Hyperuniformity of the determinantal point processes associated with the Heisenberg group

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17 March 2022

Abstract

The Ginibre point process is given by the eigenvalue distribution of a non-hermitian complex Gaussian matrix in the infinite matrix-size limit. This is a determinantal point process (DPP) on the complex plane $\mathbb{C}$ in the sense that all correlation functions are given by determinants specified by an integral kernel called the correlation kernel. Shirai introduced the one-parameter ($m \in \mathbb{N}_0$) extensions of the Ginibre DPP and called them the Ginibre-type point processes. In the present paper we consider a generalization of the Ginibre and the Ginibre-type point processes on $\mathbb{C}$ to the DPPs in the higher-dimensional spaces, $\mathbb{C}^D, D = 2, 3, \dots$, in which they are parameterized by a multivariate level $m \in \mathbb{N}_0^D$. We call the obtained point processes the extended Heisenberg family of DPPs, since the correlation kernels are generally identified with the correlations of two points in the space of Heisenberg group expressed by the Schrödinger representations. We prove that all DPPs in this large family are in Class I of hyperuniformity.

Keywords Hyperuniformity; Ginibre and Ginibre-type point processes; Determinantal point processes; Extended Heisenberg family of DPPs; Schrödinger representations of Heisenberg group

1 Introduction and Results

We consider the d-dimensional Euclid space $\mathbb{R}^d$, $d \in \mathbb{N} := \{1, 2, \dots\}$, or the D-dimensional complex space $\mathbb{C}^D$, $D \in \mathbb{N}$ as a base space S. We assume that S is associated with a reference measure λ. We consider an *infinite point process* on S, which is expressed by an infinite sum of delta measures concentrated on a set of random points $X_i, i \in \mathbb{N}$,

$$\Xi = \sum_{i:i \in \mathbb{N}} \delta_{X_i}. \tag{1.1}$$

We assume that for any bounded domain $\Lambda \subset S$, $\Xi(\Lambda) < \infty$; that is, accumulation of points does not occur. We also assume that with respect to the reference measure $\lambda(dx)$ the point process has *a finite density* $\rho_1(x) < \infty$ at almost every $x \in S$.

We consider a *homogeneous* point process in the sense that

$$\rho_1(x)\lambda(dx) = \text{const.} \times dx, \quad x \in S,$$

where dx denotes the Lebesgue measure on S. The above assumption implies that for a bounded domain $\Lambda \subset S$,

$$\mathbf{E}[\Xi(\Lambda)] \propto \text{vol}(\Lambda).$$

Now we consider the *number variance* in the domain Λ,

$$\text{var}[\Xi(\Lambda)] := \mathbf{E}[(\Xi(\Lambda) - \mathbf{E}[\Xi(\Lambda)])^2],$$

which represents local density fluctuation of the point process Ξ. The domain Λ is regarded as an *observation window* to measure the density fluctuation. If the points are non-correlated and given by a Poisson point process, then

$$\text{var}[\Xi(\Lambda)] \propto \text{vol}(\Lambda).$$

Recently in condensed matter physics and related material sciences, correlated particle systems are said to be in a *hyperuniform state* when density fluctuations are anomalously suppressed in large-scale limit. (See, for instance, [8, 23].) For an infinite random point process Ξ, the *hyperuniformity* is defined by

$$\lim_{\Lambda \to S} \frac{\text{var}[\Xi(\Lambda)]}{\mathbf{E}[\Xi(\Lambda)]} = 0.$$

This means that the number variance of points grows more slowly than the window volume in the limit such that the window covers whole of the space $\Lambda \to S$.

Torquato [23] proposed *three hyperuniformity classes* for point processes concerning asymptotics of number variances. In order to clearly assert this classification, here we assume that $S = \mathbb{R}^d$, $d \in \mathbb{N}$, and $\Lambda = \mathbb{B}_R^{(d)} := \{x \in \mathbb{R}^d : |x| < R\}$, $R > 0$, where $\text{vol}(\mathbb{B}_R^{(d)}) = \pi^{d/2} R^d / \Gamma(d/2+1)$ with the gamma function $\Gamma(z) := \int_0^\infty e^{-u} u^{z-1} du$, $\text{Re}\, z > 0$. We consider a series of balls with increasing radius R, $\{\mathbb{B}_R^{(d)}\}_{R>0}$, and the hyperuniform states are classified as follows;

$$\begin{aligned}
&\text{Class I}: && \text{var}[\Xi(\mathbb{B}_R^{(d)})] \asymp R^{d-1}, \\
&\text{Class II}: && \text{var}[\Xi(\mathbb{B}_R^{(d)})] \asymp R^{d-1} \log R, \\
&\text{Class III}: && \text{var}[\Xi(\mathbb{B}_R^{(d)})] \asymp R^{d-\alpha}, \quad 0 < \alpha < 1, \quad \text{as } R \to \infty.
\end{aligned}$$

Here $f(R) \asymp g(R)$ means that there are finite positive constants c_1 and c_2 such that $c_1 g(R) < f(R) < c_2 g(R)$. The above characterization of three classes will be similarly described for any series of windows $\{\Lambda_R\}_{R>0}$ labeled by a linear scale R of window. It is expected that the hyperuniformity and its classification are the proper properties of Ξ and do not depend on the choice of observation window Λ_R, $R > 0$.

Determinantal point processes (DPPs) studied in random matrix theory (RMT) [7] provide a variety of examples of hyperuniform systems. In general a DPP is specified by a triplet

$$(\Xi, K, \lambda(dx)),$$

where Ξ is a nonnegative-integer-valued Radon measure (1.1) representing a point process, K is a continuous function $S \times S \to \mathbb{C}$ called the *correlation kernel*, and $\lambda(dx)$ is a reference measure on S.

The most studied DPP in RMT may be the *sinc* (*sine*) *DPP*, $(\Xi_{\mathrm{sinc}}, K_{\mathrm{sinc}}, dx)$ on $S = \mathbb{R}$ with the correlation kernel

$$K_{\mathrm{sinc}}(x, y) = \frac{\sin(x-y)}{\pi(x-y)}, \quad x, y \in \mathbb{R}.$$

This DPP is obtained as the bulk scaling limit of the eigenvalue distribution of *Hermitian random matrices* in the Gaussian unitary ensemble (GUE) [7]. If $\lim_{R\to\infty} f(R)/g(R) = 1$, we will write $f(R) \sim g(R)$ as $R \to \infty$. As a classical result in RMT, it is well known that

$$\mathrm{var}[\Xi_{\mathrm{sinc}}(\mathbb{B}_R^{(1)})] \sim \frac{\log R}{\pi^2} \quad \text{as } R \to \infty.$$

(See, for instance, [5, 20, 21], [19, Remark 5.8].) That is, the sinc DPP is in Class II of hyperuniformity. Torquato [23] studied one-parameter ($d \in \mathbb{N}$) family of DPPs called the *Fermi-sphere point processes*, which is also called the *Euclidean family of DPPs* in [13]. This family gives the sinc DPP when $d = 1$. It was proved that the Fermi-sphere point processes are in Class II of hyperuniformity for general $d \in \mathbb{N}$ [24, 23].

An example of infinite DPP in Class I of hyperuniformity is also provided in RMT [7]. It is the DPP on $\mathbb{C}$ called the *Ginibre DPP*, $(\Xi_{\mathrm{Ginibre}}, K_{\mathrm{Ginibre}}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$ on $S = \mathbb{C}$, which is obtained as the bulk scaling limit of eigenvalue distribution of *non-Hermitian random matrices* [9]. There

$$K_{\mathrm{Ginibre}}(x, y) = e^{x\overline{y}}, \quad x, y \in \mathbb{C},$$
$$\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx) = \frac{e^{-|x|^2}}{\pi} dx.$$

Note that the disk on $\mathbb{C}$, $\mathbb{D}_R := \{x \in \mathbb{C} : |x| < R\}$, is identified with $\mathbb{B}_R^{(2)} \subset \mathbb{R}^2$. Shirai proved [17]

$$\mathrm{var}[\Xi_{\mathrm{Ginibre}}(\mathbb{B}_R^{(2)})] \sim \frac{R}{\sqrt{\pi}} \quad \text{as } R \to \infty. \tag{1.2}$$

In [14] the result (1.2) in $S = \mathbb{C}$ was extended to DPPs in the higher-dimensional complex spaces $S = \mathbb{C}^D, D = 2, 3 \dots$ as follows. For $S = \mathbb{C}^D, D \in \mathbb{N}$, each coordinate $x \in \mathbb{C}^D$ has D complex components; $x = (x^{(1)}, \dots, x^{(D)})$ with $x^{(\ell)} = \mathrm{Re}\, x^{(\ell)} + \sqrt{-1}\mathrm{Im}\, x^{(\ell)}$, $\ell = 1, \dots, D$. We set $x_{\mathrm{R}} := (\mathrm{Re}\, x^{(1)}, \dots, \mathrm{Re}\, x^{(D)})$, $x_{\mathrm{I}} := (\mathrm{Im}\, x^{(1)}, \dots, \mathrm{Im}\, x^{(D)}) \in \mathbb{R}^D$, and we write $x = x_{\mathrm{R}} + \sqrt{-1}x_{\mathrm{I}}$ in this paper. The Lebesgue measure on $\mathbb{C}^D$ is given by $dx = dx_{\mathrm{R}}dx_{\mathrm{I}} := \prod_{\ell=1}^{D} d\mathrm{Re}\, x^{(\ell)} d\mathrm{Im}\, x^{(\ell)}$. For $x = x_{\mathrm{R}} + \sqrt{-1}x_{\mathrm{I}}$, $y = y_{\mathrm{R}} + \sqrt{-1}y_{\mathrm{I}} \in \mathbb{C}^D$, we use the *standard Hermitian inner product*;

$$\begin{aligned} x \cdot \overline{y} &:= (x_{\mathrm{R}} + \sqrt{-1}x_{\mathrm{I}}) \cdot (y_{\mathrm{R}} - \sqrt{-1}y_{\mathrm{I}}) \\ &= (x_{\mathrm{R}} \cdot y_{\mathrm{R}} + x_{\mathrm{I}} \cdot y_{\mathrm{I}}) - \sqrt{-1}(x_{\mathrm{R}} \cdot y_{\mathrm{I}} - x_{\mathrm{I}} \cdot y_{\mathrm{R}}). \end{aligned}$$

The norm is given by $|x| := \sqrt{x \cdot \overline{x}} = \sqrt{|x_{\mathrm{R}}|^2 + |x_{\mathrm{I}}|^2}$, $x \in \mathbb{C}^D$. Notice that if $x = x_{\mathrm{R}}, y = y_{\mathrm{R}} \in \mathbb{R}^D$, then $x \cdot \overline{y} = x_{\mathrm{R}} \cdot y_{\mathrm{R}} := \sum_{\ell=1}^{D} \mathrm{Re}\, x^{(\ell)} \mathrm{Re}\, y^{(\ell)}$. The D-dimensional disk $\mathbb{D}_R^{(D)} := \{x \in \mathbb{C}^D : |x| < R\}$ is identified with $\mathbb{B}_R^{(d)}$ in $\mathbb{R}^d$ provided that $d = 2D, D \in \mathbb{N}$. On $\mathbb{C}^D$ the reference measure is given by the D-dimensional direct-product of $\lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx)$,

$$\begin{aligned} \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx) &:= \prod_{\ell=1}^{D} \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx^{(\ell)}) \\ &= \frac{e^{-(|x_{\mathrm{R}}|^2 + |x_{\mathrm{I}}|^2)}}{\pi^D} dx_{\mathrm{R}} dx_{\mathrm{I}} = \frac{e^{-|x|^2}}{\pi^D} dx, \quad x \in \mathbb{C}^D. \end{aligned} \tag{1.3}$$

In [14] the one-parameter family ($D \in \mathbb{N}$) of DPPs was studied, which is called the *Heisenberg family of DPPs* defined on $\mathbb{C}^D$ as follows.

Definition 1.1 *The Heisenberg family of DPPs is defined by* $(\Xi_{\mathsf{H}_D}, K_{\mathsf{H}_D}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx))$ *on* $\mathbb{C}^D$, $D \in \mathbb{N}$ *with the correlation kernel*

$$K_{\mathsf{H}_D}(x, y) = e^{x \cdot \overline{y}}, \quad x, y \in \mathbb{C}^D. \tag{1.4}$$

Note that K_{H_D} is hermitian; $\overline{K_{\mathsf{H}_D}(x, y)} = K_{\mathsf{H}_D}(y, x)$, $x, y \in \mathbb{C}^D$. The kernels in this form on $\mathbb{C}^D, D \in \mathbb{N}$ have been studied by Zelditch, who identified them with the Szegő kernels for the *reduced Heisenberg group* $\mathsf{H}_D^{\mathrm{red}}$ [26]. This family includes the Ginibre DPP as the lowest dimensional case; $D = 1$. The following was proved [14].

Theorem 1.2 *Any DPP in the Heisenberg family,* $(\Xi_{\mathsf{H}_D}, K_{\mathsf{H}_D}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx))$ *on* $\mathbb{C}^D$, $D \in \mathbb{N}$, *is in Class I of hyperuniformity such that*

$$\lim_{R \to \infty} R \frac{\mathrm{var}[\Xi_{\mathsf{H}_D}(\mathbb{B}_R^{(2D)})]}{\mathbf{E}[\Xi_{\mathsf{H}_D}(\mathbb{B}_R^{(2D)})]} = \frac{D}{\sqrt{\pi}}.$$

Moreover, for each $D \in \mathbb{N}$, *the following asymptotic expansion holds,*

$$\frac{\mathrm{var}[\Xi_{\mathsf{H}_D}(\mathbb{B}_R^{(2D)})]}{\mathbf{E}[\Xi_{\mathsf{H}_D}(\mathbb{B}_R^{(2D)})]} \sim \frac{D}{\sqrt{\pi}} R^{-1} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_k(D)}{(2k+1)k! 2^{4k}} R^{-2k} \quad \text{as } R \to \infty,$$

where

$$\alpha_k(D) = \begin{cases} 1, & \text{if } k = 0, \\ \displaystyle\prod_{\ell=-k+1}^{k} (2D + 2\ell - 1), & \text{if } k \in \mathbb{N}. \end{cases}$$

Instead of $\mathbb{B}_R^{(2D)} \simeq \mathbb{D}_R^{(D)}$, $D \in \mathbb{N}$, we can consider the *D-dimensional polydisk* of radius $R > 0$ in $\mathbb{C}^D$,

$$\Delta_R^{(D)} := \{x = (x^{(1)}, \dots, x^{(D)}) \in \mathbb{C}^D : |x^{(i)}| < R, i = 1, \dots, D\}, \tag{1.5}$$

as an observation window. In Remark 5 of [14], the following was proved.

Proposition 1.3 *For the Heisenberg family of DPPs, $(\Xi_{\mathsf{H}_D}, K_{\mathsf{H}_D}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx))$, $D \in \mathbb{N}$,*

$$\begin{aligned}\lim_{R\to\infty} R \frac{\mathrm{var}[\Xi_{\mathsf{H}_D}(\Delta_R^{(D)})]}{\mathbf{E}[\Xi_{\mathsf{H}_D}(\Delta_R^{(D)})]} &= \lim_{R\to\infty} R \left[1 - \left(1 - \frac{\mathrm{var}[\Xi_{\mathsf{H}_1}(\mathbb{B}_R^{(2)})]}{\mathbf{E}[\Xi_{\mathsf{H}_1}(\mathbb{B}_R^{(2)})]}\right)^D\right] \\ &= \frac{D}{\sqrt{\pi}}.\end{aligned}$$

The *Laguerre polynomial* is defined by

$$L_n^{(\alpha)}(\zeta) := \frac{\zeta^{-\alpha} e^{\zeta}}{n!} \frac{d^n}{d\zeta^n}(\zeta^{n+\alpha} e^{-\zeta}), \quad n \in \mathbb{N}_0 := \{0, 1, \dots\}, \quad \alpha, \zeta \in \mathbb{R}. \tag{1.6}$$

In particuler, we wite $L_n(\zeta) := L_n^{(0)}(\zeta)$. Note that $L_0^{(\alpha)}(\zeta) = 1$ for any $\alpha \in \mathbb{R}$. Let ${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z)$ be the hypergeometric function defined by

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n (\alpha_3)_n}{(\beta_1)_n (\beta_2)_n} \frac{z^n}{n!},$$

where $(\alpha)_n$ is the Pochhammer symbol; $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1), n \in \mathbb{N}$, $(\alpha)_0 := 1$. It is obvious that

$${}_3F_2(\alpha_1, \alpha_2, 0; \beta_1, \beta_2; z) = 1. \tag{1.7}$$

For $D = 1$, let

$$K_{\mathsf{H}_1}^{(m)}(x, y) = K_{\mathsf{H}_1}(x, y) L_m(|x - y|^2), \quad m \in \mathbb{N}_0, \quad x, y \in \mathbb{C}.$$

The one-parameter ($m \in \mathbb{N}_0$) family of DPPs, $(\Xi_{\mathsf{H}_1}^{(m)}, K_{\mathsf{H}_1}^{(m)}, \lambda_{\mathrm{N}(0,1;\mathbb{C})})$, $m \in \mathbb{N}_0$ on $\mathbb{C}$ was studied by Shirai [18], who called the DPPs in this family the *Ginibre-type point processes*. This family of DPPs was also studied by Haimi and Hedenmalm [11], where they called the DPPs the *polyanalytic Ginibre ensembles*. This family of DPPs on $\mathbb{C}$ has a physical interpretation in terms of a two-dimensional system of free electrons in a uniform magnetic field, where the electrons occupy the *m-th Landau energy level*, $m \in \mathbb{N}_0$. Shirai proved the following [18].

Proposition 1.4 *Any Ginibre-type DPP on $\mathbb{C}$, $(\Xi_{\mathsf{H}_1}^{(m)}, K_{\mathsf{H}_1}^{(m)}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$, $m \in \mathbb{N}_0$, is in Class I of hyperuniformity such that*

$$\lim_{R\to\infty} R \frac{\mathrm{var}[\Xi_{\mathsf{H}_1}^{(m)}(\mathbb{B}_R^{(2)})]}{\mathbf{E}[\Xi_{\mathsf{H}_1}^{(m)}(\mathbb{B}_R^{(2)})]} = C_{\mathsf{H}_1}^{(m)}$$

with

$$C_{\mathsf{H}_1}^{(m)} = \frac{2\Gamma(m+3/2)}{\pi m!} {}_3F_2\left(-\frac{1}{2}, -\frac{1}{2}, -m; 1, -\frac{1}{2} - m; 1\right)$$
$$\sim \frac{8}{\pi^2} m^{1/2} \quad \text{as } m \to \infty. \tag{1.8}$$

In the present paper, we consider a large family of DPPs which includes both of the Heisenberg family of DPPs and the Ginibre-type DPPs, which is parameterized by the dimensionality D of the base space $S = \mathbb{C}^D$ and the *multivariate level* expressed by a set of integers, $m = (m^{(1)}, \dots, m^{(D)}) \in \mathbb{N}_0^D$.

Definition 1.5 *The extended Heisenberg family of DPPs is defined by* $(\Xi_{\mathsf{H}_D}^{(m)}, K_{\mathsf{H}_D^{(m)}}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx))$ *on* $\mathbb{C}^D$, $D \in \mathbb{N}$ *with a multivariate level* $m \in \mathbb{N}_0^D$, *where the correlation kernel is given by*

$$K_{\mathsf{H}_D}^{(m)}(x, y) = K_{\mathsf{H}_D}(x, y) \prod_{\ell=1}^{D} L_{m^{(\ell)}}(|x^{(\ell)} - y^{(\ell)}|^2), \quad x, y \in \mathbb{C}^D. \tag{1.9}$$

Here K_{H_D} *is defined by (1.4).*

When $m = 0 := (0, \dots, 0)$, this family is reduced to the original Heisenberg family defined by Theorem 1.1. It is obvious that when $D = 1$ this family is identified with the Ginibre-type DPPs of Shirai [18].

In [18] Shirai gave a sufficient condition of Class I of hyperuniformity for DPPs on $\mathbb{R}^d$ with general dimensions $d \geq 1$. There he assumed that the observation windows are balls $\mathbb{B}_R^{(d)}$ and the DPPs are *isotropic* in a sense. We can apply this general theorem to the Heisenberg family of DPPs, $(\Xi_{\mathsf{H}_D}, K_{\mathsf{H}_D}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx))$ for all $D \in \mathbb{N}$ and the Ginibre-type DPPs on $\mathbb{C}$, $(\Xi_{\mathsf{H}_1}^{(m)}, K_{\mathsf{H}_1}^{(m)}, \lambda_{\mathrm{N}(0,1;\mathbb{C})}(dx))$ for all $m \in \mathbb{N}_0$. But, it is not applicable to the extended Heisenberg family of DPPs on $\mathbb{C}^D$, $(\Xi_{\mathsf{H}_D}^{(m)}, K_{\mathsf{H}_D}^{(m)}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx))$, if $D \geq 2$ and $m \neq 0$, since

$$\left| \sqrt{\frac{e^{-|x|^2}}{\pi^D}} K_{\mathsf{H}_D}^{(m)}(x, x') \sqrt{\frac{e^{-|x'|^2}}{\pi^D}} \right|^2 = \frac{1}{\pi^{2D}} e^{-|x-x'|^2} \prod_{\ell=1}^{D} L_{m^{(\ell)}}(|x^{(\ell)} - x'^{(\ell)}|^2)^2$$

is not a function of $|x - x'|^2 = \sum_{\ell=1}^{n} |x^{(\ell)} - x'^{(\ell)}|^2$. In other words, the DPPs in the extended Heisenberg family with $D \geq 2$ or $m \neq 0$ are not isotropic.

In the present paper, however, we prove the following result for the extended Heisenberg family of DPPs, when observation windows are given by the D-dimensional polydisks (1.5).

Theorem 1.6 *For the extended Heisenberg family of DPPs,* $(\Xi_{\mathsf{H}_D}^{(m)}, K_{\mathsf{H}_D}^{(m)}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx))$, $D \in \mathbb{N}$, $m \in \mathbb{N}_0^D$,

$$\lim_{R\to\infty} R \frac{\mathrm{var}[\Xi_{\mathsf{H}_D}^{(m)}(\Delta_R^{(D)})]}{\mathbf{E}[\Xi_{\mathsf{H}_D}^{(m)}(\Delta_R^{(D)})]} = \lim_{R\to\infty} R \left[1 - \prod_{\ell=1}^{D} \left(1 - \frac{\mathrm{var}[\Xi_{\mathsf{H}_1}^{(m^{(\ell)})}(\mathbb{B}_R^{(2)})]}{\mathbf{E}[\Xi_{\mathsf{H}_1}^{(m^{(\ell)})}(\mathbb{B}_R^{(2)})]} \right) \right]$$
$$= \sum_{\ell=1}^{D} C_{\mathsf{H}_1}^{(m^{(\ell)})}. \tag{1.10}$$

By (1.7), (1.8) gives

$$C_{\mathsf{H}_1}^{(0)} = \frac{2\Gamma(3/2)}{\pi} = \frac{1}{\sqrt{\pi}}.$$

Hence, this result can be regarded as a generalization of Proposition 1.3.

In Section 2, we show the relationship between the extended Heisenberg family of DPPs and the *Schrödinger representations of the Heisenberg group* H_D, $D \in \mathbb{N}$. There the function denoted by g specifies the representation. The correlation kernels $K_{\mathsf{H}_D}^{(m)}$, $D \in \mathbb{N}$, $m \in \mathbb{N}_0^D$ are identified with the correlations of two points in the space of H_D expressed using the Schrödinger operators acting on g. In Section 3, after giving preliminaries for point processes and comments on Theorem 1.2, the proof of Theorem 1.6 is given.

By Abreu and his coworkers [2, 3], the DPPs associated with the Schrödinger representations of H_D are named as the *Weyl–Heisenberg ensembles*, in which g is called a *window function* in the context of the *time-frequency analysis* [10]. See also Section 2.6 of [13]. It was proved by Abreu et al. [4] that the Weyl–Heisenberg ensembles are in the hyperuniform state of Class I for a general class of window functions.

2 Representations of the Heisenberg Group and Correlation Kernels

2.1 Schrödinger representations

We briefly review the representation theory of the *Heisenberg group* [6, 22, 10] in order to explain the reason why we call the DPPs defined by Definition 1.1 the *Heisenberg family* of DPPs and why we think that the DPPs defined by Definition 1.5 form its extended family.

Consider the classical and quantum kinetics of a single particle moving in $\mathbb{R}^D$, $D \in \mathbb{N}$. We note that, if $D = 3k$, $k \in \mathbb{N}$, this represents a k-particle system in the three dimensional Euclidean space. The *phase space* is given by $\mathbb{R}^{2D}$ with coordinates

$$(p, q) = (p_1, \dots, p_D, q_1, \dots, q_D).$$

In order to describe the *Heisenberg Lie algebra* h_D, we consider $\mathbb{R}^{2D+1}$ with coordinates $(p, q, \tau) = (p_1, \dots, p_D, q_1, \dots, q_D, \tau)$, in which a Lie bracket is given by

$$[(p, q, \tau), (p', q', \tau')] = (0, 0, p \cdot q' - q \cdot p') = (0, 0, [(p, q), (p', q')]).$$

The symplectic form of the *Lie bracket* $[(p,q),(p',q')] = p\cdot q' - q\cdot p'$ comes from the Poisson bracket in the classical mechanics and the commutator $[A,B] := AB - BA$ in quantum mechanics. The *Heisenberg group* H_D is the Lie group on $\mathbb{R}^{2D+1}$ satisfying the group law

$$ZZ' = Z + Z' + \frac{1}{2}[Z, Z'], \quad Z, Z' \in \mathbb{R}^{2D+1};$$

that is,

$$(p,q,\tau)(p',q',\tau') = \left(p+p', q+q', \tau+\tau' + \frac{1}{2}(p\cdot q' - q\cdot p')\right).$$

Let $L^2(\mathbb{R}^D)$ be the set of square integrable functions on $\mathbb{R}^D$, where the inner product is given by

$$\langle f, g\rangle_{L^2(\mathbb{R}^D)} := \int_{\mathbb{R}^D} f(\zeta)\overline{g(\zeta)}d\zeta, \quad f, g \in L^2(\mathbb{R}^D)$$

with the norm $\|f\|_{L^2(\mathbb{R}^D)} := \sqrt{\langle f, f\rangle_{L^2(\mathbb{R}^D)}}$, $f \in L^2(\mathbb{R}^D)$, where $\zeta = (\zeta^{(1)}, \dots, \zeta^{(D)}) \in \mathbb{R}^D$ and $d\zeta$ denotes the Lebesgue measure on $\mathbb{R}^D$. For a smooth function f, we introduce operators $X^{(\ell)}$ and $\mathcal{D}^{(\ell)}$ defined by

$$(X^{(\ell)}f)(\zeta) = \zeta^{(\ell)}f(\zeta), \quad (\mathcal{D}^{(\ell)}f)(\zeta) = \frac{1}{2\sqrt{-1}}\frac{\partial f}{\partial \zeta^{(\ell)}}(\zeta), \quad \ell = 1, \dots, D.$$

They satisfy the commutation relations

$$[X^{(\ell)}, \mathcal{D}^{(\ell')}] = \frac{\sqrt{-1}}{2}\delta_{\ell\ell'}, \quad \ell, \ell' = 1, \dots, D.$$

Note that the above will represent the *canonical commutation relations* in quantum mechanics, $[Q^{(\ell)}, P^{(\ell')}] = \sqrt{-1}\hbar\delta_{\ell\ell'}$. Here we should claim that the value of the Planck constant $\hbar$ is specially chosen to be $1/2$. This corresponds to the choice of the reference measure on $\mathbb{C}^D$ as (1.3). We consider a map from H_D to the group of unitary operators acting on $L^2(\mathbb{R}^D)$ defined by

$$\rho(p,q,\tau) = e^{2\sqrt{-1}(p\cdot\mathcal{D} + q\cdot X + \tau I)},$$

where $\mathcal{D} := (\mathcal{D}^{(1)}, \dots, \mathcal{D}^{(D)})$, $X := (X^{(1)}, \dots, X^{(D)})$ and I denotes the identity operator. By the Baker–Campbell–Hausdorff formula, we can show that

$$\rho(p,q,\tau)f(\zeta) = e^{2\sqrt{-1}(\tau + q\cdot\zeta + p\cdot q/2)}f(\zeta + p), \quad f \in L^2(\mathbb{R}^D).$$

The map ρ is called the *Schrödinger representation* of H_D. The kernel of ρ is $\{(0,0,k\pi) : k \in \mathbb{Z}\}$, since $e^{2\pi k\sqrt{-1}} = 1, k \in \mathbb{Z}$. The *reduced Heisenberg group* $\mathsf{H}_D^{\mathrm{red}}$ is defined by $\mathsf{H}_D^{\mathrm{red}} := \mathsf{H}_D/\{(0,0,k\pi) : k \in \mathbb{Z}\}$. We consider the *matrix-element functions* of $\rho(p,q,\tau)$ at $(f,g) \in L^2(\mathbb{R}^D)^2$,

$$\begin{aligned} M_{f,g}(p,q,\tau) &:= \langle \rho(p,q,\tau)f, g\rangle_{L^2(\mathbb{R}^D)} = \langle \rho(-p,q,\tau)\overline{g}, \overline{f}\rangle_{L^2(\mathbb{R}^D)} \\ &= e^{2\sqrt{-1}\tau}\int_{\mathbb{R}^D} e^{2\sqrt{-1}q\cdot\zeta} f\left(\zeta + \frac{p}{2}\right)\overline{g\left(\zeta - \frac{p}{2}\right)}d\zeta, \quad f, g \in L^2(\mathbb{R}^2), \end{aligned} \tag{2.1}$$

which is also called the *Fourier–Wigner transform* (see, for instance, [6, Section 1.4]).

2.2 Ground-state representation

We put

$$g(\zeta) = \left(\frac{2}{\pi}\right)^{D/4} \frac{e^{-\zeta^2}}{\pi^{D/2}} =: G_0(\zeta), \quad \zeta \in \mathbb{R}^D, \tag{2.2}$$

and define the complex variables

$$x = (x^{(1)}, \dots, x^{(D)}) := p + \sqrt{-1}q = (p^{(1)} + \sqrt{-1}q^{(1)}, \dots, p^{(D)} + \sqrt{-1}q^{(D)}) \in \mathbb{C}^D.$$

Then (2.1) becomes

$$M_{f,G_0}(p,q,\tau) = e^{2\sqrt{-1}\tau} \mathsf{B}[f](x) \frac{e^{-|x|^2/2}}{\pi^{D/2}}$$

with

$$\mathsf{B}[f](x) := \left(\frac{2}{\pi}\right)^{D/4} \int_{\mathbb{R}^D} f(\zeta) e^{2\zeta\cdot x - \zeta^2 - x^2/2} d\zeta, \quad f \in L^2(\mathbb{R}^D). \tag{2.3}$$

This integral is called the *Bargmann transform*. For $f \in L^2(\mathbb{R}^D)$, this integral converges uniformly for x in any compact subset of $\mathbb{C}^D$, and hence $\mathsf{B}[f]$ is an entire function on $\mathbb{C}^D$. We can prove that for $f_1, f_2 \in L^2(\mathbb{R}^D)$

$$\langle M_{f_1,G_0}, M_{f_2,G_0}\rangle_{L^2(\mathbb{R}^{2D})} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^D)} = \langle \mathsf{B}[f_1], \mathsf{B}[f_2]\rangle_{L^2(\mathbb{C}^D, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)})}, \tag{2.4}$$

where $\lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}$ is given by (1.3) and

$$\langle F_1, F_2\rangle_{L^2(\mathbb{C}^D, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)})} := \int_{\mathbb{C}^D} F_1(x)\overline{F_2(x)} \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx)$$

with the norm $\|F\|_{L^2(\mathbb{C}^D, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)})} := \sqrt{\langle F, F\rangle_{L^2(\mathbb{C}^D, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)})}}$. The Bargmann–Fock space $\mathcal{F}_D$ is defined by

$$\mathcal{F}_D := \left\{ F : F \text{ is entire on } \mathbb{C}^D \text{ and } \|F\|_{L^2(\mathbb{C}^D, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)})} < \infty \right\}.$$

The equalities (2.4) imply that the Bargmann transform is an *isometry* from $L^2(\mathbb{R}^D)$ into $\mathcal{F}_D$. Hence, if $\{f_n\}_{n\in\mathbb{N}_0}$ is a complete orthonormal system (CONS) of $L^2(\mathbb{R}^D)$, then $\{\mathsf{B}[f_n]\}_{n\in\mathbb{N}_0^D}$ makes a CONS for $\mathcal{F}_D$.

For $(p,q,\tau), (p',q',\tau') \in \mathbb{R}^{2D+1}$, we consider the correlation of these two points expressed by the Schrödinger operators acting on G_0 such that

$$\mathrm{Cor}_{G_0}((p,q,\tau),(p',q',\tau')) := \langle \rho(-p,q,\tau)G_0, \rho(-p',q',\tau')G_0\rangle_{L^2(\mathbb{R}^D)}. \tag{2.5}$$

We take a CONS $\{f_n\}_{n\in\mathbb{N}_0}$ of $L^2(\mathbb{R}^D)$, where we assume $f_n \in \mathbb{R}$, $n \in \mathbb{N}_0$. Then the two-point correlation (2.5) is expanded and expressed using the matrix-element functions

(2.1) as

$$\begin{aligned}
\mathrm{Cor}_{G_0}((p,q,\tau),(p',q',\tau')) &= \sum_{n\in\mathbb{N}_0} \langle \rho(-p,q,\tau)G_0, f_n\rangle_{L^2(\mathbb{R}^D)} \langle f_n, \rho(-p',q',\tau')G_0\rangle_{L^2(\mathbb{R}^D)} \\
&= \sum_{n\in\mathbb{N}_0} \langle \rho(p,q,\tau)f_n, G_0\rangle_{L^2(\mathbb{R}^D)} \langle G_0, \rho(p',q',\tau')f_n\rangle_{L^2(\mathbb{R}^D)} \\
&= \sum_{n\in\mathbb{N}_0} M_{f_n,G_0}(p,q,\tau)\overline{M_{f_n,G_0}(p',q',\tau')} \\
&= \frac{e^{-(|x|^2+|x'|^2)/2}}{\pi^D} \sum_{n\in\mathbb{N}_0} \mathsf{B}[f_n](x)\overline{\mathsf{B}[f_n](x')}. \qquad (2.6)
\end{aligned}$$

The Hermite polynomials on $\mathbb{R}$ are defined by

$$H_n(\zeta) := (-1)^n e^{\zeta^2} \frac{d^n}{d\zeta^n} e^{-\zeta^2}, \quad n \in \mathbb{N}_0, \quad \zeta \in \mathbb{R},$$

and then the *Hermite orthonormal functions* are given by

$$\psi_n(\zeta) := \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\zeta^2/2} H_n(\zeta), \quad n \in \mathbb{N}_0, \quad \zeta \in \mathbb{R}; \qquad (2.7)$$

$\int_{\mathbb{R}} \psi_n(\zeta)\psi_m(\zeta)d\zeta = \delta_{nm}$, $n, m \in \mathbb{N}_0$. Here we make a slight modification as

$$\widetilde{\psi}_n(\zeta) := 2^{1/4}\psi_n(\sqrt{2}\zeta), \quad n \in \mathbb{N}_0, \quad \zeta \in \mathbb{R}.$$

$\{\widetilde{\psi}_n\}_{n\in\mathbb{N}_0}$ makes a *real* CONS of $L^2(\mathbb{R})$, $\int_{\mathbb{R}} \widetilde{\psi}_n(\xi)\widetilde{\psi}_m(\xi)d\xi = \delta_{nm}$, $n, m \in \mathbb{N}_0$, as assumed above.

We extend the above results on $\mathbb{R}$ to $\mathbb{R}^D$, $D \in \mathbb{N}$ by simply considering direct products; for $n := (n^{(1)}, \dots, n^{(D)}) \in \mathbb{N}_0^D$, $\zeta := (\zeta^{(1)}, \dots, \zeta^{(D)}) \in \mathbb{R}^D$,

$$\Psi_n(\zeta) := \prod_{\ell=1}^{D} \widetilde{\psi}_{n^{(\ell)}}(\zeta^{(\ell)}). \qquad (2.8)$$

We can show that

$$\begin{aligned}
\mathsf{B}[\Psi_n](x) &:= \prod_{\ell=1}^{D} \mathsf{B}[\widetilde{\psi}_{n^{(\ell)}}](x^{(\ell)}) \\
&= \prod_{\ell=1}^{D} \phi_{n^{(\ell)}}(x^{(\ell)}) =: \Phi_n(x), \quad n \in \mathbb{N}_0^D, \quad x \in \mathbb{C}^D,
\end{aligned}$$

where

$$\phi_n(x) := \frac{x^n}{\sqrt{n!}}, \quad n \in \mathbb{N}_0, \quad x \in \mathbb{C}. \qquad (2.9)$$

Hence (2.6) is calculated as

$$\begin{aligned}\mathrm{Cor}_{G_0}((p,q,\tau),(p',q',\tau')) &= \frac{e^{-(|x|^2+|x'|^2)/2}}{\pi^D}\sum_{n\in\mathbb{N}_0^D}\Phi_n(x)\overline{\Phi_n(x')}\\ &= \frac{e^{-(|x|^2+|x'|^2)/2}}{\pi^D}\prod_{\ell=1}^{D}\sum_{n^{(\ell)}=0}^{\infty}\frac{1}{n^{(\ell)}!}\left(x^{(\ell)}\overline{x'^{(\ell)}}\right)^{n^{(\ell)}}\\ &= \sqrt{\frac{e^{-|x|^2}}{\pi^D}}K_{\mathsf{H}_D}(x,x')\sqrt{\frac{e^{-|x'|^2}}{\pi^D}}, \quad x,x'\in\mathbb{C}^D.\end{aligned}$$

By the gauge invariance of DPP (see Lemma 3.2 below), this can be identified with the correlation kernel $K_{\mathsf{H}_D}(x,x')$ of the Heisenberg family of DPPs given by (1.4). Note that this result is obtained due to the special choice of g given by (2.2), which is also written as

$$g(\zeta)=G_0(\zeta):=\frac{1}{\pi^{D/2}}\Psi_0(\zeta), \quad \zeta\in\mathbb{R}^D. \tag{2.10}$$

We think that (2.8) represents the eigenstate in a multivariate level $n\in\mathbb{N}_0^D$ of the D-dimensional direct-product system of harmonic oscillators. Since $\Psi_0(\zeta)$ gives the ground state, the above results on the choice (2.10) can be regarded as the *ground-state representation*.

2.3 Higher-level representations

Now we consider the following general choice of g,

$$\begin{aligned}&g(\zeta)=G_m(\zeta):=\frac{1}{\pi^{D/2}}\Psi_m(\zeta),\\ &\qquad m=(m^{(1)},\dots,m^{(D)})\in\mathbb{N}_0^D, \quad \zeta=(\zeta^{(1)},\dots,\zeta^{(D)})\in\mathbb{R}^D,\end{aligned}$$

which corresponds to the higher-level state of the D-dimensional system of harmonic oscillators. The matrix-element function is then given by

$$M_{f,G_m}(p,q,\tau):=\langle\rho(p,q,\tau)f,G_m\rangle_{L^2(\mathbb{R}^D)}=e^{2\sqrt{-1}\tau}\mathsf{B}_m[f](x,\overline{x})\frac{e^{-|x|^2/2}}{\pi^{D/2}},$$

where

$$\begin{aligned}\mathsf{B}_m[f](x,\overline{x}) &:= \left(\frac{2}{\pi}\right)^{D/4}\frac{1}{\sqrt{2^m m!}}\int_{\mathbb{R}^D}f(\zeta)e^{2\zeta\cdot x-\zeta^2-x^2/2}\mathbf{H}_m\left(\sqrt{2}\left(\zeta-\frac{x+\overline{x}}{2}\right)\right)d\zeta\\ &= \frac{1}{\sqrt{m!}}e^{|x|^2}\frac{\partial^m}{\partial x^m}\left[e^{-|x|^2}\mathsf{B}[f](x)\right].\end{aligned}$$

Here we have used the multivariate notations,

$$2^m:=\prod_{\ell=1}^{D}2^{m^{(\ell)}}, \quad m!:=\prod_{\ell=1}^{D}m^{(\ell)}!, \quad \frac{\partial^m}{\partial x^m}:=\prod_{\ell=1}^{D}\frac{\partial^{m^{(\ell)}}}{\partial {x^{(\ell)}}^{m^{(\ell)}}}, \quad \mathbf{H}_m(\xi):=\prod_{\ell=1}^{D}H_{m^{(\ell)}}(\xi^{(\ell)}).$$

By the definition (2.3), $\mathsf{B}_0[f](x) = \mathsf{B}[f](x)$. The integral transformation $\mathsf{B}_m[f](x, \overline{x})$, $m \in \mathbb{N}_0^D$ is called the *true-m-Bargmann transform* by Vasilevski [25], the *polyanalytic Bargmann transform* by Abreu and Feichtinger [1], and has been studied as the *coherent state transform* by Mouayn [15].

We can read from Shirai's paper [18] that, for (2.9),

$$\frac{1}{\sqrt{m!}} e^{|x|^2} \frac{\partial^m}{\partial x^m} \left[e^{-|x|^2} \phi_n(x) \right] = \sqrt{\frac{m!}{n!}} L_m^{(n-m)}(|x|^2) x^{n-m}, \quad n, m \in \mathbb{N}_0, \quad x \in \mathbb{C},$$

where $L_n^{(\alpha)}$ is defined by (1.6). Hence, we have

$$\mathsf{B}_m[\Phi_n](x, \overline{x}) = \prod_{\ell=1}^{D} \sqrt{\frac{m^{(\ell)}!}{n^{(\ell)}!}} L_{m^{(\ell)}}^{(n^{(\ell)}-m^{(\ell)})}(|x^{(\ell)}|^2) x^{n^{(\ell)}-m^{(\ell)}}$$

for $m = (m^{(1)}, \dots, m^{(D)}), n = (n^{(1)}, \dots, n^{(D)}) \in \mathbb{N}_0^D$, $x = (x^{(1)}, \dots, x^{(D)}) \in \mathbb{R}^D$. It was also shown in [18] that

$$\sum_{n=0}^{\infty} \frac{1}{n!} L_m^{(n-m)}(|x|^2) L_m^{(n-m)}(|x'|^2)(x\overline{x'})^{n-m} = \frac{1}{m!} e^{x \cdot \overline{x'}} L_m(|x - x'|^2), \quad m \in \mathbb{N}_0, \quad x, x' \in \mathbb{C}.$$

For each multivariate level $m \in \mathbb{N}_0^D$, as an extension of (2.5), the correlation of two points (p, q, τ) and (p', q', τ') in the space of H_D shall be given by

$$\mathrm{Cor}_{G_m}((p, q, \tau), (p', q', \tau')) := \langle \rho(-p, q, \tau) G_m, \rho(-p', q', \tau') G_m \rangle_{L^2(\mathbb{R}^D)}.$$

It is evaluated as

$$\begin{aligned}
\mathrm{Cor}_{G_m}((p, q, \tau), (p', q', \tau')) &= \sum_{n \in \mathbb{N}_0^D} M_{\Phi_n, G_m}(p, q, \tau) \overline{M_{\Phi_n, G_m}(p', q', \tau')} \\
&= \frac{e^{-(|x|^2 + |x'|^2)/2}}{\pi^D} \sum_{n \in \mathbb{N}_0^D} \mathsf{B}_m[\Phi_n](x, \overline{x}) \overline{\mathsf{B}_m[\Phi_n](x', \overline{x'})} \\
&= \frac{e^{-(|x|^2 + |x'|^2)/2}}{\pi^D} e^{x \cdot \overline{x'}} \prod_{\ell=1}^{D} L_{m^{(\ell)}}(|x^{(\ell)} - x'^{(\ell)}|^2) \\
&= \sqrt{\frac{e^{-|x|^2}}{\pi^D}} K_{\mathsf{H}_D}^{(m)}(x, x') \sqrt{\frac{e^{-|x'|^2}}{\pi^D}}, \quad x, x' \in \mathbb{C}^D.
\end{aligned}$$

By Lemma 3.2, this can be identified with the correlation kernel $K_{\mathsf{H}_D}^{(m)}(x, x')$ of the extended Heisenberg family of DPPs given by (1.9).

3 Preliminaries and Proofs

3.1 Preliminaries for point processes

The configuration space of *point process* $\Xi = \Xi(\cdot)$ is given by

$$\mathrm{Conf}(S) = \Big\{ \xi = \sum_i \delta_{x_i} : x_i \in S,\, \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \Big\}.$$

Let $\mathcal{B}_{\mathrm{c}}(S)$ be the set of all bounded measurable complex functions on S of compact support, and for $\xi \in \mathrm{Conf}(S)$ and $\phi \in \mathcal{B}_{\mathrm{c}}(S)$ we set

$$\langle \xi, \phi \rangle := \int_S \phi(x)\, \xi(dx) = \sum_i \phi(x_i).$$

Random variables written in this form are generally called *linear statistics* of a point process Ξ [7]. For a point process Ξ, if there exists a non-negative measurable function ρ_1 such that

$$\mathbf{E}[\langle \Xi, \phi \rangle] = \int_S \phi(x) \rho_1(x) \lambda(dx) \quad \forall \phi \in \mathcal{B}_{\mathrm{c}}(S).$$

ρ_1 is called the *first correlation function* of Ξ with respect to the reference measure $\lambda(dx)$. By definition, $\rho_1(x)$ gives the *density of point* at $x \in S$ with respect to $\lambda(dx)$. For $n \in \mathbb{N}$, from $\xi \in \mathrm{Conf}(S)$ we define

$$\xi_n := \sum_{i_1, \ldots, i_n : i_j \neq i_k, j \neq k} \delta_{x_{i_1}} \cdots \delta_{x_{i_n}},$$

and denote the n-product measure of λ as $\lambda^{\otimes n}(dx_1 \cdots dx_n) := \prod_{i=1}^n \lambda(dx_i)$. For a point process Ξ, if there exists a symmetric, non-negative measurable function ρ_n on S^n such that

$$\mathbf{E}[\langle \Xi_n, \phi \rangle] = \int_{S^n} \phi(x_1, \ldots, x_n) \rho_n(x_1, \ldots, x_n) \lambda^{\otimes n}(dx_1 \cdots dx_n) \quad \forall \phi \in \mathcal{B}_{\mathrm{c}}(S^n),$$

then we say that ρ_n is the *n-th correlation function* of Ξ with respect to $\lambda(dx)$.

Determinantal point process (DPP) is defined as follows.

Definition 3.1 *A point process Ξ on $(S, \mathcal{B}_{\mathrm{c}}(S), \lambda(dx))$ is said to be a DPP with a measurable kernel $K : S \times S \to \mathbb{C}$, if the correlation functions with respect to $\lambda(dx)$ are given by*

$$\rho_n(x_1, \ldots, x_n) = \det_{1 \leq i,j \leq n} [K(x_i, x_j)] \quad \text{for every } n \in \mathbb{N} \text{ and any } x_1, \ldots, x_n \in S. \tag{3.1}$$

The integral kernel K is called the correlation kernel. The DPP is specified by the triplet $(\Xi, K, \lambda(dx))$.

The following fact is well known.

Lemma 3.2 *Consider a non-vanishing function $f : S \to \mathbb{C}$. Even if the correlation kernel $K(x,y)$ is transformed as*

$$K(x,y) \to K_f(x,y) := f(x)K(x,y)\frac{1}{f(y)}, \quad x,y \in S, \tag{3.2}$$

all correlation functions (3.1) are the same and hence

$$(\Xi, K, \lambda(dx)) \stackrel{\text{(law)}}{=} (\Xi, K_f, \lambda(dx)).$$

The transformation (3.2) is called the *gauge transformation* and the above property of DPP is referred to *gauge invariance*. See, for instance, Lemma 3.8 in Section 3.6 of [12].

If the point process is a DPP, $(\Xi, K, \lambda(dx))$, then

$$\begin{aligned}
\mathbf{E}[\langle \Xi, \phi \rangle] &= \int_S \phi(x)K(x,x)\lambda(dx), \\
\mathrm{var}[\langle \Xi, \phi \rangle] &= \frac{1}{2}\int_{S\times S} |\phi(x)-\phi(y)|^2 K(x,y)K(y,x)\lambda^{\otimes 2}(dxdy), \quad \phi \in \mathcal{B}_{\mathrm{c}}(S).
\end{aligned}$$

In particular, when ϕ is the indicator function of a bounded domain $\Lambda \subset S$; $\phi(x) = \mathbf{1}_\Lambda(x) := 1$, if $x \in \Lambda$, and $:= 0$, otherwise,

$$\begin{aligned}
\mathbf{E}[\langle \Xi(\Lambda) \rangle] &= \int_\Lambda K(x,x)\lambda(dx), \\
\mathrm{var}[\langle \Xi(\Lambda) \rangle] &= \int_\Lambda \int_{S\setminus\Lambda} K(x,y)K(y,x)\lambda(dx)\lambda(dy).
\end{aligned}$$

3.2 Comments on Theorem 1.2

The *Bessel function of the first kind* and the *modified Bessel function of the first kind* are defined as

$$\begin{aligned}
J_\nu(z) &:= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty (-1)^n \frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)}, \\
I_\nu(z) &:= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^\infty \frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)}, \quad \nu > -1, \quad z \in \mathbb{C} \setminus (-\infty, 0],
\end{aligned}$$

respectively. The following were proved in [14].

Proposition 3.3 *For the Heisenberg family of DPPs,* $(\Xi_{\mathsf{H}_D}, K_{\mathsf{H}_D}, \lambda_{\mathrm{N}(0,1;\mathbb{C}^D)}(dx))$ *on* $\mathbb{C}^D$, $D \in \mathbb{N}$, *the following hold,*

$$
\begin{aligned}
\mathbf{E}[\Xi_{\mathsf{H}_D}(\mathbb{B}_R^{(2D)})] &= \frac{R^{2D}}{D!}, \\
\mathrm{var}[\Xi_{\mathsf{H}_D}(\mathbb{B}_R^{(2D)})] &= \frac{2R^{2D}}{(D-1)!} \int_0^\infty \frac{J_D(\kappa R)^2}{\kappa}(1 - e^{-\kappa^2/4}) d\kappa \\
&= \frac{R^{2D} e^{-2R^2}}{D!} \sum_{n=0}^{D-1} [I_n(2R^2) + I_{n+1}(2R^2)], \quad R > 0.
\end{aligned}
$$

Theorem 1.2 is concluded from the above proposition, if we use the following asymptotic formula of the modified Bessel functions (see, for instance, [16, Section 10.17]),

$$
I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_k(\nu)}{k! 2^{3k}} x^{-k}, \quad \text{as } x \to \infty.
$$

3.3 Proof of Theorem 1.6

For $R > 0, n, m \in \mathbb{N}_0$, let

$$
p_n^{(R,m)} := \frac{m!}{n!} \int_0^{R^2} u^{n-m} e^{-u} |L_m^{(n-m)}(u)|^2 du.
$$

We introduce a series of random variables $Y_{n^{(\ell)}}^{(R,m^{(\ell)})} \in \{0, 1\}$, $m^{(\ell)} \in \mathbb{N}_0$, $n^{(\ell)} \in \mathbb{N}_0$, $\ell = 1, \ldots, D$ such that they are mutually independent and

$$
Y_{n^{(\ell)}}^{(R,m^{(\ell)})} \sim \mu^{\text{Bernoulli}}_{p_{n^{(\ell)}}^{(R,m^{(\ell)})}},
$$

where the right-hand side denotes the *Bernoulli measure* of probability $p_{n^{(\ell)}}^{(R,m^{(\ell)})}$. By the general theory of the *duality relations between DPPs* [13, Theorem 2.6], we can prove that

$$
\Xi_{\mathsf{H}_D}^{(m)}(\Delta_R^{(D)}) \stackrel{\mathrm{d}}{=} \sum_{n \in \mathbb{N}_0^D} \prod_{\ell=1}^{D} Y_{n^{(\ell)}}^{(R,m^{(\ell)})}.
$$

Then it is easy to verify that

$$
\mathbf{E}[\Xi_{\mathsf{H}_D}^{(m)}(\Delta_R^{(D)})] = \sum_{n \in \mathbb{N}_0^D} \prod_{\ell=1}^{D} p_{n^{(\ell)}}^{(R,m^{(\ell)})} = \prod_{\ell=1}^{D} \sum_{k=0}^{\infty} p_k^{(R,m^{(\ell)})},
$$

and

$$\begin{aligned}\mathrm{var}[\Xi^{(m)}_{\mathsf{H}_D}(\Delta^{(D)}_R)] &= \mathrm{var}\left[\sum_{n\in\mathbb{N}_0^D}\prod_{\ell=1}^D Y^{(R,m^{(\ell)})}_{n^{(\ell)}}\right] = \sum_{n\in\mathbb{N}_0^D}\mathrm{var}\left[\prod_{\ell=1}^D Y^{(R,m^{(\ell)})}_{n^{(\ell)}}\right] \\ &= \sum_{n\in\mathbb{N}_0^D}\left[\prod_{\ell=1}^D p^{(R,m^{(\ell)})}_{n^{(\ell)}} - \left(\prod_{\ell=1}^D p^{(R,m^{(\ell)})}_{n^{(\ell)}}\right)^2\right] \\ &= \prod_{\ell=1}^D\sum_{k=0}^\infty p^{(R,m^{(\ell)})}_k - \prod_{\ell=1}^D\sum_{k=0}^\infty (p^{(R,m^{(\ell)})}_k)^2.\end{aligned}$$

Hence, we obtain

$$\frac{\mathrm{var}[\Xi^{(m)}_{\mathsf{H}_D}(\Delta^{(D)}_R)]}{\mathbf{E}[\Xi^{(m)}_{\mathsf{H}_D}(\Delta^{(D)}_R)]} = 1 - \prod_{\ell=1}^D \frac{\sum_{k=0}^\infty (p^{(R,m^{(\ell)})}_k)^2}{\sum_{k=0}^\infty p^{(R,m^{(\ell)})}_k}, \quad m\in\mathbb{N}_0^D.$$

When $D=1$, the above gives

$$\frac{\mathrm{var}[\Xi^{(m)}_{\mathsf{H}_1}(\mathbb{B}^{(2)}_R)]}{\mathbf{E}[\Xi^{(m)}_{\mathsf{H}_1}(\mathbb{B}^{(2)}_R)]} \equiv \frac{\mathrm{var}[\Xi^{(m)}_{\mathsf{H}_1}(\Delta^{(1)}_R)]}{\mathbf{E}[\Xi^{(m)}_{\mathsf{H}_1}(\Delta^{(1)}_R)]} = 1 - \frac{\sum_{k=0}^\infty (p^{(R,m)}_k)^2}{\sum_{k=0}^\infty p^{(R,m)}_k}, \quad m\in\mathbb{N}_0.$$

where we have used the obvious fact that $\Delta^{(1)}_R = \mathbb{D}_R \subset \mathbb{C}$ and it is identified with $\mathbb{B}^{(2)}_R \subset \mathbb{R}^2$. Hence the first equality in (1.10) is proved. The second equality is obtained by Proposition 1.4. The proof is hence complete.

Acknowledgements The present author would like to thank Tomoyuki Shirai and Shinji Koshida for useful discussion on this subject. This work was supported by the Grant-in-Aid for Scientific Research (C) (No.19K03674), (B) (No.18H01124), (S) (No.16H06338), (A) (No.21H04432) of Japan Society for the Promotion of Science. It was also supported by the Research Institute for Mathematical Sciences (RIMS), a Joint Usage/Research Center located in Kyoto University. The author thanks Fumio Hiroshima, Itaru Sasaki and Hayato Saigo for organizing the 2021 RIMS Workshop 'Mathematical Aspects of Quantum Fields and Related Topics', which was held online on December 6–8, 2021.

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