(x_{12})\in\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+))
$$

holds, independent of λ. Then we can easily prove the following.

Corollary 2.4. *For $\lambda > r$, for $f \in \mathcal{H}_\lambda(D_r)$, we have*

$$\|f\|^2_{\mathcal{H}_\lambda(D_r)} = \sum_{\mathbf{k}\in\mathbb{Z}^{r'}_{++}} C(\lambda,\mathbf{k})\,\|\mathcal{F}_{\lambda,\mathbf{k}}f\|^2_{\mathcal{H}_\lambda(D_{r'},V^{(r')\vee}_{\mathbf{k}})\hat{\boxtimes}\mathcal{H}_\lambda(D_{r''},V^{(r'')\vee}_{\mathbf{k}})},$$

where $C(\lambda,\mathbf{k})$ is as in (2.1).

We omit the proof of Corollary 2.4. Next we consider the decomposition of $\mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}$ for smaller λ. $\widetilde{G} = \widetilde{Sp}(r,\mathbb{R})$ acts on $\mathcal{O}_\lambda(D_r)$, and it is known that there exists a non-zero unitary subrepresentation $\mathcal{H}_\lambda(D_r) \subset \mathcal{O}_\lambda(D_r)$ if and only if

$$\lambda \in \left\{0, \frac{1}{2}, 1, \dots, \frac{r-1}{2}\right\} \cup \left(\frac{r-1}{2}, \infty\right)$$

(see, e.g., [5, Theorem XIII.2.7]). This set is called the *Wallach set*. Especially, $\mathcal{H}_\lambda(D_r)$ is a holomorphic discrete series representation (i.e., the integral (1.1) converges) for $\lambda > r$. If $\lambda > \frac{r-1}{2}$, then the decomposition of $\mathcal{H}_\lambda(D_r)|_{\widetilde{G}'}$ is again given by (1.2). On the other hand, for smaller λ the following holds. Here, $\phi(\mathbf{k})_a$ is defined as in (2.2) for $1 \le a \le 2r'$, and we set $\phi(\mathbf{k})_a := 0$ for $2r' < a \le r$.

Corollary 2.5. *For $a = 0, 1, 2, \dots, r-1$, we have*

$$\begin{aligned}\mathcal{H}_{\frac{a}{2}}(D_r)|_{\widetilde{G}'} &\simeq \sum^{\oplus}_{\substack{\mathbf{k}\in\mathbb{Z}^{r'}_{++}\\ \phi(\mathbf{k})_{a+1}=0}} \mathcal{H}_\lambda(D_{r'},V^{(r')\vee}_{\mathbf{k}})\,\hat{\boxtimes}\,\mathcal{H}_\lambda(D_{r''},V^{(r'')\vee}_{\mathbf{k}})\\ &= \sum^{\oplus}_{\substack{0\le b\le c\le r'\\ b+c\le a}} \sum^{\oplus}_{\substack{\mathbf{k}\in\mathbb{Z}^{b}_{++}\\ k_b\ge 2}} \mathcal{H}_{\frac{a}{2}}(D_{r'},V^{(r')\vee}_{(\mathbf{k},\underbrace{1,\dots,1}_{c-b},\underbrace{0,\dots,0}_{r'-c})})\,\hat{\boxtimes}\,\mathcal{H}_{\frac{a}{2}}(D_{r''},V^{(r'')\vee}_{(\mathbf{k},\underbrace{1,\dots,1}_{c-b},\underbrace{0,\dots,0}_{r''-c})}).\end{aligned}$$

Remark 2.6. *(1) Parseval–Plancherel-type formulas for Hermitian symmetric pairs (G,K), i.e., cases such that $K \subset G$ is a maximal compact subgroup, are studied by, e.g., Ørsted [23], Faraut–Korányi [4, 5], Ørsted–Zhang [24, 25], Hwang–Liu–Zhang [10] and the author [20].*

(2) Parseval–Plancherel-type formulas for general symmetric pairs of holomorphic type (G,G') are studied by, e.g., Hilgert–Krötz [7, 8], Ben Saïd [1, 2] and Kobayashi–Pevzner [18], under different realization of holomorphic discrete series representations.

(3) In this article, we treat the explicit forms of symmetry breaking operators as black boxes. Construction of differential symmetry breaking operators are studied by, e.g., Rankin [27], Cohen [3], Peng–Zhang [26], Juhl [12], Ibukiyama–Kuzumaki–Ochiai [11], Kobayashi–Ørsted–Somberg–Souček [15], Kobayashi–Pevzner [16, 17], Kobayashi–Kubo–Pevzner [14] and the author [21].

(4) Branching laws of unitary highest weight modules for discrete Wallach sets are studied by, e.g., Sekiguchi [28] and Möllers–Oshima [19]. We can also study branching laws of unitary highest weight modules by using the seesaw dual pair theory (see, e.g., [9, Section 3]) as in [19] when (G,G') is classical.

3 Proof of Theorem 2.2 and Corollary 2.5

In this section we give proofs of Theorem 2.2 and Corollary 2.5. To do this, we observe the $\widetilde{K}=\widetilde{U}(r)$-type decomposition of $\mathcal{H}_\lambda(D_r)$. The $\widetilde{K}$-finite part of $\mathcal{H}_\lambda(D_r)$ is given by

$$\mathcal{O}_\lambda(D_r)_{\widetilde{K}}=\det{}^{-\lambda}\otimes\mathcal{P}(\mathfrak{p}^+)=\det{}^{-\lambda}\otimes\mathcal{P}(\mathrm{Sym}(r,\mathbb{C})),$$

and the space of polynomials $\mathcal{P}(\mathfrak{p}^+)=\mathcal{P}(\mathrm{Sym}(r,\mathbb{C}))$ is decomposed under $K=U(r)$ as

$$\mathcal{P}(\mathfrak{p}^+)=\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)\simeq\bigoplus_{\mathbf{m}\in\mathbb{Z}^r_{++}}V^{(r)\vee}_{2\mathbf{m}}.$$

According to this decomposition, the following holds.

Theorem 3.1 (Faraut–Korányi [5, Corollary XIII.2.3])**.** *Let* $\mathbf{m}\in\mathbb{Z}^r_{++}$*. Then for* $\operatorname{Re}\lambda>r$*,* $f(x)\in\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$*, we have*

$$\left\langle f(x),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda,x}=\frac{1}{\prod_{a=1}^r\left(\lambda-\frac{a-1}{2}\right)_{m_a}}f(z).$$

Especially, for $f(x)\in\mathcal{P}(\mathfrak{p}^+)$ and for $\mathbf{l}\in\mathbb{Z}^r_{++}$,

$$\prod_{a=1}^r\left(\lambda-\frac{a-1}{2}\right)_{l_a}\left\langle f(x),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda,x}$$

is holomorphically continued for all $\lambda\in\mathbb{C}$ if and only if

$$f(x)\in\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ m_a\le l_a}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

Proof of Theorem 2.2. Under Theorem 3.1, Theorem 2.2 is equivalent to

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})\subset\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ m_a\le\phi(\mathbf{k})_a}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+),\tag{3.1}$$

and hence it is enough to prove this inclusion. Since $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})\simeq V^{(r')\vee}_{\mathbf{k}}\boxtimes V^{(r'')\vee}_{\mathbf{k}}$ as a $K'=U(r')\times U(r'')$-module and $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)\simeq V^{(r)\vee}_{2\mathbf{m}}$ as a $K=U(r)$-module, it is enough to show that

$$\mathrm{Hom}_{U(r')\times U(r'')}(V^{(r')\vee}_{\mathbf{k}}\boxtimes V^{(r'')\vee}_{\mathbf{k}},V^{(r)\vee}_{2\mathbf{m}})\neq\{0\}\quad\text{implies}\quad m_a\le\phi(\mathbf{k})_a\quad(1\le a\le r),$$

or equivalently by the definition of $\phi(\mathbf{k})_a$,

$$\mathrm{Hom}_{U(r')\times U(r'')}(V^{(r')\vee}_{\mathbf{k}}\boxtimes V^{(r'')\vee}_{\mathbf{k}},V^{(r)\vee}_{2\mathbf{m}})\neq\{0\}\quad\text{implies}\quad 2m_{i+j-1}\le k_i+k_j\quad(1\le i,j\le r'+1),$$

with $k_{r'+1}:=0$. On the other hand, for $\mathbf{k}\in\mathbb{Z}^{r'}_{++}$, $\mathbf{l}\in\mathbb{Z}^{r''}_{++}$, $\mathbf{m}\in\mathbb{Z}^r_{++}$ with $r'+r''=r$,

$$\dim\mathrm{Hom}_{U(r')\times U(r'')}(V^{(r')\vee}_{\mathbf{k}}\boxtimes V^{(r'')\vee}_{\mathbf{l}},V^{(r)\vee}_{\mathbf{m}})=\dim\mathrm{Hom}_{U(r)}(V^{(r)\vee}_{\mathbf{m}},V^{(r)\vee}_{\mathbf{k}}\otimes V^{(r)\vee}_{\mathbf{l}})$$

holds in general by [6, Theorem 9.2.3], and by the Littlewood–Richardson rule, we can show that for $\mathbf{k},\mathbf{l},\mathbf{m}\in\mathbb{Z}^r_{++}$,

$$\mathrm{Hom}_{U(r)}(V^{(r)\vee}_{\mathbf{m}},V^{(r)\vee}_{\mathbf{k}}\otimes V^{(r)\vee}_{\mathbf{l}})\neq\{0\}\quad\text{implies}\quad m_{i+j-1}\le k_i+l_j\quad(1\le i,j,\ i+j\le r+1)$$

(see [22, Lemma 3.6]). Hence the theorem follows. □

We note that this proof for $(G,G')=(Sp(r,\mathbb{R}),Sp(r',\mathbb{R})\times Sp(r'',\mathbb{R}))$ is not available for other symmetric pairs in general.

Next, to prove Corollary 2.5, we observe the $\widetilde{K}$-type formula for the discrete Wallach set. By Theorem 3.1, $\langle\cdot,\cdot\rangle_\lambda$ is meromorphically continued for all $\lambda\in\mathbb{C}$, and is positive definite on $\mathcal{P}(\mathfrak{p}^+)$ for $\lambda>\frac{r-1}{2}$. That is, $\mathcal{H}_\lambda(D_r)_{\widetilde{K}}=\det^{-\lambda}\otimes\mathcal{P}(\mathfrak{p}^+)$ holds for $\lambda>\frac{r-1}{2}$. On the other hand, $\langle\cdot,\cdot\rangle_\lambda$ has poles at $\lambda\in\frac{r-1}{2}-\frac{1}{2}\mathbb{Z}_{\geq 0}$, and $\mathcal{O}_\lambda(D_r)_{\widetilde{K}}$ becomes reducible for such λ. If the restriction of $\langle\cdot,\cdot\rangle_\lambda$ to the irreducible $(\mathfrak{g},\widetilde{K})$-submodule of $\mathcal{O}_\lambda(D_r)_{\widetilde{K}}$ is positive definite, then this becomes an infinitesimally unitary submodule. This occurs when $\lambda=0,\frac{1}{2},1,\ldots,\frac{r-1}{2}$. That is, the following holds.

Corollary 3.2 (Faraut–Korányi [5, Theorem XIII.2.7])**.** *For $a=0,1,2,\ldots,r-1$, we have*

$$\mathcal{H}_{\frac{a}{2}}(D_r)_{\widetilde{K}}=\det{}^{-\lambda}\otimes\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ m_{a+1}=0}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

Especially, $f(x)\in\mathcal{H}_{\frac{a}{2}}(D_r)_{\widetilde{K}}$ holds if and only if $\left\langle f(x),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda,x}$ is holomorphic for $\lambda>\frac{a-1}{2}$.

Proof of Corollary 2.5. We embed $\mathfrak{p}^+=\mathrm{Sym}(r,\mathbb{C})$ into $\mathfrak{g}^{\mathbb{C}}=\mathfrak{sp}(r,\mathbb{C})$, the complexified Lie algebra of $G=Sp(r,\mathbb{R})$, by $x\mapsto\left(\begin{smallmatrix}0&x\\0&0\end{smallmatrix}\right)$, and similarly embed $\mathfrak{p}^+_{11}\oplus\mathfrak{p}^+_{22}$ into $\mathfrak{g}'^{\mathbb{C}}=\mathfrak{sp}(r',\mathbb{C})\oplus\mathfrak{sp}(r'',\mathbb{C})$ compatibly, so that we have

$$\begin{array}{ccccl}\mathfrak{g}^{\mathbb{C}}&\supset&\mathfrak{p}^+&:=&\mathrm{Sym}(r,\mathbb{C})\\ \cup&&\cup&&\\ \mathfrak{g}'^{\mathbb{C}}&\supset&\mathfrak{p}^+_{11}\oplus\mathfrak{p}^+_{22}&:=&\mathrm{Sym}(r',\mathbb{C})\oplus\mathrm{Sym}(r'',\mathbb{C}).\end{array}$$

Then since $\mathfrak{p}^+_{11}\oplus\mathfrak{p}^+_{22}$ acts on $\mathcal{O}_\lambda(D_r)_{\widetilde{K}}=\mathcal{P}(\mathfrak{p}^+)$ by constant coefficient differential operators along $\mathfrak{p}^+_{11}\oplus\mathfrak{p}^+_{22}\subset\mathfrak{p}^+$, the $\mathfrak{p}^+_{11}\oplus\mathfrak{p}^+_{22}$-null part of $\mathcal{H}_{\frac{a}{2}}(D_r)_{\widetilde{K}}$ is given by

$$\mathcal{H}_{\frac{a}{2}}(D_r)_{\widetilde{K}}^{\mathfrak{p}^+_{11}\oplus\mathfrak{p}^+_{22}}=\mathcal{P}(\mathfrak{p}^+_{12})\cap\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ m_{a+1}=0}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+),$$

with $\mathfrak{p}^+_{12}:=M(r',r'';\mathbb{C})$. Since every $(\mathfrak{g}',\widetilde{K}')$-submodule in $\mathcal{H}_{\frac{a}{2}}(D_r)$ intersects the above space, it is enough to show that

$$\mathcal{P}(\mathfrak{p}^+_{12})\cap\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ m_{a+1}=0}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)=\bigoplus_{\substack{\mathbf{k}\in\mathbb{Z}^{r'}_{++}\\ \phi(\mathbf{k})_{a+1}=0}}\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})$$

holds. To prove the inclusion from right to left, suppose $\mathbf{k}\in\mathbb{Z}^{r'}_{++}$ satisfies $\phi(\mathbf{k})_{a+1}=0$. Then by (3.1), we have

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})\subset\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ m_j\leq\phi(\mathbf{k})_j}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)\subset\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ m_{a+1}=0}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

To prove the opposite inclusion, suppose $\mathbf{k}\in\mathbb{Z}^{r'}_{++}$ satisfies $\phi(\mathbf{k})_{a+1}\neq 0$, and take the smallest $a'>a$ such that $\phi(\mathbf{k})_{a'+1}=0$. Then for $f(x_{12})\in\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})$, by Theorem 2.1, we can show that

$$\left\langle f(x_{12}),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda,x}\Big|_{z_{11}=0,z_{22}=0}=C(\lambda,\mathbf{k})f(z_{12})$$

has a pole at $\lambda=\frac{a'-1}{2}$. Especially, $\left\langle f(x_{12}),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda,x}$ is not holomorphic on $\lambda>\frac{a-1}{2}$, and hence by Theorem 3.1 we have

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})\not\subset\bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^r_{++}\\ m_{a+1}=0}}\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

This completes the proof of Corollary 2.5. $\square$

4 Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. First, for $\mathbf{k} \in \mathbb{Z}^{r'}_{++}$, we define a polynomial $\Delta_{\mathbf{k}}(x_{12})$ on $\mathfrak{p}^+_{12} = M(r', r''; \mathbb{C})$ by

$$\Delta_{\mathbf{k}}(x_{12}) := \prod_{l=1}^{r'} \det(((x_{12})_{ij})_{1\le i,j\le l})^{k_l - k_{l+1}},$$

where $k_{r'+1} := 0$. Then $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})$ is generated by $\Delta_{\mathbf{k}}(x_{12})$ as a $K' = U(r') \times U(r'')$-module. Since the inner product (1.1) is K'-equivariant, it is enough to prove the theorem when $f(x_{12}) = \Delta_{\mathbf{k}}(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}^+_{12})$.

To prove the theorem, we prepare some lemmas. First, for $s < r$ we fix an inclusion $\mathrm{Sym}(s, \mathbb{C}) \hookrightarrow \mathrm{Sym}(r, \mathbb{C})$ suitably, and for $x \in \mathrm{Sym}(r, \mathbb{C})$, let $x' \in \mathrm{Sym}(s, \mathbb{C})$ denote the orthogonal projection of x. Then the following holds.

Lemma 4.1. *For* $\mathrm{Re}\,\lambda > r$, *for* $f(x') \in \mathcal{P}(\mathrm{Sym}(s, \mathbb{C})) \subset \mathcal{P}(\mathrm{Sym}(r, \mathbb{C}))$, *we have*

$$\left\langle f(x'), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\mathcal{H}_\lambda(D_r),x} = \left\langle f(x'), e^{\mathrm{tr}(x'\overline{z'})} \right\rangle_{\mathcal{H}_\lambda(D_s),x'}.$$

Proof. Let $\mathbf{m} \in \mathbb{Z}^s_{++}$. Then we have $\mathcal{P}_{\mathbf{m}}(\mathrm{Sym}(s, \mathbb{C})) \subset \mathcal{P}_{\mathbf{m}}(\mathrm{Sym}(r, \mathbb{C})) = \mathcal{P}_{(\mathbf{m},\overbrace{0,\ldots,0}^{r-s})}(\mathrm{Sym}(r, \mathbb{C}))$, and by Theorem 3.1, for $f(x') \in \mathcal{P}_{\mathbf{m}}(\mathrm{Sym}(s, \mathbb{C})) \subset \mathcal{P}_{\mathbf{m}}(\mathrm{Sym}(r, \mathbb{C}))$ we have

$$\left\langle f(x'), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\mathcal{H}_\lambda(D_r),x} = \frac{1}{\prod_{a=1}^{s} \left(\lambda - \frac{a-1}{2}\right)_{m_a}} f(z') = \left\langle f(x'), e^{\mathrm{tr}(x'\overline{z'})} \right\rangle_{\mathcal{H}_\lambda(D_s),x'}.$$

Since this holds for every $\mathbf{m} \in \mathbb{Z}^s_{++}$, we get the lemma. □

Suppose $r = r' + r''$, $r' \le r''$, and let $s = 2r'$. Then by applying the above lemma for the inclusion

$$\begin{array}{ccc} \mathrm{Sym}(2r', \mathbb{C}) & \subset & \mathrm{Sym}(r, \mathbb{C}) \\ \cup & & \cup \\ M(r', \mathbb{C}) & \subset & M(r', r''; \mathbb{C}), \end{array}$$

for every $\mathbf{k} \in \mathbb{Z}^{r'}_{++}$ we get

$$\left\langle \Delta_{\mathbf{k}}(x_{12}), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\mathcal{H}_\lambda(D_r),x} = \left\langle \Delta_{\mathbf{k}}(x'_{12}), e^{\mathrm{tr}(x'\overline{z'})} \right\rangle_{\mathcal{H}_\lambda(D_{2r'}),x'}.$$

Hence it is enough to prove Theorem 2.1 when $r' = r''$.

In the following suppose $r' = r''$, and let $\mathfrak{p}^+_{12} := M(r', \mathbb{C})$. For $x \in \mathfrak{p}^+ = \mathrm{Sym}(r, \mathbb{C})$, let

$$\det{}'(x) := \det\left(x \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right),$$

so that $\det{}'\begin{pmatrix} 0 & x_{12} \\ {}^t x_{12} & 0 \end{pmatrix} = \det(x_{12})^2$ holds. Then the following holds.

Proposition 4.2. *For* $\mathrm{Re}\,\lambda > r$, $k \in \mathbb{Z}_{\ge 0}$, $f(x) \in \mathcal{P}(\mathfrak{p}^+)$, *we have*

$$\begin{aligned} &\left\langle \det(x_{12})^k f(x), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} \\ &= \frac{1}{\prod_{i=1}^{r} \left(\lambda - \frac{i-1}{2}\right)_k} \det{}'(z)^{-\lambda + \frac{r+1}{2}} \det\left(\frac{1}{2}\frac{\partial}{\partial z_{12}}\right)^k \det{}'(z)^{\lambda + k - \frac{r+1}{2}} \left\langle f(x), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda+k,x}. \end{aligned}$$

The proof of Proposition 4.2 is given later. We also need the following.

Lemma 4.3 ([5, Proposition VII.1.6])**.** *For $\mu\in\mathbb{C}$, $k\in\mathbb{Z}_{\ge 0}$, $\mathbf{l}\in\mathbb{Z}^{r'}_{++}$, $z_{12}\in\mathfrak{p}^+_{12}=M(r',\mathbb{C})$, we have*

$$\det\left(\frac{\partial}{\partial z_{12}}\right)^k\det(z_{12})^{\mu}\Delta_{\mathbf{l}}(z_{12})=\prod_{i=1}^{r'}(\mu+l_i-k+r'-i+1)_k\det(z_{12})^{\mu-k}\Delta_{\mathbf{l}}(z_{12}).$$

Proof of Theorem 2.1. The 2nd equality of (2.1) follows from $(2\mu)_k=2^k(\mu)_{\lceil k/2\rceil}\left(\mu+\frac{1}{2}\right)_{\lfloor k/2\rfloor}$, and the 3rd equality is easy. For the 1st equality, it is enough to prove when $r'=r''$ and $f(x_{12})=\Delta_{\mathbf{k}}(x_{12})$, as explained before. We prove this by induction on $r'=r/2$. First, when $\mathbf{k}=(0,\dots,0)$ ("$r'=0$ case"), this is clear. Next we assume the theorem for $r'-1$, and prove it for r'. We write $\underline{k_{r'}}:=(\underbrace{k_{r'},\dots,k_{r'}}_{r'})$. Then by Proposition 4.2 we have

$$\begin{aligned}
&\left\langle\Delta_{\mathbf{k}}(x_{12}),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda,x}\Big|_{z_{11}=0,\,z_{22}=0}=\left\langle\det(x_{12})^{k_{r'}}\Delta_{\mathbf{k}-\underline{k_{r'}}}(x_{12}),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda,x}\Big|_{z_{11}=0,\,z_{22}=0}\\
&=\frac{1}{\prod_{i=1}^{r}\left(\lambda-\frac{i-1}{2}\right)_{k_{r'}}}\det{}'\begin{pmatrix}0&z_{12}\\ {}^tz_{12}&0\end{pmatrix}^{-\lambda+\frac{r+1}{2}}\det\left(\frac{1}{2}\frac{\partial}{\partial z_{12}}\right)^{k_{r'}}\det{}'\begin{pmatrix}0&z_{12}\\ {}^tz_{12}&0\end{pmatrix}^{\lambda+k_{r'}-\frac{r+1}{2}}\\
&\qquad\times\left\langle\Delta_{\mathbf{k}-\underline{k_{r'}}}(x_{12}),e^{\mathrm{tr}(x\overline{z})}\right\rangle_{\lambda+k_{r'},x}\Big|_{z_{11}=0,\,z_{22}=0}\\
&=\frac{C(\lambda+k_{r'},\mathbf{k}-\underline{k_{r'}})}{\prod_{i=1}^{2r'}\left(\lambda-\frac{i-1}{2}\right)_{k_{r'}}}\det(z_{12})^{2\left(-\lambda+r'+\frac{1}{2}\right)}\det\left(\frac{1}{2}\frac{\partial}{\partial z_{12}}\right)^{k_{r'}}\det(z_{12})^{2\left(\lambda+k_{r'}-r'-\frac{1}{2}\right)}\Delta_{\mathbf{k}-\underline{k_{r'}}}(z_{12})\\
&=\frac{C(\lambda+k_{r'},\mathbf{k}-\underline{k_{r'}})}{2^{k_{r'}r'}\prod_{i=1}^{2r'}\left(\lambda-\frac{i-1}{2}\right)_{k_{r'}}}\prod_{i=1}^{r'}(2\lambda+k_i-i-r')_{k_{r'}}\det(z_{12})^{k_{r'}}\Delta_{\mathbf{k}-\underline{k_{r'}}}(z_{12})\\
&=\frac{1}{2^{k_{r'}r'}\prod_{i=1}^{2r'}\left(\lambda-\frac{i-1}{2}\right)_{k_{r'}}}\frac{2^{|\mathbf{k}|-k_{r'}r'}\prod_{1\le i<j\le r'-1}(2(\lambda+k_{r'})-(i+j))_{(k_i-k_{r'})+(k_j-k_{r'})}}{\prod_{1\le i<j\le r'}(2(\lambda+k_{r'})-(i+j-1))_{(k_i-k_{r'})+(k_j-k_{r'})}}\\
&\qquad\times\frac{\prod_{i=1}^{r'-1}(\lambda+k_{r'}-i)_{k_i-k_{r'}}}{\prod_{i=1}^{r'-1}(\lambda+k_{r'}-(i-1))_{k_i-k_{r'}}}\prod_{i=1}^{r'}(2\lambda+k_i-i-r')_{k_{r'}}\Delta_{\mathbf{k}}(z_{12})\\
&=\frac{2^{|\mathbf{k}|-2k_{r'}r'}}{\prod_{i=1}^{2r'}\left(\lambda-\frac{i-1}{2}\right)_{k_{r'}}}\frac{\prod_{1\le i<j\le r'-1}(2\lambda-(i+j))_{k_i+k_j}}{\prod_{1\le i<j\le r'}(2\lambda-(i+j-1))_{k_i+k_j}}\frac{\prod_{1\le i<j\le r'}(2\lambda-(i+j-1))_{2k_{r'}}}{\prod_{1\le i<j\le r'-1}(2\lambda-(i+j))_{2k_{r'}}}\\
&\qquad\times\frac{\prod_{i=1}^{r'}(\lambda-i)_{k_i}}{\prod_{i=1}^{r'}(\lambda-(i-1))_{k_i}}\frac{\prod_{i=1}^{r'}(\lambda-(i-1))_{k_{r'}}}{\prod_{i=1}^{r'}(\lambda-i)_{k_{r'}}}\frac{\prod_{i=1}^{r'}(2\lambda-(i+r'))_{k_i+k_{r'}}}{\prod_{i=1}^{r'}(2\lambda-(i+(r'+1)-1))_{k_i}}\Delta_{\mathbf{k}}(z_{12})\\
&=\frac{2^{|\mathbf{k}|-2k_{r'}r'}}{\prod_{i=1}^{2r'}\left(\lambda-\frac{i-1}{2}\right)_{k_{r'}}}\frac{\prod_{1\le i<j\le r'}(2\lambda-(i+j))_{k_i+k_j}}{\prod_{1\le i<j\le r'+1}(2\lambda-(i+j-1))_{k_i+k_j}}\frac{\prod_{1\le i\le j\le r'-1}(2\lambda-(i+j))_{2k_{r'}}}{\prod_{1\le i<j\le r'-1}(2\lambda-(i+j))_{2k_{r'}}}\\
&\qquad\times\frac{\prod_{i=1}^{r'}(\lambda-i)_{k_i}}{\prod_{i=1}^{r'}(\lambda-(i-1))_{k_i}}\frac{\prod_{i=1}^{r'}(\lambda-(i-1))_{k_{r'}}}{\prod_{i=1}^{r'}(\lambda-i)_{k_{r'}}}(2\lambda-2r')_{2k_{r'}}\Delta_{\mathbf{k}}(z_{12})\\
&=\frac{2^{|\mathbf{k}|}\prod_{1\le i<j\le r'}(2\lambda-(i+j))_{k_i+k_j}}{\prod_{1\le i<j\le r'+1}(2\lambda-(i+j-1))_{k_i+k_j}}\frac{\prod_{i=1}^{r'}(\lambda-i)_{k_i}}{\prod_{i=1}^{r'}(\lambda-(i-1))_{k_i}}\\
&\qquad\times\frac{2^{-2k_{r'}r'}\prod_{i=1}^{r'}(2\lambda-2i)_{2k_{r'}}}{\prod_{i=1}^{r'}(\lambda-(i-1))_{k_{r'}}\prod_{i=1}^{r'}\left(\lambda-\left(i-\frac{1}{2}\right)\right)_{k_{r'}}}\frac{\prod_{i=1}^{r'}(\lambda-(i-1))_{k_{r'}}}{\prod_{i=1}^{r'}(\lambda-i)_{k_{r'}}}\Delta_{\mathbf{k}}(z_{12})\\
&=C(\lambda,\mathbf{k})\Delta_{\mathbf{k}}(z_{12}),
\end{aligned}$$

where we have used Lemma 4.1 and the induction hypothesis at the 3rd equality, and Lemma 4.3 at the 4th equality. Hence the theorem holds for all r'. □

Now the proof of Proposition 4.2 is remaining. To prove this, for $\mathfrak{p}^+ = \mathrm{Sym}(r,\mathbb{C})$ with $r = 2r'$, let $\mathfrak{n}^+ \subset \mathfrak{p}^+$ be the real form and $\Omega \subset \mathfrak{n}^+$ be the open cone given by

$$\mathfrak{n}^+ := \mathfrak{p}^+ \cap \mathrm{Herm}(r,\mathbb{C}) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (\simeq \mathrm{Sym}(r,\mathbb{R})),$$
$$\Omega := \mathfrak{p}^+ \cap \mathrm{Herm}_+(r,\mathbb{C}) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (\simeq \mathrm{Sym}_+(r,\mathbb{R})),$$

where $\mathrm{Herm}_+(r,\mathbb{C})$ is the set of $r \times r$ positive definite Hermitian matrices. Also, for $\lambda \in \mathbb{C}$ let $\Gamma_r(\lambda) := (2\pi)^{r(r-1)/4} \prod_{i=1}^r \Gamma\left(\lambda - \frac{i-1}{2}\right)$, and let $n := \dim \mathfrak{p}^+ = r(r+1)/2$. Then the following holds.

Lemma 4.4. *For* $\mathrm{Re}\,\lambda > r$, $f \in \mathcal{P}(\mathfrak{p}^+)$, $z, a \in \Omega$, *we have*

$$\left\langle f(x), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} = \det{}'(z)^{-\lambda+\frac{r+1}{2}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zw)} f(w^{-1}) \det{}'(w)^{-\lambda} dw.$$

Proof. Let $\mathbf{m} \in \mathbb{Z}^r_{++}$. Then for $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, by Theorem 3.1 we have

$$\left\langle f(x), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} = \frac{1}{\prod_{i=1}^r \left(\lambda - \frac{i-1}{2}\right)_{m_i}} f(z),$$

and by the inverse Laplace transform (Gindikin, see [5, Lemma XI.2.3, Section IX.3]), we have

$$\det{}'(z)^{-\lambda+\frac{r+1}{2}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zw)} f(w^{-1}) \det{}'(w)^{-\lambda} dw = \frac{1}{\prod_{i=1}^r \left(\lambda - \frac{i-1}{2}\right)_{m_i}} f(z).$$

Hence the both sides coincide. Since this holds for every $\mathbf{m} \in \mathbb{Z}^r_{++}$, the both sides coincide for all $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$. □

Proof of Proposition 4.2. First, let $\mathrm{Proj}_{12} \colon \mathfrak{p}^+ \to \mathfrak{p}^+_{12}$ be the orthogonal projection. Then for $w = \left(\begin{smallmatrix} w_{11} & w_{12} \\ {}^t w_{12} & w_{22} \end{smallmatrix}\right) \in \mathfrak{p}^+$, we have

$$\det(\mathrm{Proj}_{12}(w^{-1})) = \det\left(({}^t w_{12} - w_{22} w_{12}^{-1} w_{11})^{-1}\right) = \det{}'(w)^{-1} \det(w_{12}).$$

Now let $z = \left(\begin{smallmatrix} z_{11} & z_{12} \\ {}^t z_{12} & z_{22} \end{smallmatrix}\right) \in \Omega \subset \mathfrak{p}^+$ and $f(x) \in \mathcal{P}(\mathfrak{p}^+)$. Then by using lemma 4.4 twice we have

$$\begin{aligned}
&\left\langle \det(x_{12})^k f(x), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} = \left\langle \det(\mathrm{Proj}_{12}(x))^k f(x), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda,x} \\
&= \det{}'(z)^{-\lambda+\frac{r+1}{2}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zw)} \det(\mathrm{Proj}_{12}(w^{-1}))^k f(w^{-1}) \det{}'(w)^{-\lambda} dw \\
&= \det{}'(z)^{-\lambda+\frac{r+1}{2}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zw)} \det(w_{12})^k f(w^{-1}) \det{}'(w)^{-\lambda-k} dw \\
&= \frac{\det{}'(z)^{-\lambda+\frac{r+1}{2}}}{\prod_{i=1}^r \left(\lambda - \frac{i-1}{2}\right)_k} \det\left(\frac{1}{2}\frac{\partial}{\partial z_{12}}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\mathrm{tr}(zw)} f(w^{-1}) \det{}'(w)^{-\lambda-k} dw \\
&= \frac{\det{}'(z)^{-\lambda+\frac{r+1}{2}}}{\prod_{i=1}^r \left(\lambda - \frac{i-1}{2}\right)_k} \det\left(\frac{1}{2}\frac{\partial}{\partial z_{12}}\right)^k \det{}'(z)^{\lambda+k-\frac{r+1}{2}} \left\langle f(x), e^{\mathrm{tr}(x\overline{z})} \right\rangle_{\lambda+k,x}.
\end{aligned}$$

Since both sides are single-valued holomorphic with respect to $z \in \mathfrak{p}^+$, both sides coincide for all $z \in \mathfrak{p}^+$. □

5 Results for other symmetric pairs of holomorphic type

In this section we state the theorems in the preprint [22] on top terms and poles of weighted Bergman inner products $\langle\cdot,\cdot\rangle_\lambda$ on bounded symmetric domains $D\simeq G/K$ for other symmetric pairs $(G,(G^\sigma)_0)$ of holomorphic type. Let G be a connected simple Hermitian Lie group with a Cartan involution θ, that is, the maximal compact subgroup $K=G^\theta$ has a 1-dimensional center $Z(K)$. Let σ be an involution of G. Without loss of generality we may assume σ commutes with θ. Then $(G,(G^\sigma)_0)$ is called a symmetric pair of *holomorphic type* if $Z(K)\subset G^\sigma$ (see [13, Section 3.4]). Then the complexified Lie algebras $\mathfrak{g}^{\mathbb{C}}$, $(\mathfrak{g}^{\mathbb{C}})^\sigma$, $(\mathfrak{g}^{\mathbb{C}})^{\sigma\theta}$ are decomposed into the $Ad(Z(K))$-eigenspaces as

$$\begin{array}{ccccccc} \mathfrak{g}^{\mathbb{C}} & = & \mathfrak{p}^+ & \oplus & \mathfrak{k}^{\mathbb{C}} & \oplus & \mathfrak{p}^-, \\ \cup & & \cup & & \cup & & \cup \\ (\mathfrak{g}^{\mathbb{C}})^\sigma & = & (\mathfrak{p}^+)^\sigma & \oplus & (\mathfrak{k}^{\mathbb{C}})^\sigma & \oplus & (\mathfrak{p}^-)^\sigma, \\ (\mathfrak{g}^{\mathbb{C}})^{\sigma\theta} & = & (\mathfrak{p}^+)^{-\sigma} & \oplus & (\mathfrak{k}^{\mathbb{C}})^\sigma & \oplus & (\mathfrak{p}^-)^{-\sigma}. \end{array}$$

Let $(\mathfrak{p}^+)^\sigma=:\mathfrak{p}_1^+$, $(\mathfrak{p}^+)^{-\sigma}=:\mathfrak{p}_2^+$ so that $\mathfrak{p}^+=\mathfrak{p}_1^+\oplus\mathfrak{p}_2^+$, and write $x=(x_1,x_2)\in\mathfrak{p}^+=\mathfrak{p}_1^+\oplus\mathfrak{p}_2^+$. In the following, for simplicity we assume that both $\mathfrak{g}$ and the non-compact ideals of $\mathfrak{g}^{\sigma\theta}$ are of tube type, that is, there exists $e\in\mathfrak{p}_2^+\subset\mathfrak{p}^+$ such that $ad([e,\overline{e}])|_{\mathfrak{p}^+}=2I_{\mathfrak{p}^+}$ holds, where $x\mapsto\overline{x}$ is the complex conjugate with respect to the real form $\mathfrak{g}\subset\mathfrak{g}^{\mathbb{C}}$, although this assumption is not essential. Then both $\mathfrak{p}^+$ and $\mathfrak{p}_2^+$ have Jordan algebra structures with the unit element e, and we have $\operatorname{rank}\mathfrak{p}^+=\operatorname{rank}_{\mathbb{R}}G=:r$, $\operatorname{rank}\mathfrak{p}_2^+=\operatorname{rank}_{\mathbb{R}}(G^\sigma)_0=:r_2$. Let $n:=\dim\mathfrak{p}^+$, and when $r\ge 2$ let $d:=\frac{2(n-r)}{r(r-1)}$. Also, if $\mathfrak{p}_2^+$ is simple, then we define d_2 similarly for $\mathfrak{p}_2^+$. If $r_2=1$, then we cannot define d_2 by this way, and we set $d_2:=2d$. Then one of the following holds.

(1) $\mathfrak{p}_2^+=\mathfrak{p}^{+\prime}\oplus\mathfrak{p}^{+\prime\prime}$ and $r=\operatorname{rank}\mathfrak{p}^{+\prime}+\operatorname{rank}\mathfrak{p}^{+\prime\prime}=:r'+r''$,

(2) $\mathfrak{p}_2^+$ is simple, $r=2r_2$ and $d=d_2/2$,

(3) $\mathfrak{p}_2^+$ is simple, $r=r_2$ and $d=2d_2$,

(4) $\mathfrak{p}_2^+$ is simple, $r=r_2=2$ and $d>d_2$.

First we consider Case (1). Suppose $(G,(G^\sigma)_0,(G^{\sigma\theta})_0)$ is one of the following.

$$\begin{array}{ccccc} (& SO_0(2,d+2), & SO_0(2,d)\times SO(2), & SO_0(2,2)\times SO(d) &) \quad (d=d), \\ (& Sp(r,\mathbb{R}), & U(r',r''), & Sp(r',\mathbb{R})\times Sp(r'',\mathbb{R}) &) \quad (d=1), \\ (& U(r,r), & U(r',r'')\times U(r'',r'), & U(r',r')\times U(r'',r'') &) \quad (d=2), \\ (& SO^*(4r), & U(2r',2r''), & SO^*(4r')\times SO^*(4r'') &) \quad (d=4), \\ (& E_{7(-25)}, & U(1)\times E_{6(-14)}, & SL(2,\mathbb{R})\times Spin_0(2,10) &) \quad (d=8). \end{array}$$

Let $(r,r',r'')=(2,1,1)$ for the 1st case, $(r,r',r'')=(3,1,2)$ for the 5th case. Then we have the following.

Theorem 5.1. *Let* $\mathbf{k}\in\mathbb{Z}_{++}^{r'}$, $\mathbf{l}\in\mathbb{Z}_{++}^{r''}$ *and put* $k_{r'+1}=l_{r''+1}:=0$. *Let* $f(x_2)\in\mathcal{P}_{(\mathbf{k},\mathbf{l})}(\mathfrak{p}_2^+)$.

(1) For $\operatorname{Re}\lambda>\frac{2n}{r}-1$, *we have*

$$\left\langle f(x_2),e^{(x|\overline{z})}\right\rangle_{\lambda,x}\Big|_{z_1=0}=\frac{\prod_{i=1}^{r'}\prod_{j=1}^{r''}\left(\lambda-\frac{d}{2}(i+j-1)\right)_{k_i+l_j}}{\prod_{i=1}^{r'+1}\prod_{j=1}^{r''+1}\left(\lambda-\frac{d}{2}(i+j-2)\right)_{k_i+l_j}}f(z_2)$$

$$=\frac{\prod_{a=2}^{r}\prod_{i=\max\{1,a-r''\}}^{\min\{a-1,r'\}}\left(\lambda-\frac{d}{2}(a-1)\right)_{k_i+l_{a-i}}}{\prod_{a=1}^{r}\prod_{i=\max\{1,a-r''\}}^{\min\{a,r'+1\}}\left(\lambda-\frac{d}{2}(a-1)\right)_{k_i+l_{a+1-i}}}f(z_2).$$

(2) The following is holomorphically continued for all $\lambda \in \mathbb{C}$.

$$\prod_{a=1}^{r}\left(\lambda-\frac{d}{2}(a-1)\right)_{\min\{k_i+l_j \mid 1\le i\le r'+1,\,1\le j\le r''+1,\,i+j=a+1\}}\left\langle f(x_2),e^{(x|\overline{z})}\right\rangle_{\lambda,x}.$$

Next we consider Case (2). Suppose $(G,(G^\sigma)_0,(G^{\sigma\theta})_0)$ is one of the following.

$$\begin{array}{llllll}(& SO_0(2,n), & SO_0(2,n-1), & SO_0(2,1)\times SO(n-1) &) & (d=n-2),\\ (& Sp(2r_2,\mathbb{R}), & Sp(r_2,\mathbb{R})\times Sp(r_2,\mathbb{R}), & U(r_2,r_2) &) & (d=1),\\ (& SU(2r_2,2r_2), & Sp(2r_2,\mathbb{R}), & SO^*(4r_2) &) & (d=2).\end{array}$$

Let $r_2=1$ for the 1st case. Then we have the following.

Theorem 5.2. *Let* $\mathbf{k}\in\mathbb{Z}_{++}^{r_2}$ *and put* $k_{r_2+1}:=0$. *Let* $f(x_2)\in\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$.

(1) For $\operatorname{Re}\lambda>\frac{2n}{r}-1$, *we have*

$$\begin{aligned}&\left\langle f(x_2),e^{(x|\overline{z})}\right\rangle_{\lambda,x}\Big|_{z_1=0}\\ &=\frac{\prod_{1\le i<j\le r_2}\left(\lambda-\frac{d}{2}(i+j-1)\right)_{\lfloor\frac{k_i+k_j}{2}\rfloor}\prod_{1\le i\le j\le r_2}\left(\lambda-\frac12-\frac{d}{2}(i+j-1)\right)_{\lceil\frac{k_i+k_j}{2}\rceil}}{\prod_{1\le i\le j\le r_2+1}\left(\lambda-\frac{d}{2}(i+j-2)\right)_{\lfloor\frac{k_i+k_j}{2}\rfloor}\prod_{1\le i<j\le r_2+1}\left(\lambda-\frac12-\frac{d}{2}(i+j-2)\right)_{\lceil\frac{k_i+k_j}{2}\rceil}}f(z_2)\\ &=\frac{\prod_{a=3}^{2r_2-1}\prod_{i=\max\{1,a-r_2\}}^{\lceil a/2\rceil-1}\left(\lambda-\frac{d}{2}(a-1)\right)_{\lfloor\frac{k_i+k_{a-i}}{2}\rfloor}}{\prod_{a=1}^{2r_2}\prod_{i=\max\{1,a-r_2\}}^{\lceil a/2\rceil}\left(\lambda-\frac{d}{2}(a-1)\right)_{\lfloor\frac{k_i+k_{a+1-i}}{2}\rfloor}}\\ &\qquad\times\frac{\prod_{a=2}^{2r_2}\prod_{i=\max\{1,a-r_2\}}^{\lfloor a/2\rfloor}\left(\lambda-\frac12-\frac{d}{2}(a-1)\right)_{\lceil\frac{k_i+k_{a-i}}{2}\rceil}}{\prod_{a=2}^{2r_2}\prod_{i=\max\{1,a-r_2\}}^{\lfloor a/2\rfloor}\left(\lambda-\frac12-\frac{d}{2}(a-1)\right)_{\lceil\frac{k_i+k_{a+1-i}}{2}\rceil}}f(z_2).\end{aligned}$$

(2) The following is holomorphically continued for all $\lambda\in\mathbb{C}$.

$$\prod_{a=1}^{2r_2}\left(\lambda-\frac{d}{2}(a-1)\right)_{\min\left\{\lfloor\frac{k_i+k_j}{2}\rfloor\,\middle|\,1\le i\le j\le r+1,\,i+j=a+1\right\}}\left\langle f(x_2),e^{(x|\overline{z})}\right\rangle_{\lambda,x}.$$

Next we consider Case (3). Suppose $(G,(G^\sigma)_0,(G^{\sigma\theta})_0)$ is one of the following.

$$\begin{array}{llllll}(& SO^*(4r), & SO^*(2r)\times SO^*(2r), & U(r,r) &) & (d=4),\\ (& SU(r,r), & SO^*(2r), & Sp(r,\mathbb{R}) &) & (d=2),\\ (& E_{7(-25)}, & SU(2,6), & SO^*(12) &) & (d=8).\end{array}$$

Let $r=3$ for the 3rd case. Then we have the following.

Theorem 5.3. *Let* $\mathbf{k}\in\mathbb{Z}_{++}^{r}$ *and put* $k_{r+1}:=0$. *Let* $f(x_2)\in\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$.

(1) For $\operatorname{Re}\lambda>\frac{2n}{r}-1$, *we have*

$$\begin{aligned}\left\langle f(x_2),e^{(x|\overline{z})}\right\rangle_{\lambda,x}\Big|_{z_1=0}&=\frac{\prod_{1\le i<j\le r}\left(\lambda-\frac{d}{4}(i+j-2)\right)_{k_i+k_j}}{\prod_{1\le i<j\le r+1}\left(\lambda-\frac{d}{4}(i+j-3)\right)_{k_i+k_j}}f(z_2)\\ &=\frac{\prod_{a=2}^{2r-2}\prod_{i=\max\{1,a+1-r\}}^{\lfloor a/2\rfloor}\left(\lambda-\frac{d}{4}(a-1)\right)_{k_i+k_{a+1-i}}}{\prod_{a=1}^{2r-1}\prod_{i=\max\{1,a+1-r\}}^{\lceil a/2\rceil}\left(\lambda-\frac{d}{4}(a-1)\right)_{k_i+k_{a+2-i}}}f(z_2).\end{aligned}$$

(2) The following is holomorphically continued for all $\lambda \in \mathbb{C}$.

$$\prod_{a=1}^{r}\left(\lambda - \frac{d}{2}(a-1)\right)_{\min\{k_i+k_j \,|\, 1\le i<j\le r+1,\, i+j=2a+1\}} \left\langle f(x_2), e^{(x|\overline{z})}\right\rangle_{\lambda,x}.$$

Finally we consider Case (4). Suppose

$$(G, (G^\sigma)_0, (G^{\sigma\theta})_0) = (SO_0(2,n), SO_0(2,n') \times SO(n''), SO_0(2,n'') \times SO(n')),$$

with $n = n' + n''$, $n'' \geq 3$. Then we have the following.

Theorem 5.4. *Let $\mathbf{k} \in \mathbb{Z}^2_{++}$, and let $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$.*

(1) For $\operatorname{Re}\lambda > n-1$, we have

$$\left\langle f(x_2), e^{(x|\overline{z})}\right\rangle_{\lambda,x}\Big|_{z_1=0} = \frac{\left(\lambda + k_1 - \frac{n'}{2}\right)_{k_2}}{(\lambda)_{k_1+k_2}\left(\lambda - \frac{n-2}{2}\right)_{k_2}} f(z_2).$$

(2) The following is holomorphically continued for all $\lambda \in \mathbb{C}$.

$$(\lambda)_{k_1+k_2}\left(\lambda - \frac{n-2}{2}\right)_{k_2} \left\langle f(x_2), e^{(x|\overline{z})}\right\rangle_{\lambda,x}.$$

In fact, for this case we have

$$\left\langle f(x_2), e^{(x|\overline{z})}\right\rangle_{\lambda,x} = \frac{\left(\lambda + k_1 - \frac{n'}{2}\right)_{k_2}}{(\lambda)_{k_1+k_2}\left(\lambda - \frac{n-2}{2}\right)_{k_2}} {}_2F_1\left(\begin{matrix} -k_2,\ -k_1 - \frac{n''}{2} + 1 \\ -\lambda - k_1 - k_2 + \frac{n'}{2} + 1 \end{matrix}; -\frac{q(z_1)}{q(z_2)}\right) f(z_2),$$

where $q(z_1)$, $q(z_2)$ are suitable quadratic forms on $\mathfrak{p}_1^+ \simeq \mathbb{C}^{n'}$, $\mathfrak{p}_2^+ \simeq \mathbb{C}^{n''}$. By using these results, we can determine the Parseval–Plancherel-type formulas for the decomposition of $\mathcal{H}_\lambda(D)|_{(\widetilde{G}^\sigma)_0}$, and can determine the branching laws for the discrete Wallach sets. For more detail see the preprint [22].

References

[1] S. Ben Saïd, *Espaces de Bergman pondérés et série discrète holomorphe de $\widetilde{U(p,q)}$*. J. Funct. Anal. **173** (2000), no. 1, 154–181.

[2] S. Ben Saïd, *Weighted Bergman spaces on bounded symmetric domains*. Pacific J. Math. **206** (2002), no. 1, 39–68.

[3] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*. Math. Ann. **217** (1975), no. 3, 271–285.

[4] J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*. J. Funct. Anal. **88** (1990), no. 1, 64–89.

[5] J. Faraut and A. Korányi, *Analysis on symmetric cones*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.

[6] R. Goodman and N.R. Wallach, *Symmetry, representations, and invariants*. Graduate Texts in Mathematics, 255. Springer, Dordrecht, 2009. xx+716 pp.

[7] J. Hilgert and B. Krötz, *Weighted Bergman spaces associated with causal symmetric spaces*. Manuscripta Math. **99** (1999), no. 2, 151–180.

[8] J. Hilgert and B. Krötz, *The Plancherel theorem for invariant Hilbert spaces*. Math. Z. **237** (2001), no. 1, 61–83.

[9] R. Howe, E.C. Tan and J.F. Willenbring, *Stable branching rules for classical symmetric pairs*. Trans. Amer. Math. Soc. **357** (2005), no. 4, 1601–1626.

[10] S. Hwang, Y. Liu and G. Zhang, *Hilbert spaces of tensor-valued holomorphic functions on the unit ball of* $\mathbb{C}^n$. Pacific J. Math. **214** (2004), no. 2, 303–322.

[11] T. Ibukiyama, T. Kuzumaki and H. Ochiai, *Holonomic systems of Gegenbauer type polynomials of matrix arguments related with Siegel modular forms*. J. Math. Soc. Japan **64** (2012), no. 1, 273–316.

[12] A. Juhl, *Families of conformally covariant differential operators, Q-curvature and holography*. Progress in Mathematics, 275. Birkhäuser Verlag, Basel, 2009.

[13] T. Kobayashi, *Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs*. Representation theory and automorphic forms, 45–109, Progr. Math., 255, Birkhäuser Boston, Boston, MA, 2008.

[14] T. Kobayashi, T. Kubo and M. Pevzner, *Conformal symmetry breaking operators for differential forms on spheres*. Lecture Notes in Math. 2170, Springer, Singapore, 2016, ix+192 pp.

[15] T. Kobayashi, B, Ørsted, P. Somberg and V. Souček, *Branching laws for Verma modules and applications in parabolic geometry. I*. Adv. Math. **285** (2015), 1796–1852.

[16] T. Kobayashi and M. Pevzner, *Differential symmetry breaking operators: I. Genreal theory and F-method*. Selecta Math. (N.S.) **22** (2016), no. 2, 801–845.

[17] T. Kobayashi and M. Pevzner, *Differential symmetry breaking operators: II. Rankin–Cohen Operators for Symmetric Pairs*. Selecta Math. (N.S.) **22** (2016), no. 2, 847–911.

[18] T. Kobayashi and M. Pevzner, *Inversion of Rankin–Cohen operators via Holographic Transform*. Ann. Inst. Fourier (Grenoble) **70** (2020), no. 5, 2131–2190.

[19] J. Möllers and Y. Oshima, *Discrete branching laws for minimal holomorphic representations*. J. Lie Theory **25** (2015), no. 4, 949–983.

[20] R. Nakahama, *Norm computation and analytic continuation of vector valued holomorphic discrete series representations*. J. Lie Theory **26** (2016), no. 4, 927–990.

[21] R. Nakahama, *Computation of weighted Bergman inner products on bounded symmetric domains and restriction to subgroups*. SIGMA Symmetry Integrability Geom. Methods Appl. **18** (2022), 033, 105 pages.

[22] R. Nakahama, *Computation of weighted Bergman inner products on bounded symmetric domains and Parseval–Plancherel-type formulas under subgroups*. preprint (2022), arXiv:2207.11663.

[23] B. Ørsted, *Composition series for analytic continuations of holomorphic discrete series representations of* $SU(n,n)$. Trans. Amer. Math. Soc. **260** (1980), no. 2, 563–573.

[24] B. Ørsted and G. Zhang, *Reproducing kernels and composition series for spaces of vector-valued holomorphic functions on tube domains*. J. Funct. Anal. **124** (1994), no. 1, 181–204.

[25] B. Ørsted and G. Zhang, *Reproducing kernels and composition series for spaces of vector-valued holomorphic functions*. Pacific J. Math. **171** (1995), no. 2, 493–510.

[26] L. Peng and G. Zhang, *Tensor products of holomorphic representations and bilinear differential operators*. J. Funct. Anal. **210** (2004), no. 1, 171–192.

[27] R.A. Rankin, *The construction of automorphic forms from the derivatives of a given form.* J. Indian Math. Soc. (N.S.) **20** (1956), 103–116.

[28] H. Sekiguchi, *Branching rules of singular unitary representations with respect to symmetric pairs* (A_{2n-1}, D_n). Internat. J. Math. **24** (2013), no. 4, 1350011, 25 pp.

NTT Institute for Fundamental Mathematics
3-9-11, Midori-cho Musashino-shi, Tokyo, 180-8585, Japan
Email: ryosuke.nakahama@ntt.com