A Correspondence between Finite Multiple Zeta Values and Symmetric Multiple Zeta Values

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1 Introduction

Recently there has been a growing interest in finite multiple zeta values (FMZVs) and symmetric multiple zeta values (SMZVs). Kaneko and Zagier [7] conjectured that there is a certain correspondence between them and as supporting evidence many relations of the same form were found such as sum formulas, Ohno-type relations and so on (see, for example, [4, 12, 15, 16]). One of the main interests may be the reason of this correspondence. However we do not have a complete answer to this question yet. There is a trial to explain from the viewpoint of q-zeta functions: they recover FMZVs and SMZVs when q goes to 1 in an algebraic sense and an analytic sense respectively [1]. The purpose of this article is to try to give another reason of this correspondence from the viewpoint of the theory of polytopes. First, we define FMZVs associated with polytopes and give their values in terms of a generalization of Ehrhart polynomials and Bernoulli polynomials. Secondly, we construct unified multiple zeta functions (UMZFs) from the integral representation of FMZVs by changing the start point of integral and observe that special values of UMZFs recover FMZVs, and that UMZFs can be regarded as an interpolation of SMZVs. Furthermore we see that classical Ehrhart–Macdonald reciprocity means the relation between multiple zeta values (MZVs) and multiple zeta-star values (MZSVs).

Let $\mathcal{A} = \left(\prod_p \mathbb{Z}/p\mathbb{Z}\right) \Big/ \left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right)$, where p runs over all primes and $\mathcal{Z}_\mathcal{S} = \mathcal{Z}/(\zeta(2)\mathcal{Z})$, where $\mathcal{Z} = \sum_{\mathbf{k}} \mathbb{Q}\zeta(\mathbf{k})$ be the $\mathbb{Q}$-span of MZVs $\zeta(\mathbf{k})$. Then it is known that $\mathcal{A}$ and $\mathcal{Z}_\mathcal{S}$ are $\mathbb{Q}$-algebras. For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}, k_r \geq 2$, FMZVs and SMZVs are defined respectively by

$$\zeta_\mathcal{A}(k_1, \dots, k_r) = \big(\zeta_p(k_1, \dots, k_r) \bmod p\big)_p \in \mathcal{A}, \tag{1.1}$$

$$\zeta_\mathcal{S}(k_1, \dots, k_r) = \zeta^*_\mathcal{S}(k_1, \dots, k_r) \bmod (\zeta(2)) \in \mathcal{Z}_\mathcal{S}, \tag{1.2}$$

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where

$$\zeta_N(k_1,\dots,k_r)=\sum_{0<n_1<\cdots<n_r<N}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}\in\mathbb{Q}, \tag{1.3}$$

$$\zeta^*_{\mathcal{S}}(k_1,\dots,k_r)=\sum_{i=0}^{r}(-1)^{k_{i+1}+\cdots+k_r}\zeta^*(k_1,\dots,k_i)\zeta^*(k_r,\dots,k_{i+1})\in\mathbb{R}, \tag{1.4}$$

and $\zeta^*(k_1,\dots,k_r)$ denotes the harmonic regularized value.

To see what is generalized in this article, we review the previous work [11]. It is well known that the ordinary multiple zeta function defined by

$$\zeta(s_1,\dots,s_r)=\sum_{0<n_1<\cdots<n_r}\frac{1}{n_1^{s_1}\cdots n_r^{s_r}} \tag{1.5}$$

is analytically continued to a meromorphic function on $\mathbb{C}^r$ with singularities on infinitely many hyperplanes when $r\geq 2$. However the natural interpolation of $\zeta^*_{\mathcal{S}}(k_1,\dots,k_r)$ defined by

$$\zeta_{\mathcal{U}}(s_1,\dots,s_r)=\sum_{i=0}^{r}(-1)^{s_{i+1}+\cdots+s_r}\zeta(s_1,\dots,s_i)\zeta(s_r,\dots,s_{i+1}) \tag{1.6}$$

with $(-1)^s=e^{\pi is}$ has the following remarkable properties.

Theorem 1 ([11]). *$\zeta_{\mathcal{U}}(s_1,\dots,s_r)$ is entire. For $k_1,\dots,k_r\in\mathbb{Z}_{\geq 1}$,*

$$\begin{aligned}
&\zeta_{\mathcal{U}}(k_1,\dots,k_r)=\zeta^*_{\mathcal{S}}(k_1,\dots,k_r) && (k_1,\dots,k_r\geq 2),\\
&\zeta_{\mathcal{U}}(k_1,\dots,k_r)\equiv\zeta^*_{\mathcal{S}}(k_1,\dots,k_r)\pmod{\pi i\mathcal{Z}[\pi i]} && (\exists k_i=1).
\end{aligned}$$

From the above theorem and the fact that $\zeta_{\mathcal{U}}(k_1,\dots,k_r)\in\mathcal{Z}[\pi i]$ for $(k_1,\dots,k_r)\in\mathbb{Z}^r$ shown in [14], it is natural to define SMZVs on whole integers by the special values of $\zeta_{\mathcal{U}}(s_1,\dots,s_r)$, that is,

$$\zeta_{\mathcal{S}}(k_1,\dots,k_r):=\zeta_{\mathcal{U}}(k_1,\dots,k_r)\mod \pi i\mathcal{Z}[\pi i] \tag{1.7}$$

for $(k_1,\dots,k_r)\in\mathbb{Z}^r$.

Since FMZVs are naturally defined on nonpositive integers, we are led to compare them.

Theorem 2 ([11]). *For $k_1,\dots,k_r\in\mathbb{Z}_{\geq 0}$,*

$$\zeta_{\mathcal{U}}(-k_1,\dots,-k_r)=\begin{cases}0 & ((k_1,\dots,k_r)\neq(0,\dots,0)),\\ (-1)^r & ((k_1,\dots,k_r)=(0,\dots,0)).\end{cases} \tag{1.8}$$

$$\zeta_{\mathcal{A}}(-k_1,\dots,-k_r)=\begin{cases}(0)_p & ((k_1,\dots,k_r)\neq(0,\dots,0)),\\ ((-1)^r)_p & ((k_1,\dots,k_r)=(0,\dots,0)).\end{cases} \tag{1.9}$$

Furthermore their values on other integers were also shown to coincide in the double zeta cases.

Theorem 3 ([11])**.** *For $k_1 \in \mathbb{Z}_{\geq 1}$ and $k_2 \in \mathbb{Z}_{\leq 0}$,*

$$\zeta_{\mathcal{S}}(k_1,k_2) = \begin{cases} 0 & (k_1 > -k_2+1), \\ (-1)^{1-k_1-k_2}\binom{1-k_2}{1-k_1-k_2}\dfrac{B_{1-k_1-k_2}}{1-k_2} \bmod \pi i \mathcal{Z}[\pi i] & (k_1 \leq -k_2+1). \end{cases} \tag{1.10}$$

$$\zeta_{\mathcal{A}}(k_1,k_2) = \begin{cases} (0)_p & (k_1 > -k_2+1), \\ \left((-1)^{1-k_1-k_2}\binom{1-k_2}{1-k_1-k_2}\dfrac{B_{1-k_1-k_2}}{1-k_2} \bmod p\right)_p & (k_1 \leq -k_2+1). \end{cases} \tag{1.11}$$

Finally it was shown that the behavior on the whole region is reduced to that on the positive region.

Theorem 4 ([14])**.** *Under the Kaneko–Zagier conjecture, one to one correspondence between $\zeta_{\mathcal{S}}(\mathbf{k})$ and $\zeta_{\mathcal{A}}(\mathbf{k})$ holds for all indices $\mathbf{k} \in \mathbb{Z}^r$.*

To realize FMZVs and SMZVs in a function, we introduced UMZF of them, which is a further generalization of $\zeta_{\mathcal{U}}(s_1,\ldots,s_r)$. For $s_1,\ldots,s_r,t,t_1,t_2 \in \mathbb{C}$ with sufficiently large $\Re s_1,\ldots,\Re s_r$ and $\Re t, \Re t_1, \Re t_2 < 1$, we define

$$\begin{aligned} \zeta(s_1,\ldots,s_r;t) &= \sum_{0<n_1<\cdots<n_r} \frac{1}{(n_1-t)^{s_1}\cdots(n_r-t)^{s_r}}, \\ \zeta_{\widehat{\mathcal{U}}}(s_1,\ldots,s_r;t_1,t_2) &= \sum_{i=0}^{r}(-1)^{s_{i+1}+\cdots+s_r}\zeta(s_1,\ldots,s_i;t_1)\zeta(s_r,\ldots,s_{i+1};t_2). \end{aligned} \tag{1.12}$$

Theorem 5 ([11])**.** *$\zeta_{\widehat{\mathcal{U}}}(s_1,\ldots,s_r;t_1,t_2)$ is holomorphic for $s_1,\ldots,s_r \in \mathbb{C}$ and $t_1,t_2 \in \mathbb{C}\setminus\mathbb{Z}_{\geq 1}$.*

We describe $\zeta_{\mathcal{A}}(k_1,\ldots,k_r)$ and $\zeta_{\mathcal{S}}(k_1,\ldots,k_r)$ in terms of $\zeta_{\widehat{\mathcal{U}}}(s_1,\ldots,s_r;t_1,t_2)$. Here we set $t_1=0$ and explain the roles of the variables t_1,t_2 in the last section. For $t \in \mathbb{C}\setminus\mathbb{Z}_{\geq 1}$, we abbreviate

$$\begin{aligned} \zeta_{\widehat{\mathcal{U}}}(s_1,\ldots,s_r;t) &:= \zeta_{\widehat{\mathcal{U}}}(s_1,\ldots,s_r;0,t) \\ &= \sum_{i=0}^{r}(-1)^{s_{i+1}+\cdots+s_r}\zeta(s_1,\ldots,s_i)\zeta(s_r,\ldots,s_{i+1};t). \end{aligned} \tag{1.13}$$

Theorem 6 ([11])**.** *Let $\Re s_j < 0$ or $s_j = 0$ for $j=1,\ldots,r$. For $N \in \mathbb{Z}_{\geq 1}$,*

$$\zeta_{\widehat{\mathcal{U}}}(s_1,\ldots,s_r;N) := \lim_{\epsilon\to 0}\zeta_{\widehat{\mathcal{U}}}(s_1,\ldots,s_r;N-\epsilon) \text{ converges and} \tag{1.14}$$

$$\zeta_{\widehat{\mathcal{U}}}(k_1,\ldots,k_r;N) = \begin{cases} \zeta_N(k_1,\ldots,k_r) & (N \in \mathbb{Z}_{\geq 1},\ k_1,\ldots,k_r \in \mathbb{Z}_{\leq 0}), \\ \zeta_{\mathcal{U}}(k_1,\ldots,k_r) & (N = 0,\ k_1,\ldots,k_r \in \mathbb{Z}). \end{cases} \tag{1.15}$$

Theorem 7 ([11]). *For $k_1, \dots, k_r \in \mathbb{Z}$,*

$$\zeta_{\mathcal{A}}(k_1, \dots, k_r) = \left(\zeta_{\widehat{\mathcal{U}}}(-k_1^{(p)}, \dots, -k_r^{(p)}; p) \bmod p\right)_p, \qquad k^{(p)} = \begin{cases} p-1-k & (k>0), \\ -k & (k \le 0). \end{cases}$$
$$\zeta_{\mathcal{S}}(k_1, \dots, k_r) = \zeta_{\widehat{\mathcal{U}}}(k_1, \dots, k_r; 0) \bmod \pi i \mathcal{Z}[\pi i]. \tag{1.16}$$

Thus we unified all multiple zeta values $\zeta_\bullet$ ($\bullet \in \{\emptyset, \mathcal{A}, \mathcal{S}\}$) by $\zeta_{\widehat{\mathcal{U}}}$. As noted below Theorem 1, FMZVs are defined naturally on the whole integers while the existence of the corresponding SMZVs are unclear. A hint for the construction of SMZVs or UMZFs may be the following expression derived via the integral expression

$$\zeta_N(s_1, s_2) = \zeta(s_1, s_2) - \zeta(s_1)\zeta^\star(s_2; N) + \zeta^\star(s_2, s_1; N), \tag{1.17}$$

where $\zeta^\star(s_1, \dots, s_n; N)$ is the Hurwitz zeta-star function defined by

$$\zeta^\star(s_1, \dots, s_r; a) = \sum_{0 \le n_1 \le \cdots \le n_r} \frac{1}{(a+n_1)^{s_1} \cdots (a+n_r)^{s_r}}. \tag{1.18}$$

The form (1.17) strongly suggests that of the corresponding SMZVs. In the next sections, we will indeed construct SMZVs and UMZFs via the integral expression of FMZVs associated with polytopes.

Remark. The cases $\bullet \in \{\widehat{\mathcal{A}}, \widehat{\mathcal{S}}\}$ can also be described by $\zeta_{\widehat{\mathcal{U}}}$ [11].

2 Polytopes

Let $P \subset \mathbb{R}^r$ be a convex polytope, which is defined as the bounded intersection of finite half-spaces. P is called a lattice polytope if all vertices are on $\mathbb{Z}^r$. P is called a simple polytope if each vertex is the intersection of unique r hyperplanes.

Let $\{F_i\}$ be the set of all facets of P, that is, $(r-1)$ dimensional faces, and u_i be the inward-pointing primitive integral normal vector of the facet F_i. For a vertex v of P, let $G_v = \mathbb{Z}^r \big/ \bigoplus_{F_i \ni v} \mathbb{Z} u_i$ be a finite abelian group. Then a simple lattice polytope P is a Delzant polytope if $G_v = \{0\}$ for every vertex v of P.

Fix a simple lattice polytope P defined as the intersection of d half-spaces:

$$P = \{x \in V \mid (u_i, x) + \lambda_i \ge 0, 1 \le i \le d\}, \quad (\lambda = (\lambda_1, \dots, \lambda_d)). \tag{2.1}$$

For each vertex v, there are unique r facets and let $\mathcal{U}_v := \{u_i \mid F_i \ni v\}$ be the set of all normal vectors of the facet F_i containing v, $\mathcal{E}_v := \{e_v^{u_i} \mid F_i \ni v\}$, the set of their dual basis, that is, $(u_i, e_v^{u_j}) = \delta_{ij}$.

We define associated polytope $P(t)$ with P as

$$P(t) = \{x \in V \mid (u_i, x) + t_i \ge 0, 1 \le i \le d\}, \quad (t = (t_1, \dots, t_d)). \tag{2.2}$$

Then $P(\lambda) = P$ and if t is in a neighborhood of λ, $P(t)$ is indeed a polytope. Note that each vertex $v(t)$ of $P(t)$ is naturally analytically continued in t because it is determined by several linear equations of the form

$$(u_i, x) + t_i = 0. \tag{2.3}$$

We note that the faces $v(t), F_i(t)$ are dependent on t while $G_{v(t)}, \mathcal{U}_{v(t)}, \mathcal{E}_{v(t)}$, etc. are independent of t and coincide with $G_v, \mathcal{U}_v, \mathcal{E}_v$, etc. of the original polytope P.

3 Generalized Faulhaber's formula

Fix a simple lattice polytope P. For $\mathbf{z} = (z_1, \ldots, z_q) \in \mathbb{C}^q$, $\mathbf{k} = (k_1, \ldots, k_q) \in \mathbb{Z}_{\geq 0}^q$, $\mathbf{a} = (a_1, \ldots, a_q) \in \mathbb{C}^q$, $B = (b_{jk})_{1 \leq k \leq r}^{1 \leq j \leq q} \in \mathbb{C}^{q \times r}$, $t \in \mathbb{C}^d$ and $R = P$ or P^0 (interior of P), we define

$$S(\mathbf{k}; R; \mathbf{a}, B) = \sum_{(n_1, \ldots, n_r) \in R} \prod_{j=1}^{q} (a_j + b_{j1} n_1 + \cdots + b_{jr} n_r)^{k_j} \tag{3.1}$$

and

$$F(\mathbf{z}; t; P; \mathbf{a}, B) = \sum_{v:\text{vertex}} \frac{e^{-\mathbf{z}(Bv(t)+\mathbf{a})}}{|G_v|} \sum_{g \in G_v} \frac{1}{\prod_{e \in \mathcal{E}_v} (1 - e^{2\pi i (g,e)} e^{-\mathbf{z}Be})}, \tag{3.2}$$

$$F(\mathbf{z}; t; P^0; \mathbf{a}, B) = (-1)^r \sum_{v:\text{vertex}} \frac{e^{-\mathbf{z}(Bv(t)+\mathbf{a})}}{|G_v|} \sum_{g \in G_v} \frac{1}{\prod_{e \in \mathcal{E}_v} (1 - e^{2\pi i (g,e)} e^{\mathbf{z}Be})}. \tag{3.3}$$

Theorem and Definition 8.

$$F(\mathbf{z}; t; R, \mathbf{a}, B) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_q=0}^{\infty} Q(k_1, \ldots, k_q; t; R; \mathbf{a}, B) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_q^{k_q}}{k_q!}. \tag{3.4}$$

Moreover $Q(k_1, \ldots, k_q; t; R; \mathbf{a}, B)$ is a polynomial of degree at most $k_1 + \cdots + k_q + r$ in t and all coefficients are rational if $\mathbf{a}$ and B are rational.

Theorem 9 (Generalized Faulhaber's formula)**.** *If $P(t)$ is a lattice polytope,*

$$S(\mathbf{k}; R(t); \mathbf{a}, B) = \Big(\prod_{j=1}^{q} (-1)^{k_j} \Big) Q(k_1, \ldots, k_q; t; R; \mathbf{a}, B) \in \mathbb{C}[t]. \tag{3.5}$$

Remark. Although we can treat the factor of the form $e^{2\pi i (\mu_1 n_1 + \cdots + \mu_r n_r)}$ in the definition of $S(\mathbf{k}; R; \mathbf{a}, B)$, we omit it here for simplicity.

We explain that the polynomials Q are a generalization of Bernoulli polynomials and Ehrhart polynomials. To this end, for $t \in \mathbb{Z}_{\geq 1}$, let $L(R, t) = |tR \cap \mathbb{Z}^r|$, that is, the number of integer points contained in R. Then it is known that $L(R,t)$ is a polynomial in t and is called the Ehrhart polynomial. Furthermore $L(P^0,t)$ and $L(P,t)$ have the following simple relation, which is called the Ehrhart–Macdonald reciprocity.

Theorem 10 (Ehrhart–Macdonald reciprocity)**.**

$$L(P^0,t)=(-1)^r L(P,-t). \tag{3.6}$$

The following example illustrates the meaning of the Ehrhart polynomials and the Ehrhart–Macdonald reciprocity.

Example 1. $P=[0,1]^2$. Then $L(P^0,t)=(t-1)^2$, $L(P,t)=(t+1)^2$.

Thus the Ehrhart polynomials correspond to the special cases of the Faulhaber formulas for $L(R,t)=S(\mathbf{0};tR;*)$, where either $\mathbf{a}$ or B does not affect this polynomial because $\mathbf{k}=\mathbf{0}$.

The Ehrhart–Macdonald reciprocity is generalized as follows in terms of generating function.

Theorem 11 (Ehrhart–Macdonald reciprocity for generating function)**.**

$$F(\mathbf{z};t;P^0;\mathbf{a},B)=(-1)^r F(-\mathbf{z};-t;P;-\mathbf{a},B). \tag{3.7}$$

Remark. Certain special cases of the above polynomials Q were introduced and studied in a different approach in [3].

3.1 Example

We give an example of double FMZVs and observe that the Ehrhart–Macdonald reciprocity describes the relation between FMZVs and FMZSVs.

In this case, we have $\mathbf{a}=(0,0)$, $B=I_2$, $P(t)=\{(x,y)\mid 0\le x\le y\le t\}$. If $t\in\mathbb{Z}_{\ge 1}$, then

$$\begin{aligned}\zeta_t(-k_1,-k_2)&=\sum_{0<n_1<n_2<t} {n_1}^{k_1}{n_2}^{k_2}=\sum_{(n_1,n_2)\in P^0(t)} {n_1}^{k_1}{n_2}^{k_2}\\&=(-1)^{k_1+k_2}Q(k_1,k_2;t).\end{aligned} \tag{3.8}$$

We see that P is a Delzant simplex, that is, $G_v=\{0\}$. Thus the generating function is simplified and given by

$$\begin{aligned}&F(\mathbf{z};t;P^0;\mathbf{a},B)=(-1)^r\sum_{v:\text{vertex}}\frac{e^{(-z,v(t))}}{\prod_{e\in\mathcal{E}_v}(1-e^{(z,e)})}\\&=\frac{e^{-tz_1-tz_2}}{(1-e^{-z_1})(1-e^{-z_1-z_2})}+\frac{e^{-tz_2}}{(1-e^{z_1})(1-e^{-z_2})}+\frac{1}{(1-e^{z_1+z_2})(1-e^{z_2})}.\end{aligned} \tag{3.9}$$

For FMZSVs, if $k_1, k_2 \geq 1$,

$$\begin{aligned} \zeta_t^\star(-k_1,-k_2) &= \sum_{0<n_1\leq n_2\leq t} {n_1}^{k_1}{n_2}^{k_2} = \sum_{0\leq n_1\leq n_2\leq t} {n_1}^{k_1}{n_2}^{k_2} \\ &= \sum_{(n_1,n_2)\in P(t)} {n_1}^{k_1}{n_2}^{k_2} \end{aligned} \tag{3.10}$$

is obtained by the Ehrhart–Macdonald reciprocity.

4 FMZ(S)Vs to UMZFs

In this section, we construct (3.3) directly in the case of double MZVs and explain how to create the corresponding UMZF. This method can be applied to the general cases with some knowledges of polytopes and yields (3.3).

By use of the integral expression

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-z}z^{s-1}dz, \tag{4.1}$$

we have

$$\begin{aligned} &\sum_{0<n_1<n_2<N} \frac{1}{{n_1}^{s_1}{n_2}^{s_2}} = \frac{1}{\prod_{j=1,2}\Gamma(s_j)}\int_0^\infty\int_0^\infty z_1^{s_1-1}z_2^{s_2-1}dz_1dz_2 \\ &= \frac{1}{\prod_{j=1,2}\Gamma(s_j)}\int_0^\infty\int_0^\infty z_1^{s_1-1}z_2^{s_2-1}dz_1dz_2 \\ &\times \Big(\frac{e^{-z_1-2z_2}}{(1-e^{-z_1-z_2})(1-e^{-z_2})} - \frac{e^{-z_1-Nz_2}}{(1-e^{-z_1})(1-e^{-z_2})} + \frac{e^{-Nz_1-Nz_2}}{(1-e^{-z_1})(1-e^{-z_1-z_2})}\Big), \end{aligned}$$

which gives the expression (1.17). If we replace the interval $(0,\infty)$ with the Hankel contour H and set $s_1, s_2 \in \mathbb{Z}_{\leq 0}$, we have

$$\begin{aligned} &\sum_{0<n_1<n_2<N} \frac{1}{{n_1}^{s_1}{n_2}^{s_2}} = \frac{1}{\prod_{j=1,2}\Gamma(s_j)(e^{2\pi i s_j}-1)}\int_{|z_1|=\epsilon}\int_{|z_2|=\epsilon} z_1^{s_1-1}z_2^{s_2-1}dz_1dz_2 \\ &\times \Big(\frac{e^{-Nz_1-Nz_2}}{(1-e^{-z_1})(1-e^{-z_1-z_2})} - \frac{e^{-z_1-Nz_2}}{(1-e^{-z_1})(1-e^{-z_2})} + \frac{e^{-z_1-2z_2}}{(1-e^{-z_1-z_2})(1-e^{-z_2})}\Big), \end{aligned} \tag{4.2}$$

and we see that this gives FMZVs. Thus the integrand of the above leads to the generating function of polynomials Q.

To obtain the corresponding UMZF, we have only to compare (1.17) and the double case of (1.13) and note that (4.2) does not depend on the end point or the start point of the Hankel contour as long as they coincide. Hence arbitrary choice seems possible, one of which leads us to the definition

$$\begin{aligned}
\zeta_{\widehat{\mathcal{U}}}(s_1,s_2;t) &= (-1)^{s_1+s_2}\zeta(s_2,s_1;t)+(-1)^{s_2}\zeta(s_1)\zeta(s_2;t)+\zeta(s_1,s_2)\\
&= \frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{-\infty}\int_0^{-\infty}\frac{e^{-tz_1-tz_2}}{(1-e^{-z_1})(1-e^{-z_1-z_2})}z_1^{s_1-1}z_2^{s_2-1}dz_1dz_2\\
&\quad -\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\infty}\int_0^{-\infty}\frac{e^{-z_1-tz_2}}{(1-e^{-z_1})(1-e^{-z_2})}z_1^{s_1-1}z_2^{s_2-1}dz_1dz_2\\
&\quad +\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^{\infty}\int_0^{\infty}\frac{e^{-z_1-2z_2}}{(1-e^{-z_1-z_2})(1-e^{-z_2})}z_1^{s_1-1}z_2^{s_2-1}dz_1dz_2.
\end{aligned} \tag{4.3}$$

It should be emphasized that this heuristic construction is justified only after the proof that this function recovers FMZVs.

In the following, we apply the above procedure to general cases. Assume $\Re b_{jk}\geq 0$, $\Re a_j>0$. For simplicity, we assume P is Delzant. Then for $\mathbf{k}\in\mathbb{Z}_{\geq 0}^q$,

$$\begin{aligned}
&S(-\mathbf{k};R;\mathbf{a},B)\\
&=\prod_{j=1}^{q}\frac{1}{\Gamma(s_j)(e^{2\pi i s_j}-1)}\int_{\substack{|z_j|=\epsilon\\ 1\leq j\leq r}}\sum_{v:\mathrm{vertex}}\frac{e^{-\mathbf{z}(Bv(t)+\mathbf{a})}}{\prod_{e\in\mathcal{E}_v}(1-e^{-\mathbf{z}Be})}\prod_{j=1}^{q}z^{s_j-1}dz_j\Big|_{\mathbf{s}=-\mathbf{k}}.
\end{aligned} \tag{4.4}$$

Fix a vertex v and we choose the direction of the end and start point of the integral for the vertex so that the integral converges, that is, the integrand decreases rapidly in the chosen direction. Let $c_1,\dots,c_q\in\{-1,1\}$ and put $C=\mathrm{diag}(c_1,\dots,c_q)$. Let $E=(e)_{e\in\mathcal{E}_v}$ where e is regarded as a column vector. Let

$$\zeta(\mathbf{s};\mathbf{a},B):=\sum_{(n_1,\dots,n_r)\in\mathbb{Z}_{\geq 0}^r}\prod_{j=1}^{q}(a_j+b_{j1}n_1+\cdots+b_{jr}n_r)^{-s_j}. \tag{4.5}$$

Theorem 12. *We have*

$$\begin{aligned}
&\frac{1}{\Gamma(s_1)\cdots\Gamma(s_q)}\int_0^{c_1\infty}\cdots\int_0^{c_q\infty}\frac{e^{-\mathbf{z}(Bv(t)+\mathbf{a})}}{\prod_{e\in\mathcal{E}_v}(1-e^{-\mathbf{z}Be})}z_1^{s_1-1}\cdots z_q^{s_q-1}dz_1\cdots dz_q\\
&\qquad=(-1)^{\sum_{e\in\mathcal{E}_v}d_e}c_1^{s_1}\cdots c_q^{s_q}\zeta(\mathbf{s};C(\mathbf{a}+B(v(t)-Ed)),CBED),
\end{aligned} \tag{4.6}$$

if there exists $c_1,\dots,c_q\in\{-1,1\}$ such that for all $e\in\mathcal{E}_v$,

$$\Re CBe\in((\mathbb{R}_{\geq 0})^q\cup(\mathbb{R}_{\leq 0})^q)\setminus\{0\}\quad\text{and}\quad \Re C(\mathbf{a}+B(v(t)-Ed))\in(\mathbb{R}_{>0})^q. \tag{4.7}$$

Here $D=\mathrm{diag}((-1)^{d_e})$, $d=(d_e)_{e\in\mathcal{E}_v}$ is a column vector with $d_e=0$ if $\Re CBe\in(\mathbb{R}_{\geq 0})^q\setminus\{0\}$ and $d_e=1$ otherwise.

Assume that there exists C_v introduced above for each vertex v. Then we define

$$\begin{aligned}&\zeta_{\widehat{\mathcal{U}}}(\mathbf{s};t;P,\mathbf{a},B)\\&=\sum_{v:\text{vertex}}(-1)^{\sum_{e\in\mathcal{E}_v}d_{v,e}}c_{v,1}^{s_1}\cdots c_{v,q}^{s_q}\zeta(\mathbf{s};C_v(\mathbf{a}+B(v(t)-E_vd_v)),C_vBE_vD_v).\end{aligned} \tag{4.8}$$

Although that $\zeta_{\widehat{\mathcal{U}}}(\mathbf{s};t;P,\mathbf{a},B)$ depends on the choice of C_v, their special values on nonpositive integers recover FMZVs in any cases.

Theorem 13. *Assume $P(N)$ is a lattice polytope. Then we have*

$$\lim_{s_1\to-k_1}\cdots\lim_{s_q\to-k_q}\zeta_{\widehat{\mathcal{U}}}(\mathbf{s};t;P,\mathbf{a},B)\Big|_{t=N}=S(\mathbf{k};P(N);\mathbf{a},B). \tag{4.9}$$

The proof consists of the following argument: each $\lim_{s_1\to-k_1}\cdots\lim_{s_q\to-k_q}\zeta(\mathbf{s};\mathbf{a},B)$ is described by iterated residues [8, 9] and after the summation $\sum_{v:\text{vertex}}$, the iterated residues become the ordinary residues and coincide with the values for FMZVs.

However the holomorphy of $\zeta_{\widehat{\mathcal{U}}}(\mathbf{s};t;P,\mathbf{a},B)$ does not hold necessarilly.

We give an example of Mordell–Tornheim type, which is defined in [6] by

$$\zeta_{\mathcal{A},\mathrm{MT}}(k_1,k_2;k_3)=\left(\sum_{\substack{n_1,n_2>0\\n_1+n_2<p}}\frac{1}{n_1^{k_1}n_2^{k_2}(n_1+n_2)^{k_3}}\right)_p. \tag{4.10}$$

The corresponding SMZVs with q-analogues are given in [2]. In [13], $\widehat{\mathcal{A}},\widehat{\mathcal{S}}$-generalizations are studied and in [5], $\mathcal{U}$-generalization is introduced.

To realize the double Mordell–Tornheim MZVs in our framework, we set $\mathbf{a}=(0,0)$, $B=\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}$ and $P(t)=\{(x,y)\mid x,y\geq 0, x+y\leq t\}$. Then

$$\sum_{\substack{n_1,n_2>0\\n_1+n_2<t}}n_1^{k_1}n_2^{k_2}(n_1+n_2)^{k_3}=\sum_{(n_1,n_2)\in P^0(t)}{n_1}^{k_1}{n_2}^{k_2}(n_1+n_2)^{k_3} \tag{4.11}$$

and (3.3) reads

$$\begin{aligned}F(\mathbf{z};t;P^0;\mathbf{a},B)&=(-1)^r\sum_{v:\text{vertex}}\frac{e^{-\mathbf{z}Bv(t)}}{\prod_{e\in\mathcal{E}_v}(1-e^{\mathbf{z}Be})}\\&=\frac{1}{(1-e^{z_1+z_3})(1-e^{z_2+z_3})}+\frac{e^{-tz_1-tz_3}}{(1-e^{-z_1+z_2})(1-e^{-z_1-z_3})}\\&\quad+\frac{e^{-tz_2-tz_3}}{(1-e^{z_1-z_2})(1-e^{-z_2-z_3})},\end{aligned} \tag{4.12}$$

which gives one of the UMZFs as

$$\begin{aligned}\zeta_{\widehat{\mathcal{U}},\mathrm{MT}}(s_1,s_2;s_3;t) &= \zeta_{\mathrm{MT}}(s_1,s_2;s_3) + (-1)^{s_1+s_3}\zeta_{\mathrm{MT}}(s_2,s_3;s_1;t)\\ &\quad + (-1)^{s_2+s_3}\zeta_{\mathrm{MT}}(s_1,s_3;s_2;t),\end{aligned} \tag{4.13}$$

where

$$\zeta_{\mathrm{MT}}(s_1,s_2;s_3;t) = \sum_{n_1,n_2=1}^{\infty} \frac{1}{n_1^{s_1}(n_2-t)^{s_2}(n_1+n_2-t)^{s_3}}. \tag{4.14}$$

This UMZF with $t=0$ reduces to that in [5] up to factor $(-1)^{\pm s_3}$.

5 Reciprocity, Kontsevich's order and negative values

In this last section, we explain the role of t_1, t_2 in the definition (1.12) of $\zeta_{\widehat{\mathcal{U}}}(s_1,\dots,s_r;t_1,t_2)$. Let $\prec$ denotes Kontsevich's order defined on $\mathbb{Z}\setminus\{0\}$:

$$(0) \prec 1 \prec 2 \prec \cdots \prec (\infty = -\infty) \prec \cdots \prec -2 \prec -1 \prec (0). \tag{5.1}$$

Then we have the following in the sense of Theorem 6 and 7 (for the detail, see [11]):

$$\sum_{-t_1 \prec n_1 \prec \cdots \prec n_r \prec t_2} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \quad \text{“}=\text{”} \quad \begin{cases} \zeta_{\mathcal{S}}(k_1,\dots,k_r) & (t_1=0, t_2=0),\\ \zeta_N(k_1,\dots,k_r) & (t_1=0, t_2=N\in\mathbb{Z}_{\geq 1}),\\ \zeta(k_1,\dots,k_r) & (t_1=0,t_2=\infty).\end{cases} \tag{5.2}$$

From this observation, we see that $-t_1$ and t_2 are the extreme points of the sum.

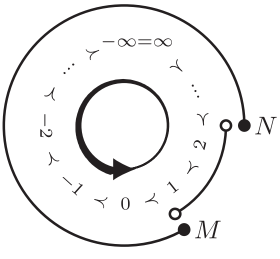
By the Ehrhart–Macdonald reciprocity, we can show that

$$\zeta_{\widehat{\mathcal{U}}}(-k_1,\dots,-k_r;-M,N) = \begin{cases} \displaystyle\sum_{M<n_1<\cdots<n_r<N} n_1^{k_1}\cdots n_r^{k_r}, & (\text{if } M<N) \quad (\zeta),\\ \displaystyle(-1)^r \sum_{N\leq n_r\leq\cdots\leq n_1\leq M} n_1^{k_1}\cdots n_r^{k_r}, & (\text{if } M\geq N) \quad (\zeta^\star),\end{cases} \tag{5.3}$$

which adds the following to the list (5.2)

$$\sum_{-t_1 \prec n_1 \prec \cdots \prec n_r \prec t_2} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \quad \text{“}=\text{”} \quad (-1)^r\zeta_N^\star(k_r,\dots,k_1) \qquad (t_1=-N\in\mathbb{Z}_{\geq 1}, t_2=0). \tag{5.4}$$

The reason why $\zeta_{\widehat{\mathcal{U}}}(s_1,\dots,s_2;t_1,t_2)$ describes both FMZVs and FMZSVs, is well understood by Kontsevich's order. To explain this phenomenon, we consider the dep $=1$ case for simplicity.



Since the sum of n^k over all integers except 0 is

$$\zeta_{\mathcal{S}}(-k) = \sum_{0 \prec n \prec 0} n^k = -\delta_{k,0} \tag{5.5}$$

we see that for $f \in \mathbb{Z}[t]$ with $f(0) = 0$,

$$\sum_{0 \prec n \preceq 0} f(n) = 0, \tag{5.6}$$

which can be regarded as a discrete analogue of Cauchy's integral theorem:

$$\int_C f(z)dz = 0. \tag{5.7}$$

The equation (5.6) implies

$$\sum_{M \prec n \prec N} f(n) = -\sum_{N \preceq n \preceq M} f(n), \tag{5.8}$$

that is, the upper extreme point is also the lower extreme point, and vice versa.

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