An Intertwining Property of Weyl Operators

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1 Introduction

The *Weyl operator* $W(u)$ associated with $u \in H$ (a separable complex Hilbert space with the norm $|\cdot|^2 = \langle \cdot | \cdot \rangle$) is defined by

$$W(u) := e^{-\frac{1}{2}|u|^2} e^{\mathfrak{a}^\dagger(u)} e^{-\mathfrak{a}(u)}$$

(see (5.2) and [22]), where $\mathfrak{a}(u) = a(\overline{u})$ with the annihilation operator $a(\xi)$ on the Boson Fock space $\Gamma(H)$ (see Section 3) and $\mathfrak{a}^\dagger(u) = a^*(u)$ is the creation operator (with the adjoint operator $a^*(\xi)$ of $a(\xi)$ with respect to the canonical complex bilinear form $\langle \cdot, \cdot \rangle = \langle \bar{\cdot} | \cdot \rangle$ on $H \times H$). Then it is well-known that the Weyl operator $W(u)$ ($u \in H$) is unitary and satisfies that for any $u, v \in H$,

$$W(u)W(v) = e^{-i\mathrm{Im}(\langle u|v\rangle)} W(u+v),$$

and so the map $u \mapsto W(u)$ is a *projective unitary representation* of the additive group H with the multiplier $\sigma(u, v) = e^{-i\mathrm{Im}(\langle u|v\rangle)}$ (see [22]).

A bijective real linear map $S : H \to H$ is called a *symplectic automorphism* if S satisfies (i) S and S^{-1} are continuous, and (ii) $\mathrm{Im}(\langle Su | Sv \rangle) = \mathrm{Im}(\langle u | v \rangle)$ for all $u, v \in H$. Then for each symplectic automorphism S, by defining unitary operator $W_S(u)$ ($u \in H$) on the Boson Fock space $\Gamma(H)$ by

$$W_S(u) = W(Su),$$

we have another projective unitary representation $W_S : u \mapsto W_S(u)$ with the multiplier $\sigma(u, v) = e^{-i\mathrm{Im}(\langle u|v\rangle)}$, i.e., we have

$$W_S(u) W_S(v) = e^{-i\mathrm{Im}(\langle u|v\rangle)} W_S(u+v)$$

for all $u, v \in H$.

We suppose that $H = H_{\mathbb{R}} + iH_{\mathbb{R}}$ the complexification of a real Hilbert space $H_{\mathbb{R}}$. Then every real linear map $S : H \to H$ is associated with an operator S_0 on $H_{\mathbb{R}} \oplus H_{\mathbb{R}}$ by defining

$$S(\xi_1 + i\xi_2) = S_{11}\xi_1 + iS_{21}\xi_1 + S_{12}\xi_2 + iS_{22}\xi_2$$

and

$$S_0 = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

In [25], Shale proved that for each symplectic automorphism S of H, there exists a unitary operator $\mathcal{U}_S$ on the Boson Fock space $\Gamma(H)$ such that

$$\mathcal{U}_S W(u) \mathcal{U}_S^{-1} = W_S(u), \quad u \in H$$

if and only if $S_0^* S_0 - I$ is a Hilbert-Schmidt operator on $H_{\mathbb{R}} \oplus H_{\mathbb{R}}$. In such a case, $\mathcal{U}_S$ is determined uniquely up to a scalar multiple of modulus unity (see Theorem 22.11 of [22]).

In this manuscript, we consider an intertwining property of the Weyl operators based on the Gelfand triples:

$$E \subset H \subset E^*, \quad (E) \subset \Gamma(H) \subset (E)^*,$$

which is a mathematical framework of the white noise theory (see [5, 6, 7, 16, 17, 20]). We consider the operators

$$V_{K,u} = e^{\frac{1}{2}\langle u, Ku\rangle} e^{a^*(u)} e^{a(Ku)} \in \mathcal{L}((E),(E)) \cap \mathcal{L}((E)^*,(E)^*),$$

where $K : E \to E$ is a real linear operator and $u \in E$. Then for each real linear continuous operator $S : E \to E$ satisfying certain conditions, we want to find an operator $U_S \in \mathcal{L}((E),(E)^*)$ satisfying that

$$U_S V_{K,u} = V_{K,Su} U_S, \quad u \in E, \tag{1.1}$$

i.e., U_S satisfies the following diagram:

$$\begin{tikzcd}
(E) \arrow[r, "U_S"] \arrow[d, "V_{K,u}"'] & (E)^* \arrow[d, "V_{K,Su}"] \\
(E) \arrow[r, "U_S"'] & (E)^*
\end{tikzcd}$$

(see Theorem 6.3). For our purpose, by applying the notion of the quantum white noise derivatives developed in [11, 12, 13, 14, 15], we derive a quantum white noise differential equation (qwnde) which is equivalent to (1.1), and then by solving the qwnde with the method developed in [14, 15], we have an operator $U_S \in \mathcal{L}((E),(E)^*)$ satisfying (1.1), which is closely related to the Bogoliubov transformation studied in [1, 8, 13, 14, 15, 23, 24].

2 White Noise Distributions

Let H be a separable complex Hilbert space with the norm $|\cdot|_0$ induced by the inner product $\langle \cdot|\cdot\rangle$. Let A be a positive, selfadjoint operator in H satisfying that there exist a complete orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for H and an increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive real numbers such that

(A0) $\lambda_1 > 1$,

(A1) for all $n \in \mathbb{N}$, $Ae_n = \lambda_n e_n$,

(A2) A^{-1} is of Hilbert-Schmidt type, i.e.

$$\left\|A^{-1}\right\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$$

For each $p \geq 0$, put

$$\begin{aligned}
E_p &= \{\xi \in H : |\xi|_p := |A^p \xi|_0 < \infty\}, \\
E_{-p} &= \overline{H}^{|\cdot|_{-p}} \quad \text{(the completion of } H \text{ with respect to the norm } |\cdot|_{-p}),
\end{aligned}$$

where $|\cdot|_{-p} = |A^{-p}\cdot|_0$. Then by identifying H^* (strong dual space) and E_p^* $(p \geq 0)$ with H and E_{-p}, respectively, we have a chain of Hilbert spaces:

$$\cdots E_q \subset E_p \subset H \cong H^* \subset E_{-p} \subset E_{-q} \subset \cdots$$

for any $0 \leq p \leq q$, and then by taking the projective limit space of E_p and the inductive limit space of E_{-p}, we have the underline Gelfand triple:

$$\operatorname*{proj\,lim}_{p\to\infty} E_p =: E \subset H \subset E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p}.$$

Then from the condition **(A2)**, the nuclearity of E is guaranteed.

The (Boson) Fock space over E_p is defined by

$$\Gamma(E_p) = \left\{ \phi = (f_n)_{n=0}^{\infty}\,;\, f_n \in E_p^{\widehat{\otimes} n},\ \|\phi\|_p^2 = \sum_{n=0}^{\infty} n!\,|f_n|_p^2 < \infty \right\}.$$

Then we obtain a chain of Fock spaces:

$$\cdots \subset \Gamma(E_p) \subset \cdots \subset \Gamma(H) \subset \cdots \subset \Gamma(E_{-p}) \cdots$$

and, as limit spaces we define

$$(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \qquad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}).$$

It is known that (E) is a countably Hilbert nuclear space. Consequently, we obtain a Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*,$$

which is referred to as the *Hida–Kubo–Takenaka space*. The dual space $\Gamma(H)$ is identified with itself through the canonical $\mathbb{C}$-bilinear form.

By the definition, the topology of (E) is generated by the norms

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!\,\,|f_n|_p^2\,, \qquad \phi = (f_n),$$

where $p \geq 0$. On the other hand, for each $\Phi \in (E)^*$ there exists $p \geq 0$ such that $\Phi \in \Gamma(E_{-p})$ and

$$\|\Phi\|_{-p}^2 \equiv \sum_{n=0}^{\infty} n!\,\,|F_n|_{-p}^2 < \infty, \qquad \Phi = (F_n).$$

The canonical $\mathbb{C}$-bilinear form on $(E)^* \times (E)$ takes the form:

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n!\,\langle F_n, f_n\rangle\,, \qquad \Phi = (F_n) \in (E)^*, \quad \phi = (f_n) \in (E).$$

3 White Noise Operators

A continuous linear operator from (E) into $(E)^*$ is called a *white noise operator*. The space of all white noise operators is denoted by $\mathcal{L}((E),(E)^*)$. The white noise operators cover a wide class of Fock space operators, for example, $\mathcal{L}((E),(E))$, $\mathcal{L}((E)^*,(E))$ and $\mathcal{L}(\Gamma(H),\Gamma(H))$ are subspaces of $\mathcal{L}((E),(E)^*)$.

For each $x \in E^*$, the *annihilation operator* $a(x) \in \mathcal{L}((E),(E))$ associated with x is defined by

$$a(x) : (E) \ni \phi = (f_n)_{n=0}^\infty \mapsto ((n+1)x \otimes_1 f_{n+1})_{n=0}^\infty \in (E),$$

where $x \otimes_1 f_n$ stands for the contraction. The adjoint operator $a^*(x) \in \mathcal{L}((E)^*,(E)^*)$ of $a(x)$ with respect to the canonical bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ is given by

$$a^*(x) : (E)^* \ni \phi = (f_n)_{n=0}^\infty \mapsto (x\hat{\otimes} f_{n-1})_{n=0}^\infty \in (E), \quad \text{(understanding } f_{-1} = 0),$$

and is called the *creation operator* associated with x. We note that $a(\zeta) \in \mathcal{L}((E)^*,(E)^*)$ and $a^*(\zeta) \in \mathcal{L}((E),(E))$. More precisely,

Lemma 3.1 *For any distribution $\zeta \in E^*$, we have $a(\zeta) \in \mathcal{L}((E),(E))$ and $a^*(\zeta) \in \mathcal{L}((E)^*,(E)^*)$. If $\zeta \in E$, then $a(\zeta)$ extends to a continuous linear operator from $(E)^*$ into itself and $a^*(\zeta)$ restricted to (E) is a continuous linear operator from (E) into itself.*

For simple notations, the extension and restriction mentioned in Lemma 3.1 are denoted by the same symbols. It is straightforward to verify the canonical commutation relation:

$$[a(\xi), a(\eta)] = 0, \quad [a^*(\xi), a^*(\eta)] = 0, \quad [a(\xi), a^*(\eta)] = \langle \xi, \eta \rangle$$

for all $\xi, \eta \in E$.

The *exponential vector* (or *coherent vector*) ϕ_ξ associated with $\xi \in H$ is defined by

$$\phi_\xi := \left(1, \xi, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right).$$

Then it is well-known that $\{\phi_\xi : \xi \in E\}$ spans a dense subspace of (E). Therefore, every white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ is uniquely determined by its *symbol* $\widehat{\Xi}$ defined by

$$\widehat{\Xi}(\xi, \eta) = \left\langle\left\langle \Xi\phi_\xi, \phi_\eta \right\rangle\right\rangle, \qquad \xi, \eta \in E.$$

The following theorem is well-known as analytic characterization theorem for symbols of white noise operators.

Theorem 3.2 ([19, 2, 10]) *Let $\Theta : E \times E \longrightarrow \mathbb{C}$ be a function. Then Θ is the symbol of some white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ if and only if for each $\xi_i, \eta_i \in E$ ($i = 1, 2$), the function*

$$\mathbb{C} \times \mathbb{C} \ni (z, w) \mapsto \Theta(\xi_1 + z\xi_2, \eta_1 + w\eta_2) \in \mathbb{C} \tag{3.1}$$

is entire holomorphic, and there exist constants $C, K \geq 0$ and $p \geq 0$ such that

$$|\Theta(\xi, \eta)| \leq Ce^{K\left(|\xi|_p^2 + |\eta|_p^2\right)}, \quad \xi, \eta \in E.$$

Furthermore, the function Θ is the symbol of some white noise operator $\Xi \in \mathcal{L}((E),(E))$ if and only if the function given as in (3.1) *is entire holomorphic, and for any $\epsilon > 0$ and $p \geq 0$, there exist $q \geq 0$ and $C > 0$ such that*

$$|\Theta(\xi, \eta)| \leq Ce^{\epsilon\left(|\xi|_{p+q}^2 + |\eta|_{-p}^2\right)}, \quad \xi, \eta \in E.$$

For each $\kappa \in (E^{\otimes(l+m)})^*$, by applying Theorem 3.2 we can see that there exists a unique operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$, called an *integral kernel operator*, such that

$$\widehat{\Xi_{l,m}(\kappa)}(\xi,\eta) = \left\langle \kappa,\, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle e^{\langle \xi,\eta \rangle}, \qquad \xi,\eta \in E,$$

where $\langle \cdot,\, \cdot \rangle$ is the canonical bilinear form on $E^* \times E$. Note that $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E))$ if and only if $\kappa \in E^{\otimes l} \otimes (E^{\otimes m})^*$. In particular, for each $x \in E^*$ we have

$$a(x) = \Xi_{0,1}(x), \quad a^*(x) = \Xi_{1,0}(x).$$

For the case of $H = L^2(\mathbb{R}, dt)$ and $\delta_t \in E^*$ (for each point $t \in \mathbb{R}$), we write

$$a_t = a(\delta_t), \quad a_t^* = a^*(\delta_t).$$

In this case, the integral kernel operator $\Xi_{l,m}(\kappa)$ is formally represented by

$$\Xi_{l,m}(\kappa) = \int_{\mathbb{R}^{l+m}} \kappa(s_l, \cdots, s_1; t_m, \cdots, t_1) a_{s_l}^* \cdots a_{s_1}^* a_{t_m} \cdots a_{t_1} dt_1 \cdots dt_m ds_1 \cdots ds_l.$$

Quadratic forms of quantum white noise are useful for applications. For each $S \in \mathcal{L}(E, E^*)$, by the kernel theorem there exists a unique $\tau_S \in E^* \otimes E^*$ such that

$$\langle \tau_S,\, \eta \otimes \xi \rangle = \langle S\xi,\, \eta \rangle, \qquad \xi,\eta \in E.$$

We put

$$\Delta_{\mathrm{G}}(S) = \Xi_{0,2}(\tau_S), \quad \Delta_{\mathrm{G}}^*(S) = \Xi_{2,0}(\tau_S), \quad \Lambda(S) = \Xi_{1,1}(\tau_S).$$

Note that $\Delta_{\mathrm{G}}(S) \in \mathcal{L}((E),(E))$, $\Delta_{\mathrm{G}}(S)^* \in \mathcal{L}((E)^*,(E)^*)$ and $\Lambda(S) \in \mathcal{L}((E),(E)^*)$. For $S = I$ (the identity operator),

$$\Delta_{\mathrm{G}} := \Delta_{\mathrm{G}}(I), \quad N := \Lambda(I)$$

are called the *Gross Laplacian* and the *number operator*, respectively. The operator $\Delta_{\mathrm{G}}(S)$, called a *generalized Gross Laplacian*, plays an important role in the study of transformation groups [3]. A linear combination of the above quadratic forms is also referred to as a *Bogoliubov Hamiltonian*, see e.g., [1].

Theorem 3.3 ([20]) *For any $\Xi \in \mathcal{L}((E),(E)^*)$ there exists a unique family of distributions $\kappa_{l,m} \in (E^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ such that*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{3.2}$$

where the right hand side converges in $\mathcal{L}((E),(E)^*)$. If $\Xi \in \mathcal{L}((E),(E))$, then so does $\Xi_{l,m}(\kappa_{l,m})$ for all l, m and the series (3.2) *converges in $\mathcal{L}((E),(E))$.*

4 Quantum white noise derivatives

The Fock expansion (see Theorem 3.3) says that every white noise operator Ξ is a "function" of quantum white noise, say, $\Xi = \Xi(a_s, a_t^*;\ s,t \in T)$. It is then natural to consider the derivatives of Ξ with respect to the coordinate variables a_t and a_t^*.

For any white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ and $\zeta \in E$ the commutators

$$[a(\zeta),\Xi] = a(\zeta)\Xi - \Xi a(\zeta), \qquad -[a^*(\zeta),\Xi] = \Xi a^*(\zeta) - a^*(\zeta)\Xi,$$

are well defined as compositions of white noise operators (see Lemma 3.1), i.e., belong to $\mathcal{L}((E),(E)^*)$. We define

$$D_\zeta^+\Xi = [a(\zeta),\Xi], \qquad D_\zeta^-\Xi = -[a^*(\zeta),\Xi].$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively. Both together are referred to as the *quantum white noise derivatives* (*qwn-derivatives* for brevity) of Ξ.

Theorem 4.1 ([12]) $(\zeta,\Xi) \mapsto D_\zeta^\pm\Xi$ *is a continuous bilinear map from* $E \times \mathcal{L}((E),(E)^*)$ *into* $\mathcal{L}((E),(E)^*)$.

As explicit examples we record the qwn-derivatives of quadratic forms. The results will be used later.

Lemma 4.2 ([13]) *For* $S \in \mathcal{L}(E,E^*)$ *and* $\zeta \in E$ *we have*

$$\begin{aligned}
&D_\zeta^+\Delta_{\mathrm{G}}(S) = 0, && D_\zeta^-\Delta_{\mathrm{G}}(S) = a(S\zeta) + a(S^*\zeta),\\
&D_\zeta^+\Delta_{\mathrm{G}}^*(S) = a^*(S\zeta) + a^*(S^*\zeta), && D_\zeta^-\Delta_{\mathrm{G}}^*(S) = 0,\\
&D_\zeta^+\Lambda(S) = a(S^*\zeta), && D_\zeta^-\Lambda(S) = a^*(S\zeta).
\end{aligned}$$

There exists a separately continuous bilinear map from $\mathcal{L}((E),(E)^*) \times \mathcal{L}((E),(E)^*)$ into $\mathcal{L}((E),(E)^*)$, denoted by $\Xi_1 \diamond \Xi_2$, uniquely specified by the following property:

$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t\,, \qquad a_t^* \diamond \Xi = \Xi \diamond a_t^* = a_t^*\Xi,$$

where the right-hand sides are well-defined compositions of white noise operators. We call $\Xi_1 \diamond \Xi_2$ the *Wick product* or *normal-ordered product*. It is more clear to define the Wick product by symbols. In fact, the Wick product $\Xi_1 \diamond \Xi_2$ is characterized by

$$(\Xi_1 \diamond \Xi_2)\widehat{\ }(\xi,\eta) = \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta)e^{-\langle\xi,\eta\rangle}, \qquad \xi,\eta \in E.$$

Equipped with the Wick product, $(\mathcal{L}((E),(E)^*),\diamond)$ becomes a commutative algebra. Also, by applying the characterization theorem for operator symbols [19, 20], we can easily see that $(\mathcal{L}((E),(E)),\diamond)$ is a subalgebra of $\mathcal{L}((E),(E)^*)$.

A continuous linear map $\mathcal{D} : \mathcal{L}((E),(E)^*) \to \mathcal{L}((E),(E)^*)$ is called a *Wick derivation* if

$$\mathcal{D}(\Xi_1 \diamond \Xi_2) = (\mathcal{D}\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond (\mathcal{D}\Xi_2), \qquad \Xi_1,\Xi_2 \in \mathcal{L}((E),(E)^*).$$

Theorem 4.3 ([13]) *The creation and annihilation derivatives* $D_\zeta^\pm$ *are Wick derivations.*

Given a Wick derivation $\mathcal{D} : \mathcal{L}((E),(E)^*) \to \mathcal{L}((E),(E)^*)$ and a white noise operator $G \in \mathcal{L}((E),(E)^*)$, we consider a linear differential equation:

$$\mathcal{D}\Xi = G \diamond \Xi. \tag{4.1}$$

The solution is described as in the case of linear ordinary differential equations. For $U \in \mathcal{L}((E),(E)^*)$ the *Wick exponential* is defined by

$$\operatorname{wexp} U = \sum_{n=0}^{\infty} \frac{1}{n!} U^{\diamond n}$$

whenever the series converges in $\mathcal{L}((E),(E)^*)$, for more details see [4].

Theorem 4.4 ([13]) *Let $G \in \mathcal{L}((E),(E)^*)$. If there is an operator $U \in \mathcal{L}((E),(E)^*)$ such that $\mathcal{D}U = G$ and* wexp $U \in \mathcal{L}((E),(E)^*)$*, then a general solution to* (4.1) *is given by*

$$\Xi = (\mathrm{wexp}\, U) \diamond F = F \diamond \mathrm{wexp}\, U$$

with a white noise operator $F \in \mathcal{L}((E),(E)^*)$ satisfying $\mathcal{D}F = 0$.

5 Weyl Operators

For each $\eta \in E$, by applying Theorem 3.2, we can easily see that

$$e^{a^*(\eta)}, e^{a(\eta)} \in \mathcal{L}((E),(E)).$$

Therefore, by applying the duality, for any $\eta, \zeta \in E$, we have

$$e^{a^*(\eta)} e^{a(\zeta)} \in \mathcal{L}((E),(E)) \cap \mathcal{L}((E)^*,(E)^*). \tag{5.1}$$

Let $K : E \to E$ be a real linear operator. Put

$$V_{K,u} : = e^{\frac{1}{2}\langle u,\, Ku\rangle} e^{a^*(u)} e^{a(Ku)}, \quad u \in E.$$

Then from (5.1), we have

$$V_{K,u} \in \mathcal{L}((E),(E)) \cap \mathcal{L}((E)^*,(E)^*),$$

and for any $\xi, \eta \in E$, we obtain that

$$\left\langle\!\left\langle V_{K,u}\phi_\xi \middle| V_{K,u}\phi_\eta \right\rangle\!\right\rangle = e^{\frac{1}{2}\left(\overline{\langle u,\, Ku\rangle} + \langle u,\, Ku\rangle\right) + \overline{\langle Ku,\, \xi\rangle} + \langle Ku,\, \eta\rangle + \langle \xi + u | \eta + u\rangle}.$$

Therefore, $\left\langle\!\left\langle V_{K,u}\phi_\xi \middle| V_{K,u}\phi_\eta \right\rangle\!\right\rangle = e^{\langle \xi | \eta\rangle}$ for all $\xi, \eta \in E$ if and only if

$$\frac{1}{2}\left(\overline{\langle u,\, Ku\rangle} + \langle u,\, Ku\rangle\right) + \langle u | u\rangle = 0, \quad \overline{\langle Ku,\, \xi\rangle} + \langle \xi | u\rangle = 0, \quad \langle Ku,\, \eta\rangle + \langle u | \eta\rangle = 0$$

for all $\xi, \eta \in E$ if and only if $Ku = -\overline{u}$ (see (1) of Example 5.3). Hence for each $u \in E$, $V_{K,u}$ has an unitary extension to $\Gamma(H)$ if and only if $K = -J$, where J is the complex conjugation, i.e., $Ju = \overline{u}$ for all $u \in E$. Then we have

$$\begin{aligned} V_{-J,u} &= e^{-\frac{1}{2}\langle u,\, \overline{u}\rangle} e^{a^*(u)} e^{-a(\overline{u})} = e^{-\frac{1}{2}|u|^2} e^{\mathfrak{a}^\dagger(u)} e^{-\mathfrak{a}(u)} \\ &=: W(u) \end{aligned} \tag{5.2}$$

for all $u \in E$, which is called the *Weyl operator* (see [22]), where the operator $\mathfrak{a}(u)$ and $\mathfrak{a}^\dagger(u)$ are defined by

$$\mathfrak{a}(u) = \overline{a(u)} = a(\overline{u}), \quad \mathfrak{a}^\dagger(u) = (\mathfrak{a}(u))^\dagger \quad \text{(the Hermitian adjoint)}.$$

Then we can easily see that $\mathfrak{a}^\dagger(u) = a^*(u)$. In fact, for any $\xi, \eta \in E$, we obtain that

$$\begin{aligned} \left\langle\!\left\langle \mathfrak{a}^\dagger(u)\phi_\xi,\, \phi_\eta \right\rangle\!\right\rangle &= \left\langle\!\left\langle \phi_{\overline{\eta}} \middle| \mathfrak{a}^\dagger(u)\phi_\xi \right\rangle\!\right\rangle = \left\langle\!\left\langle \mathfrak{a}(\overline{u})\phi_{\overline{\eta}} \middle| \phi_\xi \right\rangle\!\right\rangle = \left\langle\!\left\langle \langle \overline{u},\, \overline{\eta}\rangle \phi_{\overline{\eta}} \middle| \phi_\xi \right\rangle\!\right\rangle = \langle u,\, \eta\rangle \left\langle\!\left\langle \phi_\xi,\, \phi_\eta \right\rangle\!\right\rangle \\ &= \left\langle\!\left\langle \phi_\xi,\, a(u)\phi_\eta \right\rangle\!\right\rangle = \left\langle\!\left\langle a^*(u)\phi_\xi,\, \phi_\eta \right\rangle\!\right\rangle. \end{aligned}$$

Proposition 5.1 *If $K : E \to E$ is a real linear operator, then we have*

$$V_{K,v}V_{K,u} = e^{\frac{1}{2}(\langle Kv,u\rangle - \langle v,Ku\rangle)}V_{K,v+u}, \tag{5.3}$$

$$V_{K,v}V_{K,u} = e^{\langle Kv,u\rangle - \langle v,Ku\rangle}V_{K,u}V_{K,v} \tag{5.4}$$

for all $v, u \in E$.

Proof. By applying the Baker–Campbell–Hausdorff formula, we obtain that

$$\begin{aligned}
V_{K,v}V_{K,u} &= e^{\frac{1}{2}(\langle v,Kv\rangle + \langle u,Ku\rangle)}e^{a^*(v)}e^{a(Kv)}e^{a^*(u)}e^{a(Ku)}\\
&= e^{\frac{1}{2}(\langle v,Kv\rangle + \langle u,Ku\rangle + 2\langle Kv,u\rangle)}e^{a^*(v)}e^{a^*(u)}e^{a(Kv)}e^{a(Ku)}\\
&= e^{\frac{1}{2}(\langle v,Kv\rangle + \langle u,Ku\rangle + 2\langle Kv,u\rangle)}e^{a^*(v+u)}e^{a(Kv+Ku)}.
\end{aligned}$$

On the other hand, since K is real linear, then we obtain that

$$\begin{aligned}
V_{K,v}V_{K,u} &= e^{\frac{1}{2}(\langle v,Kv\rangle + \langle u,Ku\rangle + 2\langle Kv,u\rangle)}e^{a^*(v+u)}e^{a(K(v+u))}\\
&= e^{\frac{1}{2}(2\langle Kv,u\rangle - \langle v,Ku\rangle - \langle u,Kv\rangle)}e^{\frac{1}{2}\langle v+u,K(v+u)\rangle}e^{a^*(v+u)}e^{a(K(v+u))}\\
&= e^{\frac{1}{2}(\langle Kv,u\rangle - \langle v,Ku\rangle)}V_{K,v+u},
\end{aligned}$$

which proves the first assertion. From (5.3), we obtain that

$$\begin{aligned}
V_{K,v}V_{K,u} &= e^{\frac{1}{2}(\langle Kv,u\rangle - \langle v,Ku\rangle)}V_{K,v+u}\\
&= e^{\frac{1}{2}(\langle Kv,u\rangle - \langle v,Ku\rangle)}e^{-\frac{1}{2}(\langle Ku,v\rangle - \langle u,Kv\rangle)}V_{K,u}V_{K,v}\\
&= e^{\langle Kv,u\rangle - \langle v,Ku\rangle}V_{K,u}V_{K,v},
\end{aligned}$$

which proves (5.4). ∎

Proposition 5.2 *Let $K : E \to E$ be a real linear operator. Then for any invertible operator $S \in \mathcal{L}(E, E)$, we have*

$$\Gamma(S^{-1})V_{K,u}\Gamma(S) = V_{S^*KS,S^{-1}u}$$

Proof. For any $\xi, \eta \in E$, we obtain that

$$\left\langle\!\left\langle V_{K,u}\phi_\xi,\, \phi_\eta\right\rangle\!\right\rangle = \left\langle\!\left\langle e^{\frac{1}{2}\langle u,Ku\rangle}e^{a^*(u)}e^{a(Ku)}\phi_\xi,\, \phi_\eta\right\rangle\!\right\rangle = e^{\frac{1}{2}\langle u,Ku\rangle + \langle Ku,\xi\rangle + \langle u,\eta\rangle + \langle \xi,\eta\rangle},$$

and so we obtain that

$$\begin{aligned}
\left\langle\!\left\langle V_{K,u}\Gamma(S)\phi_\xi,\, \phi_\eta\right\rangle\!\right\rangle &= \left\langle\!\left\langle V_{K,u}\phi_{S\xi},\, \phi_\eta\right\rangle\!\right\rangle\\
&= e^{\frac{1}{2}\langle u,Ku\rangle + \langle Ku,S\xi\rangle + \langle u,\eta\rangle + \langle S\xi,\eta\rangle}\\
&= e^{\frac{1}{2}\left\langle S^{-1}u,S^*KSS^{-1}u\right\rangle + \left\langle S^*KSS^{-1}u,\xi\right\rangle + \left\langle S^{-1}u,S^*\eta\right\rangle + \langle \xi,S^*\eta\rangle}\\
&= \left\langle\!\left\langle V_{S^*KS,S^{-1}u}\phi_\xi,\, \Gamma(S^*)\phi_\eta\right\rangle\!\right\rangle,
\end{aligned}$$

from which we have the assertion. ∎

From now on we assume that there exists complete real subspace $E_{\mathbb{R}} \subset E$ such that

$$E = E_{\mathbb{R}} + iE_{\mathbb{R}}.$$

We denote $\mathcal{L}_{\mathbb{R}}(E, E)$ the (real) space of all continuous real linear operators from E into itself. For each $S \in \mathcal{L}_{\mathbb{R}}(E, E)$, define operators S_{jk} (for $1 \le j, k \le 2$) in the real nuclear space $E_{\mathbb{R}}$ by

$$S(x + iy) = S_{11}x + iS_{21}x + S_{12}y + iS_{22}y$$

for $z = x + iy \in E$ with $x, y \in E_{\mathbb{R}}$. More precisely, we define the real linear operators S_{ij} by

$$\begin{aligned} S_{11}x &= \tfrac{1}{2}\left(Sx + \overline{Sx}\right), & S_{21}x &= \tfrac{1}{2i}\left(Sx - \overline{Sx}\right), \\ S_{12}x &= \tfrac{1}{2}\left(S(ix) + \overline{S(ix)}\right), & S_{22}x &= \tfrac{1}{2i}\left(S(ix) - \overline{S(ix)}\right) \end{aligned}$$

for any $x \in E_{\mathbb{R}}$. By expressing any vector in $E_{\mathbb{R}} \oplus E_{\mathbb{R}}$ as a column $\begin{pmatrix} u \\ v \end{pmatrix}$ for some $u, v \in E_{\mathbb{R}}$, and define

$$S_0 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Example 5.3 (1) Let $J : E \to E$ be the complex conjugation, i.e., for any $\xi = \xi_1 + i\xi_2 \in E$ with $\xi_i \in E_{\mathbb{R}}$, $J\xi = \xi_1 - i\xi_2$, we have

$$J\xi_1 = \xi_1, \quad J(i\xi_2) = -i\xi_2, \quad \xi_1, \xi_2 \in E_{\mathbb{R}},$$

from which we have

$$J_{11} = I, \quad J_{21} = 0, \quad J_{12} = 0, \quad J_{22} = -I.$$

Therefore, we have $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(2) Let $L : E \to E$ be a complex linear operator. Then for any $\xi = \xi_1 + i\xi_2 \in E$ with $\xi_i \in E_{\mathbb{R}}$, we have $L\xi = L\xi_1 + iL\xi_2$ and so we have $L\xi_1 = L_{11}\xi_1 + iL_{21}\xi_1$ and

$$\begin{aligned} L_{12}\xi_2 + iL_{22}\xi_2 = L(i\xi_2) = iL\xi_2 &= i\left(L_{11}\xi_2 + iL_{21}\xi_2\right) \\ &= -L_{21}\xi_2 + iL_{11}\xi_2, \end{aligned}$$

from which we have $L_{12} = -L_{21}$ and $L_{22} = L_{11}$ and hence we have

$$L_0 = \begin{pmatrix} L_{11} & -L_{21} \\ L_{21} & L_{11} \end{pmatrix}. \tag{5.5}$$

(3) Let $M : E \to E$ be a real linear operator. Then for any $\xi = \xi_1 + i\xi_2, \eta = \eta_1 + i\eta_2 \in E$ with $\xi_i, \eta_i \in E_{\mathbb{R}}$, we obtain that

$$\begin{aligned} \langle M\xi, \eta\rangle &= \langle M_{11}\xi_1 + M_{12}\xi_2 + i(M_{21}\xi_1 + M_{22}\xi_2), \eta_1 + i\eta_2\rangle \\ &= \left\langle \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \sigma_3 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\rangle + i\left\langle \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \sigma_1 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\rangle, \end{aligned}$$

where $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices. Therefore, we have

$$\begin{aligned} \langle M\xi,\, \eta\rangle &= \left\langle (\sigma_3^* + i\sigma_1^*)\, M_0 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\rangle \\ &= \left\langle (\sigma_3 + i\sigma_1)\, M_0 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\rangle. \end{aligned} \tag{5.6}$$

(4) Let $L \in \mathcal{L}(E, E)$ be a complex linear continuous operator. Then L_0 is given as in (5.5), and from (5.6), we obtain that

$$\begin{aligned} \langle L^*\xi,\, \eta\rangle = \langle \xi,\, L\eta\rangle &= \left\langle \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, (\sigma_3 + i\sigma_1)\, L_0 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\rangle \\ &= \left\langle (L_0)^* \,(\sigma_3 + i\sigma_1) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\rangle, \end{aligned}$$

which implies that

$$(\sigma_3 + i\sigma_1)\,(L^*)_0 = (L_0)^*\,(\sigma_3 + i\sigma_1)\,. \tag{5.7}$$

Proposition 5.4 *Let $L \in \mathcal{L}(E, E)$ be a complex linear continuous operator. Then L is symmetric, i.e. $L^* = L$ if and only if L_{11} and L_{21} are symmetric, i.e., $L_{11}^* = L_{11}$ and $L_{21}^* = L_{21}$.*

Proof. From (5.5) and (5.7) we obtain that

$$\begin{aligned} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \begin{pmatrix} L_{11} & -L_{21} \\ L_{21} & L_{11} \end{pmatrix} &= (\sigma_3 + i\sigma_1)\, L_0 = (\sigma_3 + i\sigma_1)\,(L^*)_0 = (L_0)^*\,(\sigma_3 + i\sigma_1) \\ &= \begin{pmatrix} L_{11}^* & L_{21}^* \\ -L_{21}^* & L_{11}^* \end{pmatrix} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \end{aligned}$$

which is equivalent to

$$\begin{pmatrix} L_{11} + iL_{21} & -L_{21} + iL_{11} \\ -L_{21} + iL_{11} & -L_{11} - iL_{21} \end{pmatrix} = \begin{pmatrix} L_{11}^* + iL_{21}^* & -L_{21}^* + iL_{11}^* \\ -L_{21}^* + iL_{11}^* & -L_{11}^* - iL_{21}^* \end{pmatrix},$$

which is equivalent to $L_{11}^* = L_{11}$ and $L_{21}^* = L_{21}$. ∎

Let $K : E \to E$ be a real linear operator. Consider the map $\sigma_K : E \times E \to \mathbb{C}$ defined by

$$\sigma_K(u, v) = \frac{1}{2}\left(\langle Kv,\, u\rangle - \langle v,\, Ku\rangle\right), \quad u, v \in E.$$

Then for any $u_j, v_j \in E_{\mathbb{R}}$ for $j = 1, 2$, we obtain that

$$\begin{aligned} \sigma_K(u, v) &= \frac{1}{2}\left(\langle Kv,\, u\rangle - \langle v,\, Ku\rangle\right) \\ &= \frac{1}{2}\left(\left\langle (\sigma_3^* K_0 - K_0^* \sigma_3) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle + i \left\langle (\sigma_1^* K_0 - K_0^* \sigma_1) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle \right). \end{aligned} \tag{5.8}$$

In particular, if $K = -J$, then we have $K_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and so we have

$$\sigma_3^* K_0 - K_0^* \sigma_3 = 0, \quad \sigma_1^* K_0 - K_0^* \sigma_1 = 2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, we have

$$\sigma_{-J}(u,v) = \frac{1}{2}(-\langle\overline{v},\,u\rangle + \langle v,\,\overline{u}\rangle) = \frac{1}{2}(\langle u|\,v\rangle - \langle v|\,u\rangle) = i\left\langle\!\!\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right\rangle\!\!\right\rangle$$
$$= i\mathrm{Im}(\langle u|\,v\rangle)$$

Proposition 5.5 *Let $K, S : E \to E$ be real linear maps. Then S is a σ_K-symplectic map, i.e., $\sigma_K(Su, Sv) = \sigma_K(u,v)$ if and only if*

$$\begin{aligned} S_0^*(\sigma_3^*K_0 - K_0^*\sigma_3)\,S_0 &= \sigma_3^*K_0 - K_0^*\sigma_3, \\ S_0^*(\sigma_1^*K_0 - K_0^*\sigma_1)\,S_0 &= \sigma_1^*K_0 - K_0^*\sigma_1. \end{aligned} \tag{5.9}$$

In particular, S is a σ_{-J}-symplectic map if and only if

$$S_0^*\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}S_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(see (22.6) of [22]).

Proof. The proof is straightforward. From (5.8), by direct computation we have that $\sigma_K(Su, Sv) = \sigma_K(u,v)$ for all $u, v \in E$ if and only if

$$\left\langle\!\!\left\langle(\sigma_3^*K_0 - K_0^*\sigma_3)\,S_0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, S_0\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right\rangle\!\!\right\rangle + i\left\langle\!\!\left\langle(\sigma_1^*K_0 - K_0^*\sigma_1)\,S_0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, S_0\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right\rangle\!\!\right\rangle$$
$$= \left\langle\!\!\left\langle(\sigma_3^*K_0 - K_0^*\sigma_3)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right\rangle\!\!\right\rangle + i\left\langle\!\!\left\langle(\sigma_1^*K_0 - K_0^*\sigma_1)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right\rangle\!\!\right\rangle$$

for all $u, v \in E$ if and only if (5.9) holds. ∎

Corollary 5.6 *Let $K : E \to E$ be a real linear operator. For any real linear operator σ_K-symplectic operator $S : E \to E$, we have*

$$\begin{aligned} V_{K,Sv}V_{K,Su} &= e^{\sigma_K(u,v)}V_{K,S(v+u)}, \\ V_{K,Sv}V_{K,Su} &= e^{2\sigma_K(u,v)}V_{K,Su}V_{K,Sv} \end{aligned}$$

for all $v, u \in E$.

Proof. The proof is immediate from Proposition 5.1. ∎

6 An Intertwining Property of Weyl Operator

Let $S : E \to E$ be a real linear operator. We want to find an operator $U_S \in \mathcal{L}((E), (E)^*)$ such that

$$U_S V_{K,u} = V_{K,Su}U_S, \quad u \in E, \tag{6.1}$$

i.e., U_S satisfies the following diagram:

$$\begin{array}{ccc} (E) & \xrightarrow{U_S} & (E)^* \\ {\scriptstyle V_{K,u}}\big\downarrow & & \big\downarrow{\scriptstyle V_{K,Su}} \\ (E) & \xrightarrow[U_S]{} & (E)^* \end{array} \tag{6.2}$$

A family of operators $\{\Xi_\lambda\} \subset \mathcal{L}((E),(E))$ is said to be equicontinuous if for any $p \geq 0$, there exist a $q \geq 0$ and a constant $K \geq 0$ such that

$$|\Xi_\lambda \phi|_p \leq K|\phi|_q, \quad \phi \in (E)$$

for all λ (see [21, 20]).

Theorem 6.1 *Let $\{T_t\}_{t\geq 0} \subset \mathcal{L}((E),(E))$ and $\{S_t\}_{t\geq 0} \subset \mathcal{L}((E)^*,(E)^*)$ be continuous semigroups of continuous linear operators with the equicontinuous generator $T \in \mathcal{L}((E),(E))$ and $S \in \mathcal{L}((E)^*,(E)^*)$, respectively. Let $V \in \mathcal{L}((E),(E)^*)$. Then $VT_t = S_tV$ for all $t \geq 0$ if and only if $VT = SV$.*

Proof. For any $\phi \in (E)$, we obtain that

$$SV\phi = \lim_{t\to 0} \frac{S_tV\phi - V\phi}{t} = V\left(\lim_{t\to 0} \frac{T_t\phi - \phi}{t}\right) = VT\phi,$$

from which we see that $SV = VT$. Conversely, suppose that $SV = VT$. Then since S and T are equicontinuous, we construct continuous semigroups $\{T_t\}_{t\geq 0} \subset \mathcal{L}((E),(E))$ and $\{S_t\}_{t\geq 0} \subset \mathcal{L}((E)^*,(E)^*)$ with infinitesimal generators T and S by

$$T_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} T^n = e^{tT}, \quad S_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} S^n = e^{tS}, \quad t \geq 0.$$

Therefore, since $SV = VT$, for all $t \geq 0$, we obtain that

$$S_tV = \sum_{n=0}^{\infty} \frac{t^n}{n!} S^n V = V\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n\right) = VT_t,$$

which is the desired assertion. ∎

For each $t \geq 0$, put

$$V_{K,u}(t) = e^{\frac{1}{2}t^2\langle u, Ku\rangle} e^{ta^*(u)} e^{ta(Ku)},$$

where $K : E \to E$ is a real linear operator.

Proposition 6.2 *Let $u \in E$ be given. Then the family $\{V_{K,u}(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}((E),(E)) \cap \mathcal{L}((E)^*,(E)^*)$ is a differentiable one-parameter group with the infinitesimal generator $a^*(u) + a(Ku)$.*

Proof. For any $s, t \geq 0$, by applying the Baker–Campbell–Hausdorff formula, we obtain that

$$\begin{aligned} V_{K,u}(t)V_{K,u}(s) &= e^{\frac{1}{2}\left(s^2+t^2\right)\langle u, Ku\rangle} e^{ta^*(u)} e^{ta(Ku)} e^{sa^*(u)} e^{sa(Ku)} \\ &= e^{\frac{1}{2}\left(s^2+2st+t^2\right)\langle u, Ku\rangle} e^{ta^*(u)} e^{sa^*(u)} e^{ta(Ku)} e^{sa(Ku)} \\ &= e^{\frac{1}{2}(t+s)^2\langle u, Ku\rangle} e^{(t+s)a^*(u)} e^{(t+s)a(Ku)} \\ &= V_{K,u}(t+s), \end{aligned}$$

from which we see that $\{V_{K,u}(t)\}_{t\in\mathbb{R}}$ is a one-parameter group and it is easy to see that $\{V_{K,u}(t)\}_{t\in\mathbb{R}}$ is differentiable with the infinitesimal generator $a^*(u) + a(Ku)$. ∎

Therefore, by Theorem 6.1 and Proposition 6.2, we see that a white noise operator $U_S \in \mathcal{L}((E),(E)^*)$ satisfies the intertwining property given as in (6.1) if and only if U_S satisfies the intertwining property:

$$U_S\,(a^*(u) + a(Ku)) = (a^*(Su) + a(KSu))\,U_S, \quad u \in E,$$

i.e., U_S satisfies the following diagram:

$$\begin{array}{ccc} (E) & \xrightarrow{U_S} & (E)^* \\ {\scriptstyle a^*(u)+a(Ku)}\Big\downarrow & & \Big\downarrow{\scriptstyle a^*(Su)+a(KSu)} \\ (E) & \xrightarrow[U_S]{} & (E)^* \end{array}$$

which is equivalent to

$$\begin{aligned}[U_S, a^*(u)] - [a(KSu), U_S] &= -a^*(u)U_S - U_S a(Ku) + a^*(Su)U_S + U_S a(KSu) \\ &= (a^*((S-I)u) + a(K(S-I)u)) \diamond U_S, \quad u \in E.\end{aligned}$$

Therefore, we have the quantum white noise differential equation:

$$(D_u^- - D_{KSu}^+)\,U_s = (a^*((S-I)u) + a(K(S-I)u)) \diamond U_S, \quad u \in E. \tag{6.3}$$

By solving (6.3), we obtain the white noise operator $U_S \in \mathcal{L}((E),(E)^*)$ satisfying the equation (6.1).

Now, to apply Theorem 4.4 to solve the quantum white noise differential equation given as in (6.3), we want to find white noise operator $G \in \mathcal{L}((E),(E)^*)$ satisfying

$$(D_u^- - D_{KSu}^+)\,G = a^*((S-I)u) + a(K(S-I)u). \tag{6.4}$$

Consider the white noise operator $G \in \mathcal{L}((E),(E)^*)$ given as in

$$G = \Delta_G^*(L) + \Lambda(M) + \Delta_G(N), \tag{6.5}$$

where $L, M, N \in \mathcal{L}(E, E^*)$. Then from Lemma 4.2, we obtain that

$$\begin{aligned} D_u^- G &= a^*(Mu) + a(Nu) + a(N^*u), \\ D_{KSu}^+ G &= a^*(LKSu) + a^*(L^*KSu) + a(M^*KSu), \end{aligned}$$

from which we have

$$(D_u^- - D_{KSu}^+)\,G = a^*((M - LKS - L^*KS)u) + a((N + N^* - M^*KS)u).$$

On the other hand, since the operators $\Delta_G^*(L)$ and $\Delta_G(N)$ are uniquely determined by symmetric operators L and N, respectively, we may assume that L and N are symmetric, i.e., $L^* = L$ and $N^* = N$. Then we have the quantum white noise differential equation:

$$(D_u^- - D_{KSu}^+)\,G = a^*((M - 2LKS)u) + a((2N - M^*KS)u). \tag{6.6}$$

Then by comparing Equations (6.4) and (6.6), we have

$$a^*((S-I)u) + a(K(S-I)u) = a^*((M - 2LKS)u) + a((2N - M^*KS)u)$$

for all $u \in E$, which is equivalent to

$$S - I = M - 2LKS, \quad K(S-I) = 2N - M^*KS, \tag{6.7}$$

where the operators L, M and N are unknown.

Theorem 6.3 *Let $K, S : E \to E$ be real linear operators. Suppose that there exist operators $L, M, N \in \mathcal{L}(E, E^*)$ such that the equations given as in* (6.7) *hold. Then there exists a white noise operator $U_S \in \mathcal{L}((E), (E)^*)$ such that the diagram given as in* (6.2) *commutes. Furthermore, the white noise operator $U_S \in \mathcal{L}((E), (E)^*)$ is given by*

$$\begin{aligned} U_S &= (\text{wexp}\,(\Delta_{\text{G}}^*(L) + \Lambda(M) + \Delta_{\text{G}}(N))\,U) \diamond F \\ &= F \diamond \text{wexp}\,(\Delta_{\text{G}}^*(L) + \Lambda(M) + \Delta_{\text{G}}(N)) \end{aligned} \tag{6.8}$$

with a white noise operator $F \in \mathcal{L}((E), (E)^*)$ satisfying $\left(D_u^- - D_{KSu}^+\right) F = 0.$

Proof. By above discussions, we see that

$$(D_u^- - D_{KSu}^+)\,G = a^*((S - I)u) + a(K(S - I)u)$$

under the assumptions, where the white noise operator $G \in \mathcal{L}((E), (E)^*)$ is given as in (6.5). Therefore, by applying Theorem 4.4, we see that a general solution U_S of the quantum white noise differential equation given as in (6.3) is given as in (6.8), and hence U_S satisfies the intertwining property given as in (6.1). ∎

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