Semi-classical limits for the Nelson model

Zied Ammari[*], Marco Falconi[†] and Fumio Hiroshima[‡]

Abstract

We are concerned with the Nelson Hamiltonian $H_\hbar$ with semi-classical parameter $\hbar > 0$. A classical object $\mathcal{H}(q_s, p_s, u_s, \bar{u}_s)$ is defined by the solution $\{q_s, p_s, u_s\}$ to the Hamilton-Jacobi equation associated with the Nelson Hamiltonian. We show the asymptotic behaviour of

$$e^{-i\frac{t}{\hbar}H_\hbar}e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\mathrm{d}s}$$

as $\hbar \to 0$. Furthermore we introduce Wigner measures μ_0 on the particle-field phase space $X = \mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$ appearing in the semi-classical limits of a family of trace class operators $\{\rho_\hbar, \hbar \in (0,1)\}$. I.e.,

$$\lim_{\hbar\to 0} \mathrm{Tr}(\rho_\hbar \mathcal{W}(\xi')) = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_0(x)$$

for $\xi' \in X$ and $\mathcal{W}(\xi)$ denotes an exponential operator. The Wigner measure μ_t associated with the family of time evolutions of trace class operators $\{\rho_\hbar(t), \hbar \in (0,1)\}$ are given by

$$\lim_{\hbar\to 0} \mathrm{Tr}(\rho_\hbar(t) \mathcal{W}(\xi')) = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_t(x).$$

We show that $\mu_t(\cdot) = \mu_0 \circ \Phi_t^{-1}(\cdot)$, where Φ_t is the flow for the solution to the Hamilton-Jacobi equation.

1 Hamilton-Jacobi equation for the Nelson model

In the RIMS conference held on December 6-8, 2021 we gave a talk on the title "Newton Maxwell equation through semi-classical analysis". In this article, however, we are concerned with the Nelson model on coherent states for the simplicity and demonstrate a motivation why we are interested in the semiclassical analysis. This results are ultimately developed in [2] for the Pauli-Fierz model in non-relativistic QED [10] and the semi-classical limit is

[*]Rennes I University
[†]Polytechnic University of Milan
[‡]Kyushu University

investigated through the so-called Wigner measures. The Wigner measure is a probability measure on the total phase space $\mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$. The Wigner measure is applied to the semi-classical analysis in [7] for Schrödinger operators and is extended to an infinite dimensional phase space in [3]. We refer [1, 4, 6, 5] for related investigations.

1.1 Semi-classical limit of Schrödinger operators

Before going to our main results, we introduce a semi-classical limit of Schrödinger operators for the readers convenient. Let us consider $3D$-Schrödinger operator of the form:

$$h_\hbar = \frac{\hbar^2}{2m} D_x^2 + V(x),$$

where $\hbar > 0$ is a semi-classical parameter, $m > 0$ a mass of a particle, $D_x = -i\nabla_x$ and V is an external potential. Let

$$\mathcal{H} = \mathcal{H}(q,p) = \frac{p^2}{2m} + V(q),$$

where $(p,q) \in \mathbb{R}^3 \times \mathbb{R}^3$. The Hamilton-Jacobi equation associated with $h_\hbar$ is

$$\begin{cases} \dot{q}_t & = \dfrac{\delta \mathcal{H}}{\delta p_t} & = \dfrac{p_t}{m}, \\ \dot{p}_t & = -\dfrac{\delta \mathcal{H}}{\delta q_t} & = -\nabla V(q_t). \end{cases} \tag{1.1}$$

Let $(q_t, p_t) \in \mathbb{R}^3 \times \mathbb{R}^3$ be the solution to (1.1). We are interested in the asymptotic behaviour of

$$e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}$$

as $\hbar \to 0$.

Let $\xi_\hbar$ be the $3D$-dilation defined by

$$\xi_\hbar f(x) = \hbar^{3/4} f(\sqrt{\hbar} x),$$

and hence $\xi_\hbar^* f(x) = \hbar^{-3/4} f(x/\sqrt{\hbar})$. Let us define the quadratic operator Q_t^{Sch} by

$$Q_t^{Sch} = \frac{1}{2m} D_x^2 + x \cdot \nabla^2 V(q_t) x.$$

Then it actually follows that

$$\lim_{\hbar \to 0} \| e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s} \varphi - e^{\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} \xi_\hbar^* e^{-i\int_0^t Q_s^{Sch}\mathrm{d}s} \xi_\hbar e^{-\frac{i}{\hbar}(p_0 x - \hbar q_0 D_x)} \varphi \| = 0. \tag{1.2}$$

This can be proven as follows. Let

$$\gamma_t = \xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\varphi.$$

Then

$$\|e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\varphi - e^{\frac{i}{\hbar}(p_t x - \hbar q_t D_x)}\xi_\hbar^* e^{-i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_0 x - \hbar q_0 D_x)}\varphi\| = \|\gamma_t - \gamma_0\| \le \int_0^t \|\dot\gamma_s\|\mathrm{d}s.$$

We see that

$$\begin{aligned}
\dot\gamma_t =& e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\frac{i}{\hbar}\mathcal{H}(q_t,p_t)\xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar}\varphi \\
&+ e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\xi_\hbar^* i\dot{Q} e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar}\varphi \\
&+ e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar \left\{-\frac{i}{\hbar}(\dot p_t x - \hbar \dot q_t D_x)\right\} e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar}\varphi \\
&+ e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} \left\{-\frac{i}{\hbar}h_\hbar\right\} e^{-i\frac{t}{\hbar}h_\hbar}\varphi.
\end{aligned}$$

We compute $\xi_\hbar e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)}\left\{-\frac{i}{\hbar}h_\hbar\right\}$. By a shift operator $e^{\frac{i}{\hbar}(px - q\hbar D_x)}$,

$$h_\hbar \to \frac{(\hbar D_x + p_t)^2}{2m} + V(x + q_t),$$

and by a scaling $\xi_\hbar$,

$$\to \frac{(\sqrt{\hbar}D_x + p_t)^2}{2m} + V(\sqrt{\hbar}x + q_t).$$

Then the right-hand side above is

$$\begin{aligned}
\xi_\hbar e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)}h_\hbar &= \frac{(\sqrt{\hbar}D_x + p_t)^2}{2m} + V(\sqrt{\hbar}x + q_t) \\
&= \mathcal{H}(q_t,p_t) + \sqrt{\hbar}\left(\frac{p_t D_x}{m} + \nabla V(q_t)x\right) + \hbar Q_t^{Sch} + O(\hbar^{3/2}). \qquad (1.3)
\end{aligned}$$

Furthermore

$$\xi_\hbar\left\{-\frac{i}{\hbar}(\dot p_t x - \hbar \dot q_t D_x)\right\} = -\frac{i}{\sqrt{\hbar}}(\dot p_t x - \dot q_t D_x)\xi_\hbar.$$

Hence

$$\begin{aligned}
\dot\gamma_t =& \xi_\hbar^* e^{i\int_0^t Q_s^{Sch}\mathrm{d}s}\left\{\frac{i}{\hbar}\mathcal{H}(q_t,p_t) + iQ_t^{Sch} - \frac{i}{\sqrt{\hbar}}(\dot p_t x - \dot q_t D_x) - \frac{i}{\hbar}(1.3)\right\}\xi_\hbar \\
&\times e^{-\frac{i}{\hbar}(p_t x - \hbar q_t D_x)} e^{-i\frac{t}{\hbar}h_\hbar} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s)\mathrm{d}s}\varphi.
\end{aligned}$$

By (1.1) we have

$$\frac{i}{\hbar}\mathcal{H}(q_t,p_t)+iQ_t^{Sch}-\frac{i}{\sqrt{\hbar}}(\dot{p}_t x-\dot{q}_t D_x)-\frac{i}{\hbar}(1.3)=O(\sqrt{\hbar}).$$

Then (1.2) follows. In the semi-classical region we can see that

$$e^{-i\frac{t}{\hbar}h_\hbar}e^{+\frac{i}{\hbar}\int_0^t\mathcal{H}(q_s,p_s)\mathrm{d}s}\sim e^{\frac{i}{\hbar}(p_t x-\hbar q_t D_x)}\xi_\hbar^* e^{-i\int_0^t Q_s^{Sch}\mathrm{d}s}\xi_\hbar e^{-\frac{i}{\hbar}(p_0 x-\hbar q_0 D_x)}.$$

Here we emphasize that Q_s^{Sch} is independent of $\hbar$. We extend this kind of arguments to the Nelson model in quantum field theory in what follows.

1.2 Nelson model

Let $a^\dagger(f)$ and $a(f)$ be the annihilation operator and the creation operator, respectively on the boson Fock space over $L^2(\mathbb{R}^3)$:

$$\mathcal{F}=\bigoplus_{n=0}^{\infty}[\otimes_s^n L^2(\mathbb{R}^3)].$$

The adjoint relation is $a(f)^*=a^\dagger(\bar{f})$ and the CCR is given by $[a(f),a^\dagger(g)]=(\bar{f},g)\mathbb{1}$ and $[a^\sharp(f),a^\sharp(g)]=0$, where (f,g) denotes a scalar product on $L^2(\mathbb{R}^3)$ and it is linear in g and anti-linear in f. Formally we write $a^\sharp(f)=\int a^\sharp(k)f(k)\mathrm{d}k$. The field operator is given by

$$\phi(f)=\frac{1}{\sqrt{2}}(a^\dagger(f)+a(\bar{f}))$$

and its momentum conjugate by

$$\Pi(f)=\frac{i}{\sqrt{2}}(a^\dagger(f)-a(\bar{f})).$$

Thus $[\phi(f),\Pi(g)]=i\,\mathrm{Re}(f,g)$, $[\phi(f),\phi(g)]=i\,\mathrm{Im}(f,g)$ and $[\Pi(f),\Pi(g)]=i\,\mathrm{Im}(f,g)$ hold true. Let $H_\mathrm{f}=d\Gamma(\omega)$ be the second quantization of the multiplication by $\omega(k)=|k|$. Here $|k|$ denotes the energy of a massless boson with momentum $k\in\mathbb{R}^3$.

The Nelson Hamiltonian [9, 8] is defined as s self-adjoint operator on the product Hilbert space

$$\mathcal{H}=L^2(\mathbb{R}^3)\otimes\mathcal{F}$$

and is given by

$$H=(\frac{1}{2m}D_x^2+V)\otimes\mathbb{1}+\mathbb{1}\otimes H_\mathrm{f}+H_\mathrm{I},$$

and the interaction by

$$H_{\rm I}=H_{\rm I}(x)\phi(e^{-ikx}\hat{\varphi}/\sqrt{\omega})=\frac{1}{\sqrt{2}}\int\left\{\frac{e^{-ikq}\hat{\varphi}(k)}{\sqrt{\omega(k)}}a^\dagger(k)+\frac{e^{ikq}\bar{\hat{\varphi}}(k)}{\sqrt{\omega(k)}}a(k)\right\}{\rm d}k.$$

Here $\hat{\varphi}$ is a cutoff function. We assume that $\omega\sqrt{\omega}\hat{\varphi}, \sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\sqrt{\omega}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^3)$. Throughout we suppose that $V \in C^2(\mathbb{R}^3)$ and bounded. Then H is self-adjoint on $D(D_x^2)\cap D(H_{\rm f})$ and bounded from below. We introduce the semi-classical parameter $\hbar>0$ by

$$H_\hbar=(\frac{\hbar^2}{2m}D_x^2+V)\otimes 1\!\!1+\sqrt{\hbar}H_{\rm I}+\hbar 1\!\!1\otimes H_{\rm f}.$$

Let $(q,p,u)\in\mathbb{R}^3\times\mathbb{R}^3\times L^2(\mathbb{R}^3)$. The classical Nelson Hamiltonian is given by

$$\mathcal{H}(p,q,u,\bar{u})=\frac{p^2}{2m}+V(q)+\int_{\mathbb{R}^3}\omega(k)|u(k)|^2{\rm d}k+U(q,u).$$

Here

$$U(q,u)=\frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\left\{\frac{e^{-ikq}\hat{\varphi}(k)}{\sqrt{\omega(k)}}\bar{u}(k)+\frac{e^{ikq}\bar{\hat{\varphi}}(k)}{\sqrt{\omega(k)}}u(k)\right\}{\rm d}k.$$

The time evolution of $(p,q,u)\in\mathbb{R}^3\times\mathbb{R}^3\times L^2(\mathbb{R}^3)$ are governed by the Hamilton-Jacobi equation:

$$(N)\begin{cases}\dot{q}_t & =\dfrac{\delta\mathcal{H}}{\delta p_t} \quad =\dfrac{p_t}{m},\\[2ex] \dot{p}_t & =-\dfrac{\delta\mathcal{H}}{\delta q_t} \quad =-\nabla V(q_t)-\nabla U(q_t,u_t),\\[2ex] i\dot{u}_t(k) & =\dfrac{\delta\mathcal{H}}{\delta\bar{u}_t} \quad =\omega(k)u_t(k)+\dfrac{e^{-ikq_t}\hat{\varphi}(k)}{\sqrt{\omega(k)}}.\end{cases}$$

Here

$$\nabla U(q_t,u_t)=\frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\left\{-ik\frac{e^{-ikq_t}\hat{\varphi}(k)}{\sqrt{\omega(k)}}\bar{u}_t(k)+ik\frac{e^{ikq_t}\bar{\hat{\varphi}}(k)}{\sqrt{\omega(k)}}u_t(k)\right\}{\rm d}k.$$

Note that $\sqrt{\omega}\hat{\varphi}\in L^2(\mathbb{R}^3)$ and then the right-hand side above is finite.

2 Coherent states and Weyl commutation relations

Now we define coherent states for the field and the particle. In general, when $[A,B]$ is c-number, then formally

$$e^Ae^B=e^{\frac{1}{2}[A,B]}e^{A+B}$$

holds true. Let $W(f) = e^{i\Pi(f)}$. Then Weyl commutation relation holds:

$$W(f)W(g) = e^{-\frac{i}{2}\operatorname{Im}(f,g)}W(f+g).$$

Since $W(ig) =^{-\Phi(g)}$, we can see that

$$W(f)W(ig) = e^{-\frac{i}{2}\operatorname{Re}(f,g)}W(f+ig).$$

Let $z = q + ip \in \mathbb{R}^3 + i\mathbb{R}^3$. Define $T(z) = e^{i(px - q\hbar D_x)}$. Note that

$$[px - q\hbar D_x, p'x - q'\hbar D_x] = i\hbar(qp' - pq') = i\hbar \operatorname{Im}\bar{z}\cdot z'.$$

Hence

$$T(z)T(z') = e^{-\frac{i}{2}\hbar \operatorname{Im}\bar{z}\cdot z'}T(z+z')$$

and

$$T(z)T(iz') = e^{-\frac{i}{2}\hbar \operatorname{Re}\bar{z}\cdot z'}T(z+iz').$$

The coherent state smeared by u is defined by

$$W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega,$$

where $\Omega \in \mathcal{F}$ is the Fock vacuum. Note that

$$W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) = e^{-\frac{1}{\sqrt{\hbar}}(a^\dagger(u) - a(\bar{u}))}.$$

Let $(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3$ be a point in the phase space and

$$\psi_\hbar(x) = (\pi\hbar)^{-3/4}e^{-|x|^2/(2\hbar)}.$$

Thus $\|\psi_\hbar\| = 1$. The coherent state for the particle part is given by

$$\psi_{q,p}^\hbar(x) = T_{q,p}^\hbar \psi_\hbar,$$

where $T_{q,p}^\hbar = T\left(\frac{z}{\hbar}\right)$ for $z = q + ip$, i.e.,

$$T_{q,p}^\hbar = \exp\left(\frac{i}{\hbar}(px - \hbar q D_x)\right).$$

Note that $\psi_{q,p}^\hbar$ is normalized in $L^2(\mathbb{R}^3)$ for each $(q,p) \in \mathbb{R}^3 \times \mathbb{R}^3$. We see that

$$T_{q,p}^\hbar = e^{-\frac{i}{2}\frac{1}{\hbar}pq}e^{\frac{i}{\hbar}px}e^{-iqD_x} = e^{\frac{i}{2}\frac{1}{\hbar}pq}e^{-iqD_x}e^{\frac{i}{\hbar}px}.$$

Let (q_t, p_t, u_t) be the solution to (N). Define

$$\Phi_t^\hbar = T^\hbar_{q_t,p_t,u_t}(\psi_\hbar \otimes \Omega), \quad t \geq 0.$$

Here

$$T^\hbar_{q,p,u} = T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right), \quad z = q + ip,$$

is unitary. The unitary operator $T^\hbar_{q_t,p_t,u_t}$ is the shift operator such that

$$\begin{aligned}
&T^{\hbar*}_{q_t,p_t,u_t} x T^\hbar_{q_t,p_t,u_t} = x + q_t, \\
&T^{\hbar*}_{q_t,p_t,u_t} \hbar D_x T^\hbar_{q_t,p_t,u_t} = \hbar D_x + p_t, \\
&T^{\hbar*}_{q_t,p_t,u_t} \sqrt{\hbar} a(k) T^\hbar_{q_t,p_t,u_t} = \sqrt{\hbar} a(k) + u_t(k), \\
&T^{\hbar*}_{q_t,p_t,u_t} \sqrt{\hbar} a^\dagger(k) T^\hbar_{q_t,p_t,u_t} = \sqrt{\hbar} a^\dagger(k) + \bar{u}_t(k).
\end{aligned}$$

From these relations we can see that

$$\begin{aligned}
&(x + i\hbar D_x)\Phi_t^\hbar = (q_t + ip_t)\Phi_t^\hbar, \\
&\sqrt{\hbar} a(k)\Phi_t^\hbar = u_t(k)\Phi_t^\hbar, \\
&\sqrt{\hbar} a^\dagger(k)\Phi_t^\hbar = \bar{u}_t(k)\Phi_t^\hbar.
\end{aligned}$$

The classical objects appear as the eigenvalues.

3 Semi-classical limits

In this section we shall prove that

$$\lim_{\hbar \to 0} \| e^{-i\frac{t}{\hbar}H_\hbar}\Phi_\hbar - e^{-\frac{i}{\hbar}\int_0^t \mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\mathrm{d}s} T^\hbar_{q_t,p_t,u_t} e^{-\frac{i}{\hbar}\int_0^t Q_{\hbar,s}\mathrm{d}s} T^{\hbar*}_{q_0,p_0,u_0}\Phi_\hbar \| = 0. \tag{3.1}$$

Here $\int_0^t Q_{\hbar,s}\mathrm{d}s$ is a quadratic operator derived from $H_\hbar$. The strategy to see (3.1) is due to the fact

$$T^{\hbar*}_{q_t,p_t,u_t} H_\hbar T^\hbar_{q_t,p_t,u_t} = \mathcal{H}(q_t,p_t,u_t,\bar{u}_t) + Q_{\hbar,t} + \text{reminder} + O(\sqrt{\hbar}).$$

This corresponds to (1.3) for Schrödinger operators. See (3.3). The quadratic term is given by

$$Q_{\hbar,t} = \frac{\hbar^2}{2m}D_x^2 + \frac{1}{2}x \cdot \left(\nabla^2 V(q_t) + \nabla^2 U(q_t,u_t)\right) x + \sqrt{\hbar}\nabla H_\mathrm{I}(q_t)x + \hbar H_\mathrm{f}.$$

Here $\nabla H_{\rm I}(q_s) = \phi(-ike^{-ikq_s}\hat\varphi/\sqrt{\omega})$, $\nabla^2 V(q_t) = (\nabla_\alpha\nabla_\beta V(q_t))_{1\le\alpha,\beta\le 3}$ and $\nabla^2 U(q_t,u_t) = (\nabla_\alpha\nabla_\beta U(q_t,u_t))_{1\le\alpha,\beta\le 3}$ with

$$\nabla_\alpha\nabla_\beta U(q_t,u_t) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\left\{-k_\alpha k_\beta\frac{e^{-ikq_t}\hat\varphi(k)}{\sqrt{\omega(k)}}\bar u_t(k) - k_\alpha k_\beta\frac{e^{ikq_t}\bar{\hat\varphi}(k)}{\sqrt{\omega(k)}}u_t(k)\right\}{\rm d}k.$$

The main theorem is as follows.

Theorem 3.1 *Let $(q,p,u)\in\mathbb{R}^3\times\mathbb{R}^3\times L^2(\mathbb{R}^3)$. Suppose that $(q_t,p_t,u_t)\in\mathbb{R}^3\times\mathbb{R}^3\times L^2(\mathbb{R}^3)$ is the solution to (N) with initial condition $(q_0,p_0,u_0)=(q,p,u)$. Then*

$$\|e^{-i\frac{t}{\hbar}H_\hbar}e^{\frac{i}{\hbar}\int_0^t\mathcal{H}(q_s,p_s,u_s,\bar u_s){\rm d}s}\Phi_\hbar - T^\hbar_{q_t,p_t,u_t}e^{-\frac{i}{\hbar}\int_0^t Q_{\hbar,s}{\rm d}s}T^{\hbar*}_{q,p,u}\Phi_\hbar\|\le C\sqrt{\hbar}. \tag{3.2}$$

Proof: Let $\xi_\hbar$ be the dilation defined by $\xi_\hbar f(x) = \hbar^{3/4}f(\sqrt{\hbar}x)$, and hence $\xi_\hbar^* f(x) = \hbar^{-3/4}f(x/\sqrt{\hbar})$. In particular we have

$$\xi_\hbar^*\psi_1(x) = \psi_\hbar(x).$$

To show (3.2), the Cook method is applied. Let

$$\nu_t = \xi_\hbar e^{\frac{i}{\hbar}\int_0^t Q_{\hbar,s}{\rm d}s}T^{\hbar*}_{q_t,p_t,u_t}e^{-i\frac{t}{\hbar}H_\hbar}e^{\frac{i}{\hbar}\int_0^t\mathcal{H}(q_s,p_s,u_s,\bar u_s){\rm d}s}\xi_\hbar^*\Phi_1,$$

where $\Phi_1=\psi_1\otimes\Omega$. Since $\nu_0 = \xi_\hbar T^*_{q,p,u}\xi^*_\hbar$, we have $\nu_t-\nu_0 = \int_0^t\dot\nu_s{\rm d}s$ and then the left-hand side of (3.2) can be written as $\|\nu_t-\nu_0\|$. We note that

$$\nu_t-\nu_0 = \int_0^t \xi_\hbar e^{\frac{i}{\hbar}\int_0^s Q_{\hbar,r}{\rm d}r}\left(\frac{i}{\hbar}\mathcal{H}(q_s,p_s,u_s,\bar u_s) + \frac{i}{\hbar}Q_{\hbar,s} + C_s - \frac{i}{\hbar}H_\hbar(s)\right)T^{\hbar*}_{q_s,p_s,u_s}e^{-i\frac{s}{\hbar}H_\hbar}\xi_\hbar^*\Phi_1{\rm d}s,$$

where

$$\frac{d}{ds}T^{\hbar*}_{q_s,p_s,u_s} = C_sT^{\hbar*}_{q_s,p_s,u_s}.$$

Since $T^\hbar_{q_t,p_t,u_t}$ acts as the shift by $x\to x+q_t$, $\hbar D_x\to\hbar D_x+p_t$ and $\sqrt{\hbar}a(k)\to\sqrt{\hbar}a(k)+u(k)$, we used the intertwining property:

$$T^{\hbar*}_{q_t,p_t,u_t}H_\hbar = H_\hbar(t)T^{\hbar*}_{q_t,p_t,u_t},$$

where

$$\begin{aligned}H_\hbar(t) =&\frac{(\hbar D_x+p_t)^2}{2m} + V(x+q_t) + \int_{\mathbb{R}^3}\omega(k)(\sqrt{\hbar}a^\dagger(k)+\bar u_t(k))(\sqrt{\hbar}a(k)+u_t(k)){\rm d}k\\ &+\int_{\mathbb{R}^3}\left\{\frac{e^{-ik(x+q_t)}}{\sqrt{\omega(k)}}\hat\varphi(k)(\sqrt{\hbar}a^\dagger(k)+\bar u_t(k)) + \frac{e^{+ik(x+q_t)}}{\sqrt{\omega(k)}}\bar{\hat\varphi}(k)(\sqrt{\hbar}a(k)+u_t(k))\right\}{\rm d}k\\ =&\frac{(\hbar D_x+p_t)^2}{2m} + V(x+q_t) + U(x+q_t,u_t) + \hbar H_{\rm f} + \int_{\mathbb{R}^3}\omega(k)|u_t(k)|^2{\rm d}k\\ &+\sqrt{\hbar}\sqrt{2}\phi(\omega u_t) + \sqrt{\hbar}H_{\rm I}(x+q_t).\end{aligned}\tag{3.3}$$

We shall estimate the term $\xi_\hbar\left(\frac{i}{\hbar}\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)+\frac{i}{\hbar}Q_{\hbar,s}+C_s-\frac{i}{\hbar}H_\hbar(s)\right)\xi_\hbar^*$. Since

$$\xi_\hbar\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\xi_\hbar^*=\mathcal{H}(q_s,p_s,u_s,\bar{u}_s),$$

we investigate

$$\frac{i}{\hbar}\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)+\frac{i}{\hbar}\xi_\hbar Q_{\hbar,s}\xi_\hbar^*+\xi_\hbar\left(C_s-\frac{i}{\hbar}H_\hbar(s)\right)\xi_\hbar^*.$$

We can directly compute C_t as

$$C_t=-\frac{i}{\hbar}\left(\dot{p}_tx-\hbar\dot{q}_tD_x\right)-\frac{1}{\sqrt{\hbar}}\left\{a^\dagger(\dot{u}_t)-a(\overline{\dot{u}_t})\right\}.$$

Note that $\xi_\hbar x\xi_\hbar^*=x\sqrt{\hbar}$ and $\xi_\hbar D_x\xi_\hbar^*=D_x/\sqrt{\hbar}$. Then

$$\xi_\hbar C_t\xi_\hbar^*=-\frac{i}{\sqrt{\hbar}}\left\{\dot{p}_tx-\dot{q}_tD_x-(a^\dagger(i\dot{u}_t)+a(\overline{i\dot{u}_t}))\right\}.$$

Next we compute $\xi_\hbar H_\hbar(t)\xi_\hbar^*$. By (3.3) we have

$$\begin{aligned}\xi_\hbar H_\hbar(t)\xi_\hbar^*=&\frac{(\sqrt{\hbar}D_x+p_t)^2}{2m}+V(\sqrt{\hbar}x+q_t)+U(\sqrt{\hbar}x+q_t,u_t)\\&+\sqrt{\hbar}H_{\mathrm{I}}(\hbar x+q_t)+\sqrt{\hbar}\sqrt{2}\phi(\omega u_t)+\int_{\mathbb{R}^3}\omega(k)|u_t(k)|^2\mathrm{d}k+\hbar H_{\mathrm{f}}.\end{aligned}$$

By

$$\begin{aligned}&V(\sqrt{\hbar}x+q_t)=V(q_t)+\sqrt{\hbar}\nabla V(q_t)x+\frac{1}{2}\hbar x\cdot\nabla^2V(q_t)x+O(\hbar^{3/2}),\\&U(\sqrt{\hbar}x+q_t,u_t)=U(q_t,u_t)+\sqrt{\hbar}\nabla U(q_t,u_t)x+\frac{1}{2}\hbar x\cdot\nabla^2U(q_t,u_t)x+O(\hbar^{3/2}),\\&\sqrt{\hbar}H_{\mathrm{I}}(\sqrt{\hbar}x+q_t)=\sqrt{\hbar}H_{\mathrm{I}}(q_t)+\hbar\nabla H_{\mathrm{I}}(q_t)x+O(\hbar^{3/2}),\end{aligned}$$

we see that

$$\begin{aligned}&\xi_\hbar\left(C_t-\frac{i}{\hbar}H_\hbar(t)\right)\xi_\hbar^*\\&=-i\left\{\frac{1}{2m}D_x^2+\frac{1}{2}x\cdot\nabla^2V(q_t)x+\frac{1}{2}x\cdot\nabla U(q_t,u_t)x+\nabla H_{\mathrm{I}}(q_t)x+H_{\mathrm{f}}\right\}\\&-\frac{i}{\sqrt{\hbar}}\left\{\frac{1}{m}pD_x+\nabla V(q_t)x+\nabla U(q_t,u_t)x+\sqrt{2}\phi(\sqrt{\omega}u_t)+H_{\mathrm{I}}(q_t)\right.\\&\qquad\qquad\left.+\dot{p}_tx-\dot{q}_tD_x-(a^\dagger(i\dot{u}_t)+a(\overline{i\dot{u}_t}))\right\}\\&-\frac{i}{\hbar}\left\{\frac{1}{2m}p_t^2+V(q_t)+\int\omega(k)|u_t(k)|^2\mathrm{d}k+U(q_t,u_t)\right\}+O(\sqrt{\hbar}).\end{aligned}$$

The second term of the right-hand side above is identically zero by equation (N). Hence

$$\xi_\hbar \left(C_t - \frac{i}{\hbar} H_\hbar(t) \right) \xi_\hbar^* = -\frac{i}{\hbar} \xi_\hbar Q_{\hbar,t} \xi_\hbar^* - \frac{i}{\hbar} \mathcal{H}(q_t, p_t, u_t, \bar{u}_t) + O(\sqrt{\hbar}). \tag{3.4}$$

It follows that

$$\begin{aligned}
&\|\nu_t - \nu_0\| \\
&\leq \int_0^t \left\| \left(\frac{i}{\hbar} \mathcal{H}(q_s, p_s, u_s, \bar{u}_s) + \frac{i}{\hbar} \xi_\hbar Q_{\hbar,s} \xi_\hbar^* + \xi_\hbar (C_s - \frac{i}{\hbar} H_\hbar(s)) \xi_\hbar^* \right) \xi_\hbar T^*_{q_s,p_s,u_s} e^{-i\frac{s}{\hbar} H_\hbar} \xi_\hbar^* \Phi_1 \right\| \mathrm{d}s \\
&\leq tC\sqrt{\hbar} \|\Phi_1\|
\end{aligned}$$

with some constant $C > 0$ by (3.4). Then the theorem follows. ■

4 Wigner measures

In this section we introduce Wigner measures on the phase space $\mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$ appearing in the semi-classical limits of a family of trace class operators $\{\rho_\hbar, \hbar \in (0,1)\}$. This has been studied in e.g., [7, 3].

4.1 Examples

We recall that

$$T^\hbar_{q_t,p_t,u_t} = T\left(\frac{z_t}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u_t}{\sqrt{\hbar}}\right),$$

where $z_t = q_t + ip_t \in \mathbb{R}^3 + i\mathbb{R}^3$ and $u_t \in L^2(\mathbb{R}^3)$ are the solution to (N). In the previous section we consider the asymptotic behavior of $T^\hbar_{q_t,p_t,u_t}$ as $\hbar \to 0$ in the sense of Theorem 3.1. Note that $\|T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar\| = 1$ but $(\Phi_\hbar, T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar) \to 0$ as $\hbar \to 0$.

In this section the following strategy is taken to analyze the asymptotic behavior of coherent vector $T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar$ as $\hbar \to 0$. For each $z = q + ip \in \mathbb{R}^3 + i\mathbb{R}^3$ and $u \in L^2(\mathbb{R}^3)$, we define the trace class operator $\mathcal{C}_\hbar(z, u)$ by

$$\mathcal{C}_\hbar = \mathcal{C}_\hbar(z, u) = |T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar\rangle\langle T\left(\frac{z}{\hbar}\right) \otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right) \Phi_\hbar|.$$

This is a one-rank operator. Let $z' = q' + ip' \in \mathbb{R}^3 + i\mathbb{R}^3$ nd $u' \in L^2(\mathbb{R}^3)$. We prepare the operator

$$\mathcal{W} = \mathcal{W}(z', u') = T(2\pi i z') \otimes W(\sqrt{2}\pi i \sqrt{\hbar} u') = e^{2\pi i (q' x + p' \hbar D_x)} \otimes e^{-\sqrt{2}\pi i \sqrt{\hbar} \phi(u')}.$$

We consider the asymptotic behaviour of the trace $\mathrm{Tr}(\mathcal{C}_\hbar \mathcal{W})$.

Lemma 4.1 *Let* $z = q + ip, z' = q' + ip' \in \mathbb{R}^3 + i\mathbb{R}^3$ *and* $u, u' \in L^2(\mathbb{R}^3)$. *Then it follows that*

$$\lim_{\hbar\to 0} \mathrm{Tr}(\mathcal{C}_\hbar(z,u)\mathcal{W}(z',u')) = e^{2\pi i\,\mathrm{Re}((u,u')+\bar{z}\cdot z')}.$$

Proof: The formulae $W(f)^* = W(-f)$ and $T(z)^* = T(-z)$, and

$$(W(f)\Omega, W(ig)W(f)\Omega) = (\Omega, W(ig)\Omega)e^{i\,\mathrm{Re}(f,g)}$$

and

$$(T(z)\psi, T(iz')T(z)\psi) = (\psi, T(iz')\psi)e^{i\,\mathrm{Re}\bar{z}\cdot z'}$$

are useful. We see that $\mathrm{Tr}(\mathcal{C}_\hbar(z,u)\mathcal{W}(z',u'))$ can be decomposed into two factors:

$$\begin{aligned}&\mathrm{Tr}(\mathcal{C}_\hbar(z,u)\mathcal{W}(z',u'))\\ &= (T\left(\frac{z}{\hbar}\right)\psi_\hbar, T(2\pi iz')T\left(\frac{z}{\hbar}\right)\psi_\hbar)\cdot(W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega, W(\sqrt{2}\pi i\sqrt{\hbar}u')W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega).\end{aligned}$$

Then the field part turns out to be

$$(W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega, W(\sqrt{2}\pi i\sqrt{\hbar}u')W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right)\Omega) = (\Omega, W(\sqrt{2}\pi i\sqrt{\hbar}u')\Omega)e^{2\pi i\,\mathrm{Re}(u,u')}$$

and the particle part

$$(T\left(\frac{z}{\hbar}\right)\psi_\hbar, T(2\pi iz')T\left(\frac{z}{\hbar}\right)\psi_\hbar) = (\psi_\hbar, T(2\pi iz')\psi_\hbar)e^{2\pi i\,\mathrm{Re}\bar{z}\cdot z'}.$$

We also see that

$$\lim_{\hbar\to 0}(\Omega, W(\sqrt{2}\pi i\sqrt{\hbar}u')\Omega)e^{2\pi i\,\mathrm{Re}(u,u')} = e^{2\pi i\,\mathrm{Re}(u,u')}.$$

Since $\psi_\hbar^2 \to \delta(x)$ and $T(2\pi iz') \to e^{2\pi i q' x}$ as $\hbar \to 0$, we can see that

$$\lim_{\hbar\to 0}(\psi_\hbar, T(2\pi iz')\psi_\hbar)e^{2\pi i\,\mathrm{Re}\bar{z}\cdot z'} = e^{2\pi i\,\mathrm{Re}\bar{z}\cdot z'}.$$

Then the lemma is proven. ■

4.2 Wigner measures

Let $X = \mathbb{R}^3 \times \mathbb{R}^3 \times L^2(\mathbb{R}^3)$. Set

$$(\xi, \xi')_X = qq' + pp' + i(qp' - pq') + (u, u')$$

for $\xi = (q, p, u) \in X$ and $\xi' = (q', p', u') \in X$. We define $\mathcal{W}(\xi') = \mathcal{W}(z', u') = \mathcal{W}(q', p', u')$ and $\mathcal{C}_\hbar(\xi) = \mathcal{C}_\hbar(z, u) = \mathcal{C}_\hbar(q, p, u)$. Then the statements of Lemma 4.1 can be rewritten as

$$\lim_{\hbar\to 0} \mathrm{Tr}(\mathcal{C}_\hbar(\xi)\mathcal{W}(\xi')) = e^{2\pi i \,\mathrm{Re}(\xi,\xi')_X}.$$

Furthermore

$$e^{2\pi i \,\mathrm{Re}(\xi,\xi')_X} = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_\xi(x),$$

where $\mu_\xi(x)$ is the Dirac measure $\delta_\xi(x)$ on the phase space X with mass at $x = \xi$. This is called the Wigner measure associated with $\{\mathcal{C}_\hbar(\xi), \hbar \in (0,1)\}$. In [2] we consider Wigner measures μ_0 associated with a general family of trace class operators $\{\rho_\hbar, \hbar \in (0,1)\}$ on the total Hilbert space $L^2(\mathbb{R}^3) \otimes \mathcal{F}$. I.e.,

$$\lim_{\hbar\to 0} \mathrm{Tr}(\rho_\hbar \mathcal{W}(\xi')) = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_0(x).$$

The existence and the uniqueness of the measure μ_0 associated with $\{\rho_\hbar, \hbar \in (0,1)\}$ are established in [2] but for the Pauli-Fierz model which is rather complicated than the Nelson model.

We can show that any Borel probability measure μ on X is a Wigner measure. We define the family of trace class operators by

$$\rho_\hbar = \int_X \mathcal{C}_\hbar(\xi) d\mu(\xi), \quad \hbar \in (0,1).$$

Proposition 4.2 [2, Lemma 4.3] *The Wigner measure of* $\{\rho_\hbar, \hbar \in (0,1)\}$ *is* μ.

Proof: It is straightforward to see that

$$\mathrm{Tr}[\rho_\hbar \mathcal{W}(\xi')] = \int_X \mathrm{Tr}(\mathcal{C}_\hbar(\xi)\mathcal{W}(\xi')) d\mu(\xi) \to \int_X e^{2\pi i \,\mathrm{Re}(\xi,\xi')_X} d\mu(\xi).$$

Then the proposition follows. ■

4.3 Time evolution of Wigner measures and flows

The time evolution of the Wigner measure is given by

$$\lim_{\hbar\to 0} \mathrm{Tr}(\rho_\hbar(t) \mathcal{W}(\xi')) = \int_X e^{2\pi i \,\mathrm{Re}(x,\xi')_X} d\mu_t(x),$$

where

$$\rho_\hbar(t) = e^{-i\frac{t}{\hbar}H_\hbar} \rho_\hbar e^{i\frac{t}{\hbar}H_\hbar}.$$

Here we give an example. Fix $\xi=(z,u)=(q,p,u)\in X$. Let

$$\mathcal{C}_\hbar(\xi)(t)=e^{-i\frac{t}{\hbar}H_\hbar}\mathcal{C}_\hbar(\xi)e^{i\frac{t}{\hbar}H_\hbar}.$$

We set

$$T_\xi=T\left(\frac{z}{\hbar}\right)\otimes W\left(\frac{\sqrt{2}u}{\sqrt{\hbar}}\right),$$
$$T_{\xi_t}=T\left(\frac{z_t}{\hbar}\right)\otimes W\left(\frac{\sqrt{2}u_t}{\sqrt{\hbar}}\right).$$

Here $\xi_t=(z_t,u_t)=(q_t,p_t,u_t)\in X$ is the solution to (N) with the initial condition $\xi_0=\xi=(z,u)=(q,p,u)\in X$. Since $\mathcal{C}_\hbar(\xi)=|T_\xi\Phi_\hbar\rangle\langle T_\xi\Phi_\hbar|$, we have

$$\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi'))=(T_\xi\Phi_\hbar,e^{i\frac{t}{\hbar}H_\hbar}\mathcal{W}(\xi')e^{-i\frac{t}{\hbar}H_\hbar}T_\xi\Phi_\hbar), \tag{4.1}$$
$$\mathrm{Tr}(\mathcal{C}_\hbar(\xi_t)\mathcal{W}(\xi'))=(T_{\xi_t}\Phi_\hbar,\mathcal{W}(\xi')T_{\xi_t}\Phi_\hbar). \tag{4.2}$$

By Theorem 3.1, we can see that

$$e^{-i\frac{t}{\hbar}H_\hbar}e^{\frac{i}{\hbar}\int_0^t\mathcal{H}(q_s,p_s,u_s,\bar{u}_s)\mathrm{d}s}\sim T_{\xi_t}e^{-\frac{i}{\hbar}\int_0^t Q_{\hbar,s}\mathrm{d}s}T_\xi^* \tag{4.3}$$

in a semi-classical region. Let us define

$$\hat{H}_\hbar=H_\hbar-\mathcal{H}(q_s,p_s,u_s,\bar{u}_s),$$
$$Q_t=\frac{1}{2m}D_x^2+\frac{1}{2}x\cdot(\nabla^2V(q_t)+\nabla^2U(q_t,u_t))x+\phi\left(-ike^{-ikq_t}\frac{\hat{\varphi}}{\sqrt{\omega}}\right)x+H_{\mathrm{f}}.$$

Thus Q_t is quadratic and independent of $\hbar$. Note that

$$\xi_\hbar e^{-\frac{i}{\hbar}\int_0^t Q_{\hbar,s}\mathrm{d}s}\xi_\hbar^*=e^{-i\int_0^t Q_s\mathrm{d}s}$$

and in particular $e^{-i\int_0^t Q_s\mathrm{d}s}$ is independent of $\hbar$. By (4.3) we have a corollary.

Corollary 4.3 *It follows that*

$$e^{-i\int_0^t\frac{1}{\hbar}\hat{H}_\hbar\mathrm{d}s}\sim T_{\xi_t}\xi_\hbar^*e^{-i\int_0^t Q_s\mathrm{d}s}\xi_\hbar T_{\xi_0}^*,\quad \hbar\to 0. \tag{4.4}$$

Here $A\sim B$ *means that* $\lim_{\hbar\to 0}\|A\Phi-B\Phi\|=0$.

Hence

$$\begin{aligned}\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) &\sim (T_{\xi_t}\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{d}s}\xi_\hbar\Phi_\hbar, \mathcal{W}(\xi')T_{\xi_t}\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{d}s}\xi_\hbar\Phi_\hbar) \\ &= (\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{d}s}\Phi_1, T_{\xi_t}^*\mathcal{W}(\xi')T_{\xi_t}\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{d}s}\Phi_1) \\ &= (e^{-i\int_0^t Q_s \mathrm{d}s}\Phi_1, \xi_\hbar\mathcal{W}(\xi')\xi_\hbar^* e^{-i\int_0^t Q_s \mathrm{d}s}\Phi_1)e^{2\pi i\,\mathrm{Re}(\xi_t,\xi)}.\end{aligned}$$

Furthermore

$$\xi_\hbar\mathcal{W}(\xi')\xi_\hbar^* = e^{2\pi i\sqrt{\hbar}(q'x+p'D_x)} \otimes e^{-\sqrt{2}\pi i\sqrt{\hbar}\phi(u')} \to \mathbb{1}$$

as $\hbar \to 0$. Then

$$\lim_{\hbar\to 0}\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) = \|\Phi_1\|^2 e^{2\pi i\,\mathrm{Re}(\xi_t,\xi)} = e^{2\pi i\,\mathrm{Re}(\xi_t,\xi)}. \tag{4.5}$$

(4.5) has been rigorously proven and ultimately generalized in [2, Theorem 1.4].

A relationship between μ_0 and μ_t is given through solutions to (N). Let $\Phi_t : X \to X$ be such that $\xi_t = \Phi_t(\xi)$ is the solution to (N) with the initial condition $\xi_0 = \xi$.

Theorem 4.4 [2, Theorem 1.4] *It follows that* $\mu_t(\cdot) = \mu_0 \circ \Phi_t^{-1}(\cdot)$.

By this we can see that

$$\lim_{\hbar\to 0}\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) = \int_X e^{2\pi i\,\mathrm{Re}(x,\xi')_X} d\mu_\xi \circ \Phi_t^{-1}(x) \tag{4.6}$$

and hence

$$\int_X e^{2\pi i\,\mathrm{Re}(x,\xi')_X} d\mu_\xi \circ \Phi_t^{-1}(x) = e^{2\pi i\,\mathrm{Re}(\xi_t,\xi')_X}.$$

Then (4.5) follows. As a corollary we can see that

$$\lim_{\hbar\to 0}\mathrm{Tr}(\mathcal{C}_\hbar(\xi)(t)\mathcal{W}(\xi')) = \lim_{\hbar\to 0}\mathrm{Tr}(\mathcal{C}_\hbar(\xi_t)\mathcal{W}(\xi')).$$

Acknowledgements: FH thanks kind hospitality of Rennes I University in France in September of 2022, Aalborg University in Denmark in May of 2022 and Polytechnic University of Milano in April of 2022. This work has been partially done there. This work is also financially supported by JSPS KAKENHI 16H03942 and JSPS KAKENHI 20H01808.

References

[1] Z. Ammari, S. Breteaux, and F. Nier. Quantum mean-field asymptotics and multiscale analysis. *Tunisian J. Math.*, 1:221–272, 2019.

[2] Z. Ammari, M. Falconi, and F. Hiroshima. Towards a derivation of classical electrodynamics of charges and fields from qed. *arXiv:2202.05015*, 2022.

[3] Z. Ammari and F. Nier. Mean field limit for bosons and infinite dimensional phase-space analysis. *Ann. Henri Poincaré*, 9(8):1503–1574, 2008.

[4] Z. Ammari and M. Zerzeri. On the classical limit of self-interacting quantum field hamiltonians with cutoffs. *Hokkaido Math. J.*, 43:385–425, 2014.

[5] J. Ginibre and G. Velo. The classical field limit of scattering theory for nonrelativistic many-boson systems i. *Commun. Math. Phys.*, 66:37–76, 1979.

[6] K. Hepp. The classical limit for quantum mechanical correlation functions. *Commun. Math. Phys.*, 35:265–277, 1974.

[7] P.L. Lions and T. Paul. Sur les mesures de wigner. *Rev. Mat. Iberoamericana*, 9(3):553–618, 1993.

[8] E. Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Math. Phys.*, 5:1990–1997, 1964.

[9] E. Nelson. Schrödinger particles interacting with a quantized scalar field. In *Proc. of a conference on analysis in function space, W. T. Martin and I. Segal (eds.)*, page 87. MIT Press, 1964.

[10] W. Pauli and M. Fierz. Zur Theorie der Emission langwelliger Lichtquanten. *Nuovo Cimento*, 15:167–188, 1938.