Classification theory of planar p-elasticae

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This paper is a summary and an announcement of some recent results about classification of planar p-elasticae, obtained in joint work with K. Yoshizawa [8, 9, 10]. The detailed versions will be submitted to somewhere.

1 Introduction of p-elastica

Euler's elastica is introduced by Daniel Bernoulli and studied by Leonhard Euler in the 18$^{\text{th}}$ century for modelling elastic rods. It is defined as a critical point of the bending energy

$$B[\gamma] := \int_\gamma k^2 ds$$

among a class of fixed-length curves γ, where k and s denote the curvature and the arclength parameter of γ. The admissible curves typically lie in the Euclidean plane $\mathbf{R}^2$ and satisfy some boundary conditions. It is an old but very important fact, worked out by Louis Saalschütz in the 19$^{\text{th}}$, that all planar elasticae are completely classified on the level of smooth critical points and explicitly represented by Jacobian elliptic integrals and functions; see [3] for more details of the history. In addition, by now it is also well known that the natural energy space for elastica theory is of $W^{2,2}$ Sobolev class, since the curvature basically corresponds to the second derivative. In fact, a critical point in the $W^{2,2}$ class is always smooth (in fact analytic) by a standard bootstrap argument.

For $p \in (1,\infty)$, the notion of p-elastica is defined as an L^p-counterpart of Euler's elastica, namely a critical point of the p-bending energy

$$B_p[\gamma] := \int_\gamma |k|^p ds$$

in the class of fixed-length $W^{2,p}$-curves. By the Lagrange multiplier method, it is equivalent to define as a critical point of the energy of the form

$$E_{p,\lambda}[\gamma] := B_p[\gamma] + \lambda L[\gamma] = \int_\gamma |k|^p ds + \lambda \int_\gamma ds.$$

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More precisely, an immersed curve $\gamma \in W^{2,p}(0,1;\mathbf{R}^2)$ is called p-elastica if there is $\lambda \in \mathbf{R}$ such that for any $\eta \in C_c^\infty(0,1;\mathbf{R}^2)$,

$$\frac{d}{d\varepsilon} E_{p,\lambda}[\gamma + \varepsilon\eta]\Big|_{\varepsilon=0} = 0.$$

Such a curve weakly solves the Euler–Lagrange equation of the form

$$p|k|^{p-2}\partial_s^2 k + p(p-2)|k|^{p-4}k(\partial_s k)^2 + |k|^p k - \lambda k = 0.$$

If $p = 2$, this equation reduces to the classical one for Euler's elastica, i.e.,

$$2\partial_s^2 k + k^3 - \lambda k = 0,$$

which can be explicitly solved by Jacobian elliptic functions.

The p-bending energy and related critical points are also widely studied by many authors. Apart from their own analytical and geometrical interest, those research objects appear in several contexts both from theoretical and applied aspects; see [8] and references therein for where they come from and what is known about them.

Among many other previous studies, here we mention a remarkable result of Watanabe [19], which provides an explicit family of planar p-elasticae.

An important fact found in [19] is that some special cases of p-elliptic integrals introduced by Takeuchi [17] directly appear in the representation of p-elasticae. This fact suggests that there is a possibility to extend the known explicit formulae of Saalschütz type to p-elasticae by introducing appropriate notions of p-elliptic functions. However this has not been achieved, the reason of which seems to be that although some p-elliptic functions are already defined in several ways [16, 17], they are well suited for analyzing equations involving the p-Laplacian $(|u'|^{p-2}u')'$ but do not directly fit into the equation for the curvature of p-elasticae whose leading order term is expressed as $(|u|^{p-2}u)''$.

Another important fact is that if $p > 2$, the degeneracy of the equation (in the sense that the prefactor $|k|^{p-2}$ in front of $\partial_s^2 k$ can vanish) yields a new family of non-periodic critical points, called flat-core. More precisely, a flat-core p-elastica may have some interval-type zero sets of the curvature with arbitrary length, which particularly means that the curve may not be analytic. This fact indicates that there is a substantial analytical challenge to extend the classification of Euler's elasticae, in which such non-periodic solutions do not exist.

2 Classification of planar p-elasticae

Overcoming the aforementioned difficulties, in our first paper [8], we succeeded in classifying all planar p-elasticae with a general $p \in (1,\infty)$ in form of a complete extension of the Saalschütz-type formulae. In particular, we introduced new types of p-elliptic functions; if $p = 2$, they agree with the standard Jacobian elliptic functions so that we can directly recover the classical case. In addition, our formulae tell us that the flat-core p-elasticae obtained by Watanabe can be understood as a nontrivial generalization of the so-called borderline elastica.

In what follows, we state our classification results in terms of p-elliptic integrals and functions, whose definitions are briefly given in Appendix A below; see [8] for more details.

2.1 Classification and Saalschütz-type formulae

For clarity we consider the two cases of $p \leq 2$ and $p > 2$ separately. The first case $p \leq 2$ can be stated in a very parallel way to the classical case.

Theorem 2.1 (Classification of planar p-elasticae: $p \leq 2$)**.** *Let $p \in (1, 2]$ and γ be a p-elastica in $\mathbf{R}^2$. Then up to similarity (i.e., translation, rotation, reflection, and dilation) and reparameterization, the curve γ is represented by $\gamma(s) = \gamma_*(s + s_0)$ with some $s_0 \in \mathbf{R}$, where $\gamma_* : \mathbf{R} \to \mathbf{R}^2$ is one of the following five arclength parameterizations:*

- (Case I — Linear p-elastica) $\gamma_\ell(s) = (s, 0)$, *where* $k_\ell \equiv 0$.
- (Case II — Wavelike p-elastica) *For some* $q \in (0, 1)$,

$$\gamma_w(s,q) = \begin{pmatrix} 2\mathrm{E}_{1,p}(\mathrm{am}_{1,p}(s,q),q) - s \\ -q\frac{p}{p-1}|\operatorname{cn}_p(s,q)|^{p-2}\operatorname{cn}_p(s,q) \end{pmatrix}. \tag{2.1}$$

In this case, $\theta_w(s) = 2\arcsin(q\operatorname{sn}_p(s,q))$ *and* $k_w(s) = 2q\operatorname{cn}_p(s,q)$.

- (Case III — Borderline p-elastica)

$$\gamma_b(s) = \begin{pmatrix} 2\tanh_p s - s \\ -\frac{p}{p-1}(\operatorname{sech}_p s)^{p-1} \end{pmatrix}. \tag{2.2}$$

In this case, $\theta_b(s) = 2\,\mathrm{am}_{1,p}(s,1) = 2\,\mathrm{am}_{2,p}(s,1)$ *and* $k_b(s) = 2\operatorname{sech}_p s$.

- (Case IV — Orbitlike p-elastica) *For some* $q \in (0, 1)$,

$$\gamma_o(s,q) = \frac{1}{q^2}\begin{pmatrix} 2\mathrm{E}_{2,\frac{p}{p-1}}(\mathrm{am}_{2,p}(s,q),q) + (q^2-2)s \\ -\frac{p}{p-1}\operatorname{dn}_p(s,q)^{p-1} \end{pmatrix}. \tag{2.3}$$

In this case, $\theta_o(s) = 2\,\mathrm{am}_{2,p}(s,q)$ *and* $k_o(s) = 2\operatorname{dn}_p(s,q)$.

- (Case V — Circular p-elastica) $\gamma_c(s) = (\cos s, \sin s)$, *where* $k_c \equiv 1$.

Here θ_* denotes the tangential angle $\partial_s\gamma_* = (\cos\theta_*, \sin\theta_*)$, and k_* the (counterclockwise) signed curvature $k_* = \partial_s\theta_*$.

Now we turn to the case of $p > 2$. For describing flat-core p-elasticae in a concise way, it is convenient to introduce the following 'concatenation' of curves: For $\gamma_j : [a_j, b_j] \to \mathbf{R}^2$ with $L_j := b_j - a_j \geq 0$, we define $\gamma_1 \oplus \gamma_2 : [0, L_1 + L_2] \to \mathbf{R}^2$ by

$$(\gamma_1 \oplus \gamma_2)(s) := \begin{cases} \gamma_1(s + a_1), & s \in [0, L_1], \\ \gamma_2(s + a_2 - L_1) + \gamma_1(b_1) - \gamma_2(a_2), & s \in [L_1, L_1 + L_2], \end{cases}$$

and inductively define $\gamma_1 \oplus \cdots \oplus \gamma_N := (\gamma_1 \oplus \cdots \oplus \gamma_{N-1}) \oplus \gamma_N$. We also write

$$\bigoplus_{j=1}^{N} \gamma_j := \gamma_1 \oplus \cdots \oplus \gamma_N.$$

Theorem 2.2 (Classification of planar p-elasticae: $p > 2$)**.** *Let $p \in (2, \infty)$ and γ be a p-elastica in $\mathbf{R}^2$. Then up to similarity and reparameterization, the curve γ is represented by $\gamma(s) = \gamma_*(s + s_0)$ with some $s_0 \in \mathbf{R}$, where either $\gamma_* : \mathbf{R} \to \mathbf{R}^2$ is one of the four arclength parameterizations in Cases I, II, IV, and V of Theorem 2.1, or $\gamma_* = \gamma_f : [0, L] \to \mathbf{R}^2$ is the following arclength parameterization:*

- (Case III' — Flat-core p-elastica) *For some integer $N \geq 1$, signs $\sigma_1, \ldots, \sigma_N \in \{+, -\}$, and nonnegative numbers $L_1, \ldots, L_N \geq 0$,*

$$\gamma_f = \bigoplus_{j=1}^{N} (\gamma_\ell^{L_j} \oplus \gamma_b^{\sigma_j}), \tag{2.4}$$

where $\gamma_b^{\pm} : [-\mathrm{K}_p(1), \mathrm{K}_p(1)] \to \mathbf{R}^2$ and $\gamma_\ell^{L_j} : [0, L_j] \to \mathbf{R}^2$ are defined by

$$\gamma_b^{\pm}(s) = \begin{pmatrix} 2\tanh_p s - s \\ \mp\frac{p}{p-1}(\mathrm{sech}_p\, s)^{p-1} \end{pmatrix}, \quad \gamma_\ell^{L_j}(s) = \begin{pmatrix} -s \\ 0 \end{pmatrix}. \tag{2.5}$$

The curves $\gamma_b^{\pm}(s)$ have $\theta_b^{\pm}(s) = \pm 2\,\mathrm{am}_{1,p}(s, 1) = \pm 2\,\mathrm{am}_{2,p}(s, 1)$ and $k_b^{\pm}(s) = \pm 2\,\mathrm{sech}_p\, s$ for $s \in [-\mathrm{K}_p(1), \mathrm{K}_p(1)]$.

The above statement does not contain any information about the multiplier λ. This is just for simplicity, and in fact the precise relation between λ and our classification is obtained in [8]. In particular, if $\lambda = 0$, then the corresponding p-elastica (called free p-elastica) is either linear or wavelike with the special modulus $q = 1/\sqrt{2}$. In the latter case, the profile curve is represented as the graph of an antiperiodic function whose slope is vertical at each zero. In particular, there is no closed planar free p-elastica.

2.2 Optimal regularity

Based on the previous classification we also clarified optimal regularity of planar p-elasticae [8]. In general, any planar p-elastica with $p \in (1,\infty)$ is of class $W^{3,1} \subset C^2$ and hence has continuous curvature. In addition, the (arclength parameterized) curvature has optimal regularity which is roughly speaking "same" as the function $x \mapsto |x|^{\frac{2-p}{p-1}}x$ $(= \mathrm{sign}(x)|x|^{\frac{1}{p-1}})$ around $x = 0$. In particular, if $\frac{1}{p-1}$ is an odd integer, or equivalently if

$$p \in \left\{2, \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \dots \right\},$$

then every p-elastica is smooth (analytic). If otherwise, then p-elasticae may not be smooth but always belong to some optimal Sobolev class, which depends on p and gets worse as p increases.

The above classification and regularity results play fundamental and important roles in our successive works.

3 Classification of closed planar p-elasticae

Using our classification we could also classify all p-elasticae of closed planar curves for the first time [8]. In the classical case $p = 2$ it is well known that any closed planar elastica is either a circle or a figure-eight elastica. This fact is extended to all $p \in (1,\infty)$ by suitably introducing the notion of "figure-eight p-elastica". More precisely, the figure-eight p-elastica is defined as a wavelike p-elastica with a unique modulus $q_* = q_*(p) \in (0,1)$ such that

$$2\mathrm{E}_{1,p}(q_*) - \mathrm{K}_{1,p}(q_*) = 0,$$

or in other words, such that the curve $\gamma_w(\cdot, q_*)$ defines a closed curve.

Theorem 3.1 (Classification of closed planar p-elastica)**.** *Let $p \in (1,\infty)$ and γ be a closed planar p-elastica. Then γ is either a circle or a figure-eight p-elastica, possibly multiply covered.*

In particular, any flat-core p-elastica is ruled out in the class of closed planar p-elasticae.

4 Classification of pinned planar p-elasticae

In the next paper [9] we addressed a boundary value problem. We classified all the possible critical points among fixed-length planar curves subject to the so-called pinned boundary condition, which means that the endpoints are prescribed only up to zeroth order. In this setting we deduce from the standard first variation argument that in addition to the fixed boundary condition we also have the natural boundary condition (or zero Navier boundary condition), i.e., the curvature vanishes at the endpoints. Thanks to this fact with some additional computations, we can completely extend the known classification [20, 7] for $p = 2$ to a general $p \in (1, \infty)$ (within the framework of planar curves).

Given $P_0, P_1 \in \mathbf{R}^2$ and $L_0 > |P_0 - P_1|$, we let A_{pin} denote the set of all immersed curves $\gamma \in W^{2,p}(0,1;\mathbf{R}^2)$ such that $\gamma(0) = P_0$, $\gamma(1) = P_1$, and $L[\gamma] = L_0$. A curve is called a pinned planar p-elastica if it is a critical point of B_p in the class A_{pin}. The classification of pinned planar p-elasticae are briefly summerized as follows (see [9] for more details):

(i) If $|P_0 - P_1| = 0$, then any pinned planar p-elastica is an $\frac{N}{2}$-fold figure-eight p-elastica with some integer $N \geq 1$, whose endpoints are both placed at the crossing point.

(ii) If $|P_0 - P_1| > 0$, and if either $p \leq 2$ or $\frac{|P_0-P_1|}{L_0} < \frac{1}{p-1}$, then any pinned planar p-elastica is given by either a convex arc, a locally convex loop, or their suitable periodic extensions (with rescaling).

(iii) If $|P_0-P_1| > 0$, and if $p > 2$ and $\frac{|P_0-P_1|}{L_0} \geq \frac{1}{p-1}$, then any pinned planar p-elastica is given by either a convex arc, its periodic extensions, or a flat-core p-elasticae in a suitable class.

In summary, the case of $p \leq 2$ or small $|P_0 - P_1|$ turns out to be a very parallel generalization of the classical case $p = 2$; on the other hand, if $p > 2$ then flat-core p-elasticae emerge and yield various new phenomena. For example, the number of critical points changes from countable to uncountable (up to invariances).

5 Uniqueness of minimal pinned planar p-elasticae and Li–Yau type inequality

In [9] we also obtained unique existence of global minimizers under the pinned boundary condition. In case (i) in the previous section, any global minimizer of B_p in A_{pin} is (up to isometry and reparameterization) uniquely given by a half-fold figure-eight p-elastica. In both cases (ii) and (iii), such a unique global minimizer is given by a convex arc.

As an application, we can obtain a new Li–Yau type inequality. We first recall that, in [7], the author obtained an optimal form of a Li–Yau type inequality, which bounds the bending energy from below in terms of multiplicity for closed curves in Euclidean space of any codimension. The proof relies on the known classification of classical elasticae. In [9], based on our new result on the global minimality of a half-fold figure-eight p-elastica, we could apply the strategy in [7] to extend the Li–Yau type inequality to all $p \in (1, \infty)$ in codimension one. To state this result, we introduce the normalized p-bending energy $\bar{B}_p$ by

$$\bar{B}_p[\gamma] := L[\gamma]^{p-1} B_p[\gamma].$$

This energy is invariant with respect to rescaling.

Theorem 5.1 (Li–Yau type inequality)**.** *Let $p \in (1, \infty)$. If $\gamma \in W^{2,p}(\mathbf{S}^1; \mathbf{R}^2)$ is an immersed closed curve with a point of multiplicity $m \geq 2$, then*

$$\bar{B}_p[\gamma] \geq m^p.$$

This inequality is optimal in the 'more than half' case in the sense that there is a closed curve attaining equality for any $p \in (1, \infty)$ and any even integer $m \geq 2$. On the other hand, for odd integers $m \geq 3$, the optimality sensitively depends on the value of p. If $p = 2$, it is shown in [7] that for any odd integer $m \geq 3$ the inequality is not optimal in codimension one due to an algebraic reason (but optimal in codimension more than one). Our extension to a general exponent $p \in (1, \infty)$ reveals that infinitely many exponents p recover the optimality for all but finite (small) multiplicities. More interestingly, there is a unique exponent in which our inequality is fully optimal.

Theorem 5.2 (Unique exponent for full optimality)**.** *There exists a unique exponent $p \in (1, \infty)$ with the following property: For any integer $m \geq 2$ there is an immersed closed curve $\gamma \in W^{2,p}(\mathbf{S}^1; \mathbf{R}^2)$ with a point of multiplicity m such that*

$$\bar{B}_p[\gamma] = m^p.$$

The above exponent is given by a unique solution to a transcendental equation, and numerically computed to be

$$p = 1.5728...$$

The uniqueness property relies on the following new monotonicity result.

Theorem 5.3 (Monotonicity of the crossing angle)**.** *The crossing angle of figure-eight p-elasticae is strictly monotone with respect to $p \in (1, \infty)$ and varies from 0 to π.*

Furthermore, in [9] we also applied our Li–Yau inequality to prove existence of minimal p-elastic networks, thus extending the recent result for $p = 2$ by Dall'Acqua–Novaga–Pluda [2]. Our proof mainly follows the strategies of [2, 7] but also needs some monotonicity involving p-elliptic integrals.

6 Stability of planar p-elasticae

Finally we discuss stability issues, in particular announcing the contents of our forthcoming third paper [10].

Stability analysis is complicated even for classical elasticae in general. Usually, in order to know stability (resp. minimality) one needs to compute the second variation (resp. the value) of the bending energy but such computations are far from easy except for some special cases. In addition, the second variation approach relies on a certain fine functional analytic structure and hence it is not clear if it can be extended to more general functionals including the p-bending energy.

In our forthcoming paper [10] we develop a general theory on rigidity of stable as well as minimal critical points for a very general class of functionals depending on the curvature of planar curves. Combined with our previous classification results, this theory is quite well applicable to planar p-elasticae.

A particular consequence of our theory is a far-reaching extension of Sachkov's optimal rigidity principles for clamped planar elasticae (cf. [12, 11, 14, 13, 15]). Sachkov's principles are very roughly summarized as follows:

- If a clamped planar elastica is minimal, then it does not exceed one period.
- If a clamped planar elastica is stable, then it does not contain three inflection points (in its interior).

We extend those facts to a wide class of critical points in a completely different way. Our proof is based on a very simple 'cut-and-paste' trick. In fact, this trick has been introduced in the author's unpublished notes [6] in order to answer a question by Glen Wheeler, who asked the author whether there is a more geometric way (a là Avvakmov–Karpenkov–Sossinsky [1]) to understand Sachkov's theory. In [10] this method is opened for the first time and, more importantly, extended to general functionals, to other boundary conditions, and even to non-periodic curves such as flat-core p-elasticae, with various new techniques.

In this paper we do not write down the precise statements of the general results obtained in [10] but instead we state two typical important consequences for p-elasticae.

The first one is classification of stable closed planar p-elasticae, which turns out to be a natural extension of the classical case $p = 2$.

Theorem 6.1 (Uniqueness of stable closed planar p-elasticae)**.** *Let $p \in (1, \infty)$. For each C^1-regular homotopy class of closed planar curves there is a unique stable closed planar p-elastica (up to invariances).*

In fact, this result is an almost direct consequence of our general principles for (possibly non-closed) clamped stable solutions.

The second one is about pinned p-elasticae. Here the situation depends on whether p exceeds 2, but at least for $p \leq 2$ we can obtain a very simple uniqueness theorem.

Theorem 6.2 (Uniqueness of stable pinned planar p-elasticae)**.** *Let $p \in (1,2]$. In the class A_{pin} there is a unique stable critical point of B_p (up to invariances), which is nothing but a global minimizer.*

To the authors' knowledge, our result seems to be the first to explicitly and rigorously claim such a uniqueness property even in the classical case $p = 2$. Note however that such a property would be quite expected in view of, and also at least formally follow by, classical studies based on linear stability analysis [4, 5]. Anyhow, the case of $p \neq 2$ is completely new, and our (simpler) argument would also be new even for $p = 2$.

We finally remark that the above rigidity also extends to $p > 2$ in the absence of flat-core p-elasticae (where the distance of the endpoints is sufficiently small). Even for flat-core p-elasticae, our method yields some partial instability results, but at this moment we do not reach the complete classification of stability of pinned planar p-elasticae with $p > 2$. In fact, we expect that flat-core p-elasticae may have different stability properties from the classical case. We plan to address this issue in a future work.

A p-Elliptic integrals and functions

In this appendix we only write down all the necessary definitions. The interested reader can find their basic properties more precisely in [8] (and also [9]).

Definition A.1 (p-Elliptic integrals of the first kind)**.** Let $p \in (1,\infty)$. We define the incomplete p-elliptic integrals of the first kind $\mathrm{F}_{1,p}(x,q)$ and $\mathrm{F}_{2,p}(x,q)$ of modulus $q \in [0,1)$, where $x \in \mathbf{R}$, by

$$\mathrm{F}_{1,p}(x,q) := \int_0^x \frac{|\cos\theta|^{1-\frac{2}{p}}}{\sqrt{1-q^2\sin^2\theta}}\,d\theta,$$
$$\mathrm{F}_{2,p}(x,q) := \int_0^x \frac{1}{\sqrt[p]{1-q^2\sin^2\theta}}\,d\theta,$$

and the corresponding complete p-elliptic integrals $\mathrm{K}_{1,p}(q)$ and $\mathrm{K}_{2,p}(q)$ by

$$\mathrm{K}_{1,p}(q) := \mathrm{F}_{1,p}(\pi/2,q), \quad \mathrm{K}_{2,p}(q) := \mathrm{F}_{2,p}(\pi/2,q).$$

For $q = 1$, we define

$$\mathrm{F}_{1,p}(x,1) = \mathrm{F}_{2,p}(x,1) := \int_0^x \frac{d\theta}{|\cos\theta|^{\frac{2}{p}}},$$

where $x \in (-\frac{\pi}{2},\frac{\pi}{2})$ if $1 < p \leq 2$ and $x \in \mathbf{R}$ if $p > 2$. In addition,

$$\mathrm{K}_{1,p}(1) = \mathrm{K}_{2,p}(1) = \mathrm{K}_p(1) := \begin{cases} \infty & \text{if } 1 < p \leq 2, \\ \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{(\cos\theta)^{\frac{2}{p}}} < \infty & \text{if } p > 2. \end{cases}$$

Definition A.2 (p-Elliptic integrals of the second kind)**.** Let $p \in (1, \infty)$. We define the incomplete p-elliptic integrals of the second kind $\mathrm{E}_{1,p}(x, q)$ and $\mathrm{E}_{2,p}(x, q)$ of modulus $q \in [0, 1]$, where $x \in \mathbf{R}$, by

$$\mathrm{E}_{1,p}(x, q) := \int_0^x \sqrt{1 - q^2 \sin^2 \theta}\, |\cos\theta|^{1-\frac{2}{p}}\, d\theta,$$
$$\mathrm{E}_{2,p}(x, q) := \int_0^x \sqrt[p]{1 - q^2 \sin^2 \theta}\, d\theta,$$

and the corresponding complete p-elliptic integrals $\mathrm{E}_{1,p}(q)$ and $\mathrm{E}_{2,p}(q)$ by

$$\mathrm{E}_{1,p}(q) := \mathrm{E}_{1,p}(\pi/2, q), \quad \mathrm{E}_{2,p}(q) := \mathrm{E}_{2,p}(\pi/2, q).$$

Remark A.3. The above p-elliptic integrals can be regarded as special cases of Takeuchi's generalization [17, 18], and have already been used by Watanabe [19]. The p-elliptic functions below are newly introduced in [8].

Definition A.4 (p-Elliptic functions)**.** Let $p \in (1, \infty)$ and $q \in [0, 1]$. We define $\mathrm{am}_{1,p}(x, q)$ by the inverse function of $\mathrm{F}_{1,p}(x, q)$, so that

$$x = \int_0^{\mathrm{am}_{1,p}(x,q)} \frac{|\cos\theta|^{1-\frac{2}{p}}}{\sqrt{1 - q^2 \sin^2 \theta}}\, d\theta \qquad \text{for } x \in \mathbf{R}.$$

We define $\mathrm{sn}_p(x, q)$, *p-elliptic sine function* with modulus q, by

$$\mathrm{sn}_p(x, q) := \sin \mathrm{am}_{1,p}(x, q), \quad x \in \mathbf{R},$$

and define $\mathrm{cn}_p(x, q)$, *p-elliptic cosine function* with modulus q, by

$$\mathrm{cn}_p(x, q) := |\cos \mathrm{am}_{1,p}(x, q)|^{\frac{2}{p}-1} \cos \mathrm{am}_{1,p}(x, q), \quad x \in \mathbf{R}.$$

In addition, we also define $\mathrm{am}_{2,p}(x, q)$ by the inverse function of $\mathrm{F}_{2,p}(x, q)$,

$$x = \int_0^{\mathrm{am}_{2,p}(x,q)} \frac{1}{\sqrt[p]{1 - q^2 \sin^2 \theta}}\, d\theta \qquad \text{for } x \in \mathbf{R},$$

and define $\mathrm{dn}_p(x, q)$, *p-delta amplitude function* with modulus q, by

$$\mathrm{dn}_p(x, q) := \sqrt[p]{1 - q^2 \sin^2 \big(\mathrm{am}_{2,p}(x, q)\big)}, \quad x \in \mathbf{R}.$$

Definition A.5 (p-Hyperbolic secant function)**.** Let $p \in (1, \infty)$. We define

$$\mathrm{sech}_p\, x := \begin{cases} \mathrm{cn}_p(x, 1) = \mathrm{dn}_p(x, 1), & x \in (-\mathrm{K}_p(1), \mathrm{K}_p(1)), \\ 0, & x \in \mathbf{R} \setminus (-\mathrm{K}_p(1), \mathrm{K}_p(1)). \end{cases}$$

When $1 < p \leq 2$, we regard $(-\mathrm{K}_p(1), \mathrm{K}_p(1))$ as $\mathbf{R}$.

Definition A.6 (p-Hyperbolic tangent function)**.** Let $p \in (1, \infty)$. We define

$$\tanh_p x := \int_0^x (\mathrm{sech}_p\, t)^p dt, \qquad x \in \mathbf{R}.$$

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