# Comparison of convergence theorems for a complete geodesic space
# 完備測地距離空間における収束定理の比較

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Abstract

In this work, we propose a new iterative method on Hadamard spaces, a modified version of the shrinking projection method. It can be regarded as a corresponding method to Mann's. We compare the proposed method with Mann's and discuss the efficiency of approximating a fixed point of mapping.

## 1 Introduction

Fixed point theory is one of the central research topics in nonlinear analysis, and we mainly consider the approximation techniques of fixed points of nonexpansive mappings in this work.

Approximating fixed points of nonlinear operators has been investigated for many years. There are several popular methods to generate an iterative sequence converging to a fixed point of given mapping. For example, Mann's iterative sequence is guaranteed to converge weakly, whereas Halpern's iterative method converges strongly to the nearest fixed point from a given anchor point. These methods use the convex combination between two or more points to obtain the next point $x_{n+1}$.

On the other hand, several types of projection methods are also proposed. For instance, a sequence generated by the CQ projection method converges strongly to the same point as Halpern's.

In this work, we deal with the shrinking projection method, which was proved by Takahashi, Takeuchi, and Kubota in 2008 as the following theorem. Notice that the

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actual result shows the generated sequence converges to a common fixed point of a family of nonexpansive mappings.

**Theorem 1** (Takahashi, Takeuchi, and Kubota [9]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $T\colon C \to C$ be a nonexpansive mapping such that the set $\operatorname{Fix} T$ of its fixed points is nonempty. Let $\{\alpha_n\}$ be a nonnegative real sequence such that $\sup_{n\in\mathbb{N}} \alpha_n < 1$. For an arbitrary point $u \in H$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{aligned}
&y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\
&C_{n+1} = \{z \in H : \|y_n - z\| \le \|x_n - z\|\} \cap C_n,\\
&x_{n+1} = P_{C_{n+1}}u
\end{aligned}$$

*for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{\operatorname{Fix} T}u \in C$, where $P_K\colon H \to K$ is the metric projection onto a nonempty closed convex subset $K$ of $H$.*

This method has been generalized to the setting of a Banach space [4, 6], that of a Hadamard space [3], and that of the unit sphere of a Hilbert space [5].

We know that the limit point of the sequence generated by the shrinking method coincides with the limit of Halpern's sequence, and they both converge strongly. At the same time, it is unknown what projection methods are analogous to Mann's type method. If there is, it must converge weakly to a fixed point.

In this work, we propose a new iterative method on Hadamard spaces, which is $\Delta$-convergent to a fixed point. This method is a modified version of the shrinking projection method and can be regarded as a corresponding method to Mann's. We remark that $\Delta$-convergence on Hadamard spaces corresponds to weak convergence on Hilbert spaces.

Moreover, we compare the proposed method with Mann's and discuss the efficiency of approximating a fixed point of mapping.

## 2 Preliminaries

In this work, we deal with a geodesic space, which is defined as follows: Let $X$ be a metric space. For $x, y \in X$ with $l = d(x, y)$, we define a geodesic between $x$ and $y$ by a mapping $c_{xy}\colon [0, l] \to X$ such that $c_{xy}(0) = x$, $c_{xy}(l) = y$, and $d(c_{xy}(s), c_{xy}(t)) = |s - t|$ for every $s, t \in [0, l]$. We say $X$ to be a geodesic space if, for every pair $x, y \in X$, a geodesic $c_{xy}$ between $x$ and $y$ exists. In particular, if $c_{xy}$ is unique for each choice of $x, y \in X$, then $X$ is called a uniquely geodesic space.

In a uniquely geodesic space $X$, convex combination is naturally defined: For $x, y \in X$ and $t \in [0, 1]$, a point $z = c_{xy}((1-t)d(x, y))$ is called a convex combination between $x$ and $y$ with a ratio $t$, and we use the notation

$$z = c_{xy}((1-t)d(x, y)) = tx \oplus (1-t)y.$$

A CAT(0) space is a uniquely geodesic space having a specific geometrical structure. It is usually defined by using a model space and comparison triangles. However, we also know the following equivalent condition. Namely, a uniquely geodesic space $X$ is a CAT(0) space if and only if the inequality

$$d(tx \oplus (1-t)y, z)^2 \leq td(x,z)^2 + (1-t)d(y,z)^2 - t(1-t)d(x,y)^2$$

holds for every $x, y, z, \in X$ and $t \in [0,1]$. A complete CAT(0) space is called a Hadamard space. For more details, see [2, 1] for instance.

Let $\{x_n\}$ be a sequence in a Hadamard space $X$. An asymptotic center of $\{x_n\}$ is a point $p \in X$ such that

$$\limsup_{n\to\infty} d(x_n, p) = \inf_{y\in X} \limsup_{n\to\infty} d(x_n, y).$$

It is known that if $\{x_n\}$ is bounded, then its asymptotic center is unique. We say that $\{x_n\}$ is $\Delta$-convergent to $x_0 \in X$ if $x_0$ is the asymptotic center of every subsequence of $\{x_n\}$. For more details, see [7, 8]

Let $X$ be a metric space and $T$ a mapping from $X$ into itself. We say $T$ to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for any $x, y \in X$. A point $z \in X$ is called a fixed point of $T$ if $Tz = z$. $\operatorname{Fix} T$ denotes the set of all fixed points of $T$. We know that a nonexpansive mapping $T$ defined on a Hadamard space has a closed convex set of fixed points.

Let $C$ be a nonempty closed convex subset of a Hadamard space $X$. Then, for each $x \in X$, there exists a unique point $y_x \in C$ such that

$$d(x, y_x) = \inf_{y\in C} d(x, y).$$

Using this point, we define a metric projection $P_C\colon X \to C$ by $P_C x = y_x$ for $x \in X$. We know that metric projections defined on a Hadamard space are nonexpansive.

## 3 Modified shrinking projection method

**Theorem 2.** *Let $X$ be a Hadamard space and suppose that a subset $\{z \in X \mid d(x,z) \leq d(y,z)\}$ is convex for every $x, y \in X$. Let $T\colon X \to X$ be a nonexpansive mapping such that $\operatorname{Fix} T \neq \emptyset$. Generate a sequence $\{x_n\} \subset X$ and a sequence $\{C_n\}$ of subsets of $X$ as follows: $x_1 \in X$, $C_1 = X$, and*

$$\begin{aligned} C_{n+1} &= \{z \in X \mid d(Tx_n, z) \leq d(x_n, z)\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x_n \end{aligned}$$

*for every $n \in \mathbb{N}$. Then, $\{x_n\}$ is well defined and is $\Delta$-convergent to some point in $\operatorname{Fix} T$.*

*Proof.* We first show that the following conditions hold for every $n \in \mathbb{N}$ by induction:

- $\operatorname{Fix} T \subset C_n$;

- $x_n \in X$ is well defined.

For $n=1$, it is obvious that $\operatorname{Fix} T \subset C_1 = X$. Since $x_1 \in X$ is given, these conditions hold for $n=1$.

Suppose the case $n=k$; $\operatorname{Fix} T \subset C_k$ and $x_k$ is well defined. Then, since $T$ is nonexpansive, $z \in \operatorname{Fix} T$ implies

$$d(Tx_k, z) \leq d(x_k, z)$$

and thus $z \in C_{k+1}$. This shows that $\operatorname{Fix} T \subset C_{k+1}$ and thus $C_{k+1}$ is nonempty. We also know that $C_{k+1}$ is convex by assuption, and $C_{k+1}$ is closed by the continuity of $d$. Thus $P_{C_{n+1}}$ is defined as a single point and $x_{n+1} = P_{C_{n+1}} x_n$ is well defined. Hence we have $\operatorname{Fix} T \subset C_n$ for every $n \in \mathbb{N}$ and the sequence $\{x_n\}$ is well defined.

Let $u \in \operatorname{Fix} T$. Then, since all metric projections on Hadamard spaces are nonexpansive, and $u \in \operatorname{Fix} T \subset C_{n+1} = \operatorname{Fix} P_{C_{n+1}}$, we have

$$d(x_{n+1}, u) = d(P_{C_{n+1}} x_n, u) \leq d(x_n, u)$$

for $n \in \mathbb{N}$. This shows that a real sequence $\{d(x_n, u)\}$ is nonincreasing, and $\{x_n\}$ is bounded. Since $\{d(x_n, u)\}$ is bounded below, it has a limit $c_u$. Further, since $tu \oplus (1-t) P_{C_{n+1}} x_n \in C_{n+1}$ for every $t \in ]0,1[$, we have

$$\begin{aligned}
d(x_{n+1}, x_n)^2 &= d(P_{C_{n+1}} x_n, x_n)^2 \\
&\leq d(tu \oplus (1-t) P_{C_{n+1}} x_n, x_n)^2 \\
&\leq t d(u, x_n)^2 + (1-t) d(P_{C_{n+1}} x_n, x_n)^2 - t(1-t) d(u, P_{C_{n+1}} x_n)^2 \\
&= t d(u, x_n)^2 + (1-t) d(x_{n+1}, x_n)^2 - t(1-t) d(u, x_{n+1})^2.
\end{aligned}$$

It follows that $d(x_{n+1}, x_n)^2 \leq d(u, x_n)^2 - (1-t) d(u, x_{n+1})^2$, and letting $t \to 0$, we have

$$d(x_{n+1}, x_n)^2 \leq d(u, x_n)^2 - d(u, x_{n+1})^2$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we have

$$\lim_{n \to \infty} d(x_{n+1}, x_n)^2 \leq c_u^2 - c_u^2 = 0$$

and hence $d(x_{n+1}, x_n) \to 0$. Since $x_{n+1} \in C_{n+1}$, we have

$$\begin{aligned}
0 \leq d(Tx_n, x_n) &\leq d(Tx_n, x_{n+1}) + d(x_{n+1}, x_n) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_n) \to 0,
\end{aligned}$$

which implies $d(Tx_n, x_n) \to 0$.

Let $x_0$ be a unique asymptotic center of $\{x_n\}$. To show $\{x_n\}$ is $\Delta$-convergent to $x_0$, take an arbitrary subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with its asymptotic center $y_0$, and we will show that $x_0 = y_0$. We have

$$\limsup_{k \to \infty} d(x_{n_k}, Ty_0) \leq \limsup_{k \to \infty} (d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Ty_0))$$

$$\le \limsup_{k\to\infty} d(x_{n_k}, Tx_{n_k}) + \limsup_{k\to\infty} d(Tx_{n_k}, Ty_0)$$
$$\le \limsup_{k\to\infty} d(x_{n_k}, y_0).$$

Hence, from the uniqueness of the asymptotic center of $\{x_{n_k}\}$, we have $y_0 \in \operatorname{Fix} T$. It follows that

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, y_0) &= \lim_{n\to\infty} d(x_n, y_0) = \lim_{k\to\infty} d(x_{n_k}, y_0)\\ &\le \limsup_{k\to\infty} d(x_{n_k}, x_0) \le \limsup_{n\to\infty} d(x_n, x_0).\end{aligned}$$

Using the uniqueness of the asymptotic center of $\{x_n\}$, we have $x_0 = y_0$. Therefore, $\{x_n\}$ is $\Delta$-convergent to $x_0 \in \operatorname{Fix} T$. □

By reading the proof of this theorem closely, you will find that taking an intersection with the previous convex set is not effective. That is, even if we define

$$C_{n+1} = \{z \in X \mid d(Tx_n, z) \le d(x_n, z)\},$$

the corresponding sequence is convergent to a fixed point of $T$. Moreover, this sequence is identical to that generated by the Mann iterative method with the coefficient $\alpha_n = 1/2$. Indeed, suppose that

$$y = \frac{1}{2}x \oplus \frac{1}{2}Tx$$

and

$$C = \{z \in X \mid d(Tx, z) \le d(x, z)\}.$$

Then, since $y$ is the midpoint of $x$ and $Tx$, we have $d(Tx, y) = d(x, y)$ and thus $y \in C$. It follows that $d(P_C x, x) \le d(y, x)$. On the other hand, since $P_C x \in C$, we have $d(Tx, P_C x) \le d(x, P_C x)$. Thus we have

$$\begin{aligned}d(y, P_C x)^2 &= d\left(\frac{1}{2}x \oplus \frac{1}{2}Tx, P_C x\right)^2\\ &\le \frac{1}{2}d(x, P_C x)^2 + \frac{1}{2}d(Tx, P_C x)^2 - \frac{1}{4}d(x, Tx)^2\\ &\le \frac{1}{2}d(y, x)^2 + \frac{1}{2}d(y, x)^2 - d(x, y)^2\\ &= 0,\end{aligned}$$

and hence $y = P_C x$.

From this fact, you may evaluate that the proposed method is more complicated than the Mann type method and is ineffectual. However, the following example shows that the proposed method will be more effective than the Mann type method under some settings.

**Example.** Let $X = \mathbb{C}$ be the set of all complex numbers and define $T\colon X \to X$ by

$$Tx = e^{i\pi/3}x$$

for $x \in X$. Then, $T$ is a nonexpansive mapping on $X$ with $\operatorname{Fix} T = \{0\}$. Let $x_1 = y_1 = 1 \in X$. We generate a sequence $\{y_n\}$ by

$$\begin{aligned} D_{n+1} &= \{z \in X \mid |Ty_n - z| \leq |y_n - z|\}, \\ y_{n+1} &= P_{D_{n+1}} y_n \end{aligned}$$

Then, it is identical to the sequence generated by the Mann iterative method with the coeficient sequence $\alpha_n = 1/2$:

$$y_{n+1} = \frac{1}{2} y_n + \frac{1}{2} T y_n$$

for every $n \in \mathbb{N}$. Then, we get

$$y_n = \left( \frac{\sqrt{3}}{2} \right)^{n-1} e^{(n-1)\pi i/6}$$

for $n \in \mathbb{N}$ and it is convergent to $0 \in \operatorname{Fix} T$ as $n \to \infty$.

On the other hand, generate $\{x_n\}$ by the method in Theorem 2:

$$\begin{aligned} C_{n+1} &= \{z \in X \mid |Tx_n - z| \leq |x_n - z|\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x_n \end{aligned}$$

for $n \in \mathbb{N}$. Then, we have

$$C_n = \begin{cases} \left\{ z \in X \,\middle|\, \dfrac{(n-1)\pi}{6} \leq \arg z \leq \dfrac{7\pi}{6} \right\} & (2 \leq n \leq 8), \\ \{0\}. & (8 < n) \end{cases}$$

Thus, the sequence $\{x_n\}$ satisfies $x_n = y_n$ for $1 \leq n \leq 8$, and $x_n = 0 \in \operatorname{Fix} T$ for $8 < n$. This fact shows that it is adequate to take an intersection with the previous convex set when generating a sequence of subsets $\{C_n\}$.

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