The spectral theory of the Neumann–Poincaré operator on convex domains*

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Abstract

The Neumann–Poincaré operator (abbreviated by NP) is a boundary integral operator naturally arising when solving classical boundary value problems using layer potentials. If the boundary of the domain, on which the NP operator is defined, is $C^{1,\alpha}$ smooth, then the NP operator is compact. Thus, the Fredholm integral equation, which appears when solving Dirichlet or Neumann problems, can be solved using the Fredholm index theory. Regarding spectral properties of the NP operator, the NP spectrum depends heavily on geometry of the surface (or the curve) on which the operator is defined. Our main purpose is to introduce recent selected properties of the NP spectrum on convex domains. Then we discuss relationships among the NP spectrum and PDEs.

1 Introduction

Let us consider a bounded Lipschitz domain Ω in $\mathbb{R}^n$ $(n = 2, 3)$. The (electro-static) Neumann–Poincaré (abbreviated by NP) operator $\mathcal{K}_{\partial\Omega} : L^2(\partial\Omega) \ni \psi \to L^2(\partial\Omega) \ni \mathcal{K}_{\partial\Omega}[\psi]$ is defined by

$$\mathcal{K}_{\partial\Omega}[\psi](x) \equiv \int_{\partial\Omega} \psi(y) \frac{\partial}{\partial n_y} E(x, y)\, dS_y,$$

where

$$E(x, y) = \begin{cases} \frac{1}{2\pi} \log \frac{1}{|x-y|}, & \text{if } n = 2, \\ \frac{-1}{4\pi} \frac{1}{|x-y|}, & \text{if } n = 3, \end{cases}$$

ds_y is the line or surface element and $\frac{\partial}{\partial n_y}$ is the outer normal derivative on $\partial\Omega$. If the boundary of the domain $\partial\Omega$ is smooth, then it is known that $\mathcal{K}_{\partial\Omega}$ is a compact operator on $L^2(\partial\Omega)$ (even on $H^s(\partial\Omega)$) and consists of at most a countable number of eigenvalues, with 0 as the only possible limit point (See e.g. [M, MS] and references therein for the details.). It is also known that the eigenvalues of the NP operator (the so-called double layer potential integral operator) lie in the interval $(-1/2, 1/2]$ and that the eigenvalue $1/2$ corresponds to constant eigenfunctions. Here it is worth mentioning that if the boundary has corners (i.e. Lipschitz) then absolute continuous spectrum appears on a suitable Hilbert space $\mathcal{H}$ (e.g. [HKL, PP]). Especially for the case of smooth boundaries, the spectrum of the NP oerator (NP spectrum) on $\mathcal{H}$ is the same as one on L^2. Thus we shall denote the spectrum of the NP operator as $\sigma(\mathcal{K}_{\partial\Omega})$ unless stated otherwise.

Our purpose here is to introduce some structural properties of the NP spectrum. Micellous properties of the NP spectrum are ongoing topics and they are significant not only in modern physics but also in pure mathematics. However, to avoid the condition of being tedious, we shall mention only a few selected properties and meanings instead of mentioning enomous results (See references and therein for the details.). To do so, let us recall Harmonic Bergman space $A^2(\Omega)$, namely, harmonic L^2 functions:

$$A^2(\Omega) \equiv \{f(x) \in L^2(\Omega) \mid \triangle f = 0 \text{ in } \Omega\ \}. \tag{1.1}$$

When we denote a single layer potential operator as

$$\mathcal{S}_{\partial\Omega}[\psi](x) \equiv \int_{\partial\Omega} \psi(y) E(x, y)\, dS_y, \tag{1.2}$$

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the operator $\mathcal{S}_{\partial\Omega} : H^{-1/2}(\partial\Omega) \to A^2(\Omega)$ is known to be bijective (See e.g. [AKM2].). Thus an arbitrary operator $T_{\partial\Omega}$ on $H^{-1/2}(\partial\Omega)$ is comprehended as the corresponding operator T_Ω on $A^2(\Omega)$:

$$T_\Omega \quad \text{on } A^2(\Omega) \cong \mathcal{S}_{\partial\Omega} \cdot T_{\partial\Omega} \cdot \mathcal{S}_{\partial\Omega}^{-1} \quad \text{on } H^{-1/2}(\partial\Omega). \tag{1.3}$$

This formulation is a toy model of holographic principles, that is, correspondence between boundary behaviors and interior behaviors. It is easily seen that the NP operator $\mathcal{K}_{\partial\Omega}$ is the typical one of boundary integral operators. Then the corresponding operator T_Ω is the so-called Ahlfors–Beuling operator in the case of two dimensions [Ahl, PP]. In fact, M. Perfekt and M. Putinar have shown from (1.3) that the NP operator of two-dimensional curvilinear polygon has an essential spectrum, which depends only on the angles of the corners [PP]. So we may consider the calculation of the NP spectrum as an example of getting familiar with the principle (1.3).

2 A resent result of the NP spectrum in two dimensions

Miscellous properties of the NP spectrum have been shown even in two dimensions (See e.g. [AKM2, MS] and references therein.). Here we shall introduce one of the interesting properties for thin domains: As in section 1, it is proved lately in [PP] that if a two-dimensional domain Ω has corners on its boundary, then $\mathcal{K}_{\partial\Omega}$ has essential spectrum which is a connected interval symmetric with respect to 0, and the end points of the interval are completely determined by the smallest angle of the corners. In particular, if Ω is a rectangle, then the essential spectrum is known to be the interval $[-1/4, 1/4]$. It is also known that $\sigma(\mathcal{K}_{\partial\Omega}) \setminus \{1/2\}$ is a closed subset of $(-1/2, 1/2)$. In recent work [HKL], a classification method to distinguish eigenvalues from essential spectrum has been proposed and implemented numerically to investigate existence of eigenvalues on various domains with corners. The numerical experiments reveal that on rectangles more and more eigenvalues of the NP operator appear outside the interval $[-1/4, 1/4]$ of the essential spectrum as the aspect ratio of the rectangle gets larger. It is also proved that if the aspect ratio is large enough, there is at least one eigenvalue outside $[-1/4, 1/4]$. In [AKM1] we improve this result drastically and prove that the spectra actually fill up the whole interval $(-1/2, 1/2)$ in some sense as the aspect ratio gets larger.

To be more precise, the two-dimensional domains to be considered here are not just rectangles. The long sides are lines, but the short sides do not have to be lines, they can be curves. Since the NP operator is dilation invariant, we define planar thin domains as follows: for $R \geq 1$, let Ω_R be a rectangle-shaped domain whose boundary consists of three parts, say

$$\partial\Omega_R = \Gamma_R^+ \cup \Gamma_R^- \cup \Gamma_R^s, \tag{2.1}$$

where the top and bottom are

$$\Gamma_R^+ = [-R, R] \times \{1\}, \quad \Gamma_R^- = [-R, R] \times \{-1\}, \tag{2.2}$$

and the side Γ_R^s consists of the left and right sides, namely, $\Gamma_R^s = \Gamma_R^l \cup \Gamma_R^r$, where Γ_R^l and Γ_R^r are curves connecting points $(\mp R, 1)$ and $(\mp R, -1)$, respectively. We assume that Γ_R^l and Γ_R^r are of any but fixed shape independent of R. In other words, Γ_R^l and Γ_R^r are of the form $\Gamma_R^l = \Gamma^l - (R, 0)$ and $\Gamma_R^r = \Gamma^r + (R, 0)$, where Γ^l and Γ^r are curves connecting points $(0, 1)$ and $(0, -1)$. If both Γ^l and Γ^r are line segments, Ω_R is a rectangle. The boundary $\partial\Omega_R$ is assumed to be Lipschitz continuous. We say that the domain Ω_R is of the aspect ratio R even if it is not necessarily a rectangle. It is worthwhile to emphasize that $\partial\Omega_R$ is allowed to be smooth in which case the associated NP operator is compact and has eigenvalues accumulating to 0.

The following theorem is one of our results in two dimensions:

Theorem 1 ([AKM1]). *If $\{R_j\}$ be an increasing sequence such that $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\bigcup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})} = [-1/2, 1/2]. \tag{2.3}$$

3 Recent results of the NP spectrum in three dimensional convex domains

Three-dimensional bounded domains exhibit the NP spectral structure different from two-dimensional ones. In two dimensions, the NP spectrum (spectrum of the NP operator) always appears in pair $\pm\lambda$ except 1/2. In fact, we know that the NP eigenvalues on a sphere are $1/(4k+2)$ for $k = 0, 1, 2 \ldots$ [Poi], and they are all positive even on prolate spheroids [AA]. Thus, the property (2.3) can not hold for prolate spheroids. It is worth mentioning that, as far as we are aware of, prolate spheroids are the only domains without negative NP eigenvalues. It is an intriguing question to find geometric conditions which allow only positive NP eigenvalues. It is proved in [MR] that the NP operator on the boundary of strictly convex domains in three dimensions can have at most finitely many negative eigenvalues. If the boundary of the domain has a concave part, then there are (infinitely) many negative eigenvalues (see [AJKKM, JK, MR]).

Let us begin with the prolate spheroids. Let Π_R be a prolate spheroid, namely, for $R \geq 1$,

$$\Pi_R := \left\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + \frac{x_3^2}{R^2} < 1\right\}. \tag{3.1}$$

Then we obtain the following theorem for prolate spheroids.

Theorem 2 ([AKLM])**.** *Let Π_R be the prolate spheroid defined by* (3.1)*. If R_j is a sequence of numbers such that $R_j \geq 1$ for all j and $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\bigcup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial \Pi_{R_j}})} = [0, 1/2]. \tag{3.2}$$

Theorem 2 shows that totality of eigenvalues of $\mathcal{K}_{\partial \Pi_{R_j}}$ is dense in $[0, 1/2]$ regardless of choice of the sequence R_j. A natural question arises: whether the property (3.2) is generic for thin, long domains, e.g., long cylinders. (If we dilate Π_R by R^{-1}, Π_R becomes thin. That is why we call them 'thin' domains. The spectrum of the NP operator is invariant under dilation.)

There are significant work on the NP spectrum on ellipsoids [Ahn2, AA, ADR, Ma, Ritt]. However, it is unlikely that Theorem 2 (and Theorem 4 below) are derived from those results. However, we are able to prove the following theorem based on those results, which is in good comparison with Theorem 2: The following theorem shows that the totality (in continuum) of the NP eigenvalues on prolate spheroids covers the interval $(0, 1/2]$ while Theorem 2 shows that the NP eigenvalues on a sequence of prolate spheroids, which is countable, are dense in $[0, 1/2]$ regardless of the choice of the sequence.

Theorem 3 ([AKLM])**.** *Let Π_R be the prolate spheroid defined by* (3.1)*. It holds that for any $R_0 \geq 1$,*

$$\bigcup_{R \geq R_0} \sigma(\mathcal{K}_{\partial \Pi_R}) = (0, 1/2]. \tag{3.3}$$

The property for oblate spheroids seems a generic property of thin, flat domains as in Theorem 1. To demonstrate it, we consider typical thin, flat domains other than oblate ellipsoids. To define such a domain, let U be a bounded planar domain with the Lipschitz continuous boundary ∂U. Let Φ be the domain in $\mathbb{R}^3$ whose boundary consists of three pieces, namely,

$$\partial \Phi = \Sigma^+ \cup \Sigma^- \cup \Sigma^s \tag{3.4}$$

where the top and bottom are given by $\Sigma^{\pm} = U \times \{\pm 1\}$ and Σ^s is a surface connecting $\partial U \times \{+1\}$ and $\partial U \times \{-1\}$. We assume that $\partial \Phi$ is Lipschitz continuous. For $R > 0$ let

$$\Phi_R := \{(Rx_1, Rx_2, x_3) : (x_1, x_2, x_3) \in \Phi\}. \tag{3.5}$$

Then we obtain the following theorem.

Theorem 4 ([AKLM])**.** *Let Φ_R be the domain defined by* (3.5)*. If R_j is a sequence such that $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\bigcup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial \Phi_{R_j}})} = [-1/2, 1/2]. \tag{3.6}$$

Here we can also give a property as a consequence of Theorem 3 and Theorem 4:

Corollary 5. Let $-1/2 < c < 0$. There exists a smooth convex domain $\Omega \subset \mathbb{R}^3$ such that the minimum NP eigenvalue satisfies

$$\min \sigma(\mathcal{K}_{\partial \Omega}) = c. \tag{3.7}$$

In fact, the smooth perturbation of domains yields the continuity of minimum eigenvalue [AKMU]. It then follows that the oblate-like perturbation of a sphere gives the satisfactory minimum eigenvalue. It is worth mentioning that to construct an explicit domain is a formiddable task.

Thus even in convex regions we can find negative NP spectrum (See also [AJKKMR].). However, we couldn't find the domain except for prolate spheroids, in which there are no negative eigenvalues.

4 Unsolved problems and conjectures as future prospects

In section 1 we introduce a toy model of holographic principles, that is, the correspondency between boundary behaviors and interior behaviors. The Bergman space $A^2(\Omega)$ consists of harmonic functions and the correspondency (1.3) can be denote as the product of single layer potential operators. Here we employed the Bergman space as harmonic functions. When we can define single layer potential operators via fundamental solutions of linear PDEs, one can expect the analogy of the Bergman space as the solutions of PDEs. In fact, the so-called Dirichlet–Neumann map can be denoted by NP operators and single layer potential operators. So one can expect a rigorous theory of holographic principles other than 2-dimensional NP operators (See e.g. [LBM] for the idea of correspondencies).

Many applications of NP operators also can be found in mathematical physics. As a significant applicantion, we disproved the so-called cloakings by anomalous resonance (abbreviated by CALR), which is one of electromagnetic effects, if the boundary $\partial\Omega$ is convex smooth. We emphasize that such applications deeply depend the NP spectrum. In other words, if CALR happens in three dimensions then the NP operator has infinitely many negative eigenvalues (See e.g. [AKMN] and references therein.).

They are ongoing subjects and so we end this article by proposing problems and conjectures:

Problem 6. Can one find CALR for concave regions such as a torus ?

We don't have such an example of concave regions at present. The infinitely many negative NP eigenvalues are essential. At this point, it is emphasized that we can't find satisfactory conditions for the nonexistence of negative eigenvalues (See [AJKKMR].):

Problem 7. Can one find the rigion except prolate spheloids in which the corresponding NP operator has positive eigenvalues only ?

Corollary 5 similarly allows us to ask problem 8:

Problem 8. Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of regions in $\mathbb{R}^3$. For $-1/2 < c < 0$, can one construct a sequence such that

$$\overline{\bigcup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_j})} = [c, 1/2]? \tag{4.1}$$

The maximal eigenvalue other than 1/2 corresponds the so-called Fredholm eigenvalue (See e.g. [Ahl] for two dimensions and [MS].). Details of such eigenvalue are significant in PDEs. However they are still unknown especially in dimension 3:

Conjecture 9 ([MS]). Let $\Omega \subset \mathbb{R}^3$ and $\overline{\lambda} \equiv \max \sigma_p(\mathcal{K}_{\partial\Omega})\backslash\{1/2\}$. Can one prove

$$\inf_{\partial\Omega} \overline{\lambda} = \frac{1}{6},$$

where the infimum is taken over all C^∞ simply connected closed surfaces? Is the infimum achieved if and only if $\partial\Omega = S^2$?

The present author confirmed the validity of conjecture 9. When C^∞ closed surfaces are replaced by ellipsoids, this problem is proved in [MS]. Related results can be also found in [AKMU].

Conjecture 10 ([M, MR]). Let Ω_1 and Ω_2 be three dimensional bounded regions. If each NP spectrum coincides, namely, $\sigma_p(\mathcal{K}_{\partial\Omega_1}) = \sigma_p(\mathcal{K}_{\partial\Omega_1})$ (i.e. isospectral), then are Ω_1 and Ω_2 similar figures?

Conjecture 10 is proven only for the case of $\partial\Omega = S^2$ or $\partial\Omega = T^2$ (Clifford Torus). We suspect that conjecture 10 holds true for all of Willmore surfaces.

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