COMMON FIXED POINT THEOREMS FOR ASYMPTOTIC MAPPINGS IN COMPLETE METRIC SPACES

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ABSTRACT. In this paper we consider an asymptotic version of α-ψ contractive mappings in vector metric spaces taking value in Riesz spaces. and prove ?xed point theorem on this spaces.

1. INTRODUCTION

In [31], Samet and Vetro-Vetro intoroduced the notion of α-ψ mapping and consider the fixed point theorem. In [34], we introduced the notion of α-ψ_n mapping which is a generalization of the α-ψ mapping and consider the fixed point theorem. It is a generalization of the the mapping in Caccioppoli's fixed point theorem and also the generalization of the (c)-comparison operator. In this paper we consider the common fixed point theorem for the α-ψ and α-ψ_n mappings under the ordered vector metric spaces settings. For the mappings of the metric spaces and vector metric spaces, a lot of authors study common fixed theorem. For instance in [4, 11], Altun and Cevik introduce an vector metric spaces and proved Banach contraction theorem. In [30], Rahimi generalized fixed point theorem and they prove some common fixed point theorems for four mappings in ordered vector metric spaces. They also extend and generalize well-known comparable results in the literature.

2. PRELIMINARIES

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and $\mathbb{R}$ the real number. In this section we give sevral preliminaries for the ordered vector metric spaces settings, α-ψ and α-ψ_n mappings,etc.

Let E be a non-empty set. A relation $\leq$ on E is called:

(i) reflexive if $x \leq x$ for all $x \in E$
(ii) transitive if $x \leq y$ $y \leq z$ imply $x \leq z$
(iii) antisymmetric if $x \leq y$ and $y \leq x$ imply $x = y$
(iv) preorder if it is reflexive and transitive. A preorder is called a partial order if it is antisymmetric.
(v) translation invariant if $x \leq y$ implies $(x + z) \leq (y + z)$ for any $z \in E$
(vi) scale invariant if $x \leq y$ implies $(\lambda x) \leq (\lambda y)$ for any $\lambda > 0$. A preorder $\leq$ is called partial order or an order relation if it is antisymmetric.

Given a partially ordered set $(E, \leq)$, that is, the set E equipped with a partial order $\leq$, the notation $x < y$ stands for $x \leq y$ and $x \neq y$. An order interval $[x, y]$ in E is the set $\{z \in E : x \leq z \leq y\}$. A real linear space E equipped with an order relation $\leq$ on E which is compatible with the algebraic structure of E is called an

2010 *Mathematics Subject Classification.* Primary 46A40, 47B50, 47H10.
Key words and phrases. Fixed point theorem, contractive mappings, asymptotic contraction.

ordered linear space or ordered vector space. The ordered vector space $(E, \leq)$ is called a Riesz space (vector lattice or linear lattice) if for every $x, y \in E$, there exist $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$. If we denote $x_+ = 0 \vee x$, $x_- = 0 \wedge (-x)$ and $|x| = x \vee (-x)$, then $x = x_+ - x_-$ and $|x| = x_+ + x_-$.

A closed convex set K in E is called closed convex cone, if for any $\lambda > 0$ and $x \in K$, $\lambda x \in K$. A convex cone K is called pointed if $K \cap (-K) = \{0\}$. From now on we shall call a closed convex pointed cone simply cone.

Let E be a Riesz space. The cone $\{x \in E : x \geq 0\}$ of nonnegative elements in an ordered vector space E is denoted by E_+. E is said to be Archimedean if $\frac{a}{n} \downarrow 0$ holds for every $a \in E_+$.

Let K be a cone in E. Denote $x \leq y$ if $y - x \in K$. Then, order $\leq$ defines a partial order on E called the order induced by K. Conversely, if order $\leq$ is a partial order on E, then E is called ordered vector space and the set $K = \{x \in E : 0 \leq x\}$ is a cone called the positive cone of E. In this case it is easy to see that $x < y$ if and only if $y - x \in E$. Note that if $a \leq ha$ where $a \in K$ and $h \in (0, 1)$, then $a = 0$.

Definition 1. *A sequence of vectors $\{x_n\}$ in E is said to: (i) decrease to an element $x \in E$ if $x_{n+1} \leq x_n$ for every $n \in N$ (set of natural numbers) and $x = \inf\{x_n : n \in N\} = \wedge_{n \in N} x_n$. We denote it by $x_n \downarrow x$. (ii) increase to an element $x \in E$ if $x_n \leq x_{n+1}$ for every $n \in N$ and $x = \sup\{x_n : n \in N\} = \vee_{n \in N} x_n$. We denote it by $x_n \uparrow x$.*

Definition 2. *A sequence of vectors $\{x_n\}$ in E is said to be order convergent to $x \in E$ if there exist sequences $\{y_n\}$ and $\{z_n\}$ in E such that $y_n \downarrow x$, $z_n \uparrow x$ and $z_n \leq x_n \leq y_n$. We denote this by $x = o\text{-}\lim_{n\to\infty} x_n$. If the sequence is order convergent, then its order limit is unique.*

Definition 3. *A sequence of vectors $\{x_n\}$ in E is said to be order Cauchy sequence in E if the sequence $\{x_m - x_n\}$ in cone K is order convergent to* 0.

Definition 4. *Let E and F be two Riesz spaces. A mapping $f : E \to F$ is called order continuous at x_0 in E if for any sequence $\{x_n\}$ in E such that $x = o\text{-}\lim_{n\to\infty} x_n$, we have $f(x) = o\text{-}\lim_{n\to\infty} f(x_n)$.*

Remark 5. *If $x_n \in K$ for every $n \in N$ and $x = o\text{-}\lim_{n\to\infty} x_n$, then $x \in K$. Also, if $x_n \in K$ for every $n \in N$ and $\{y_n\}$ is any sequence for which $y_n - x_n \in K$ with $o\text{-}\lim_{n\to\infty} y_n = 0$, then $o\text{-}\lim_{n\to\infty} x_n = 0$.*

Definition 6. *A cone $K \subset E$ is called regular if every decreasing sequence of elements in K is convergent.*

Definition 7. *Riesz space E is complete if there exists $\sup A$ and $\inf A$ for each bounded countable subset A of E. For more details on Riesz space, order convergence, and order continuity, we refer to* [25] *and references mentioned therein.*

Definition 8. *If $(E, \leq)$ is a Riesz space and $f : E \to E$ is such that $f(x) \leq f(y)$ whenever $x, y \in E$ and $x \leq y$, then f is said to be nondecreasing.*

Definition 9. *Let $(E, \leq)$ be a Riesz space. The set $(UF)f = \{x \in E : x - f(x) \in K\}$ is called upper fixed point set of f, $(LF)f = \{x \in E : f(x) - x \in K\}$ is called lower fixed point set of f and $(F)f = \{x \in E : f(x) = x\}$ is called the set of all fixed points of f.*

Definition 10. *Let $(E, \leq)$ be a Riesz space. The self map f on E is called: (i) dominated on E if $(UF)f = E$. (ii) dominating on E if $(LF)f = E$.*

Example 1. *Let* $E = [0, 1]$ *be endowed with the usual ordering. Let* $f : E \to E$ *be defined by* $f(x) = x$. *Then* $(LF)f = E$.

Example 2. *Let* $E = [0, \infty)$ *be endowed with the usual ordering. Define* $f : E \to E$ *by*

$$f(x) = \begin{cases} x^{1/n} \text{ for } x \in [0, 1), \\ x^n \text{ for } x \in [1, \infty), \end{cases}$$

$n \in N$, *then* $(LF)f = E$.

Definition 11. *Let* $(E, \leq)$ *be a Riesz space. Two mappings* $f, g : E \to E$ *are said to be mutually dominated if* $f(x) \in (UF)g$ *and* $g(x) \in (UF)f$ *for all* $x \in E$. *That is,* $f(x) \geq g(f(x))$ *and* $g(x) \geq f(g(x))$ *for all* $x \in E$.

Definition 12. *Let* $(E, \leq)$ *be a Riesz space. Two mappings* $f, g : E \to E$ *are said to be mutually dominating if* $f(x) \in (LF)g$ *and* $gx \in (LF)f$ *for all* $x \in E$. *That is,* $f(x) \leq g(f(x))$ *and* $g(x) \leq f(g(x))$ *for all* $x \in E$. *The following two examples show that there exist discontinuous and mutually dominating mappings which are not nondecreasing mappings*

The examples of mutually dominating maps which are not non decreasing maps and mutually dominating maps but not non decreasing. These examples are given in [27, example 15,16].

Definition 13. *Let* $(E, \leq)$ *be a Riesz space and* K *be its positive cone. A monotone increasing mapping* : $K \to K$ *is called comparison operator if* $\lim_{n\to\infty} \Phi_n(t) = 0$ *for each* $t \in K$.

Definition 14. *Let* $(E, \leq)$ *be a Riesz space and* $\{x_n\}$ *be a sequence in* E. *If* $o\text{-}\lim_{j\to\infty} x_j$ *exists, then we say that the series* $\sum_{n=1}^{\infty} x_n$ *is order convergent.*

Next we consider the extention of α-ψ_n and α-ψ mappings.

Definition 15. [31] *Let* (X, d) *be a metric space. We say that mapping* $T : X \to X$ *is* α-ψ *contractive if there exist a mapping* $\alpha : X \times X \to [0, 1)$ *and a sequence of nondecreasing mappings* ψ *of* $[0, 1)$ *into itself such that the series* $\sum_{n=1}^{\infty} \psi^n(t)$ *converges for all* $t > 0$ *and for any* $x, y \in X, n \in N$, *we have*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)).$$

Definition 16. [34] *Let* (X, d) *be a metric space. We say that mapping* $T : X \to X$ *is* α-ψ_n *contractive if there exist a mapping* $\alpha : X \times X \to [0, 1)$ *and a sequence of nondecreasing mappings* ψ_n *of* $[0, 1)$ *into itself such that the series* $\sum_{n=1}^{\infty} \psi_n(t)$ *converges for all* $t > 0$ *and for any* $x, y \in X, n \in N$, *we have*

$$\alpha(x, y)d(T^n x, T^n y) \leq \psi_n(d(x, y)).$$

Definition 17. *Let* (X, d) *be a metric space. We say that two mapppings* $f, g : X \to X$ *are said to mutually* α-ψ *contractive if there exist a mapping* $\alpha : X \times X \to [0, 1)$ *and a sequence of nondecreasing mappings* ψ *of* $[0, 1)$ *into itself such that the series* $\sum_{n=1}^{\infty} \psi^n(t)$ *converges for all* $t > 0$ *and for any* $x, y \in X, n \in N$, *we have*

$$\alpha(x, y)d(f(x), f(y)) \leq \psi(d(g(x), g(y))).$$

And we say that two mapppings $f, g : X \to X$ *are said to mutually* α-ψ_n *contractive if there exist a mapping* $\alpha : X \times X \to [0, 1)$ *and a sequence of nondecreasing* *mappings* ψ_n *of* $[0,1)$ *into itself such that the series* $\sum_{n=1}^{\infty}\psi_n(t)$ *converges for all* $t>0$ *and for any* $x,y\in X, n\in N$, *we have*

$$\alpha(x,y)d(f^n(x),f^n(y))\leq \psi_n(d(g(x),g(y))).$$

We also give a definiton of α-admissible mappping.

Definition 18. [31] *We say mapping* f *is* α*-admissible if*

$$\alpha(x,y)>1 \text{ implies } \alpha(f(x),f(y))>1.$$

Example 3. $E=(C([0,1],R^2)$

$$f(x_1,x_2)=\left(\frac{3x_1}{5},\frac{3x_2}{5}\right), g(x_1,x_2)=\left(\frac{4x_1}{5},\frac{4x_2}{5}\right)$$

Define order $x=(x_1,x_2)\succ y=(y_1,y_2)$ *iff* $x_1\geq y_1$ *and* $x_2\geq y_2$. *Define* $\alpha: X\times X\to[0,\infty)$ *by the following.*

$$\alpha(x,y)=\begin{cases}1 \text{ if } x\succeq y,\\ 0 \text{ if } x\prec y.\end{cases}$$

Let $x,y\in R^2$ *such that* $y=(y_1,y_2)\preceq x=(x_1,x_2)$. *Then,* $y_1\leq x_1$ *and* $y_2\leq x_2$. *In this case since*

$$f(x)-f(y)=\left(\frac{3}{5}(x_1-y_1),\frac{3}{5}(x_2-y_2)\right)\succeq 0,$$

Then we have $\alpha(f(x),f(y))\geq 1$ *Thus* f *is* α*-admissible. Moreover if we take* $x_0=(1,1)$, *then* $f(x_0)=(\frac{3}{5},\frac{3}{5})$ *and* $x_0-f(x_0)=(\frac{2}{5},\frac{2}{5})$. *thus* $\alpha(f(x_0,f(x_0))\geq 1$. $d(x,y)=\|x-y\|=(|x_1-y_1|^2+|x_2-y_2|^2)^{1/2}$. $\psi_n(t)=\left(\frac{3}{4}\right)^n t$. *Then*

$$\begin{aligned}\alpha(x,y)d(f^n(x),f^n(y))&=\left(\frac{3}{5}\right)^n\|x-y\|\leq\left(\frac{3}{4}\right)^n\left(\left(\frac{4}{5}\right)^n\|x-y\|\right)\\&=\psi_n((d(g(x),g(y))).\end{aligned}$$

Definition 19. *If* $(E,\leq)$ *is a Riesz space and* $f:E\to E$ *is such that* $f(x)\leq f(y)$ *whenever* $x,y\in E$ *and* $x\leq y$, *then* f *is said to be nondecreasing.*

Definition 20. *Let* $(E,\leq)$ *be a Riesz space. The set* $(UF)f=\{x\in E: x-f(x)\in K\}$ *is called upper fixed point set of* f, $(LF)f=\{x\in E: f(x)-x\in K\}$ *is called lower fixed point set of* f *and* $(F)f=\{x\in E: f(x)=x\}$ *is called the set of all fixed points of* f.

Definition 21. *Let* X *be a nonempty set and* E *a Riesz space. A mapping* $d: X\times X\to E$ *is said to be a vector metric or* E*-metric if it satisfies the following conditions:*

(E 1) $d(x,y)=0$ *if and only if* $x=y$;
(E 2) $d(x,y)\leq d(x,z)+d(y,z)$; *for all* $x,y,z\in X$.

We call (X,d,E) *a vector metric space.*

Definition 22. *(See [24]). Let* $f,g: X\to X$ *be mappings of a set* X. *If* $f(w)=g(w)=z$ *for some* $z\in X$, *then* w *is called a coincidence point of* f *and* g, *and* z *is called a point of coincidence of* f *and* g.

Definition 23. *(See [24]). Paire of self-mappings (f, g) on an ordered vector metric space (X, d, E) is said to be compatible if, for arbitrary sequence $(x_n) \subset X$, such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) \in X$, and for arbitrary $c \in E$, there exists $n_0 \in N$ such that*

$$d(fg(x_n), gf(x_n)) \leq c,$$

whenever $n > n_0$. It is said to be weakly compatible if $f(x) = g(x)$ implies $fg(x) = gf(x)$. It is clear that, as in the case of metric space, the pair (f, i_X) (i_X is the identity mapping) is both compatible and weakly compatible, for each self-map f.

Lemma 24. *(See [2]). Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $z = f(w) = g(w)$, then z is the unique common fxed point of f and g.*

For arbitrary elements x, y, z and w of a vector metric space, the following holds true:

(Em 1) $0 \leq d(x, y)$;
(Em 2) $d(x, y) = d(y, x)$;
(Em 3) $|d(x, z) - d(y, z)| \leq d(x, y)$;
(Em 4) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$.

A Riesz space E is a vector metric space with $d : E \times E \to E$ defined by $d(x, y) = |x - y|$. This vector metric is called absolute valued metric on E.

Definition 25. *(See [6, 11]).*

(i) *A sequence $\{x_n\}$ in X is vectorial converges or E-converges to some $x \in X$ (we write $x_n \to^{d,E} x$), if there is a sequence $\{a_n\}$ in E satisfying $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all n;*
(ii) *A sequence (x_n) is called E-Cauchy sequence if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ holds for all n and p;*
(iii) *A vector metric space X is called E-complete if each E-Cauchy sequence in X E-converges to a limit in X.*

Lemma 26. *(See [6, 11]). We have following properties in vector metric space X:*

(a) *The limit x is unique;*
(b) *Every subsequence of (x_n) E-converges to x;*
(c) *If $x_n \to^{d,E} x$ and $y_n \to^{d,E} y$, then $d(x_n, y_n) \to^o d(x, y)$.*

Definition 27. *(See [6, 11]) An ordered set is σ-complete if $\sup A$ and $\inf A$ exists in numerable subset A in X.*

3. Main results

A fixed point problem is to find some $x \in E$ such that $f(x) = x$ and we denote it by $FP(f, E)$. Let $f, g : E \to E$. A common fixed point problem is to find some $x \in E$ such that $x = f(x) = g(x)$ and we denote it by $CFP(f, g, E)$. The equation $f(x) = g(x)$ $(f(x) = g(x) = x)$ is called coincidence point equation (resp. common fixed point equation).

In [4, Corollary 1], the following results obtained, see also [1].

Theorem 28. *Let (X, d) be a complete metric space. Suppose mappings $f, g : X \to X$ satisfy*

(1) $$d(f(x), f(y)) \leq kd(g(x), g(y)), \text{ for all } x, y \in X,$$

where $k \in [0,1)$ is a constant. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Inspirering the above result, we give a fixed point theorem versions of α-ψ and α-ψ_n contractive mappings.

Definition 29. *Let f, g be mappings such that the range of f is contained in the range of g. We say that f is g-continous at $x_0 \in X$ if $g(x) \to g(x_0)$ imples $f(x) \to f(x_0)$.*

Proposition 30. *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = f(x) = g(x)$, then w is the unique common fixed point of f and g.*

Proof. Since $w = f(x) = g(x)$ and f and g are weakly compatible, we have $f(w) = fg(x) = gf(x) = g(w)$: i.e., $f(w) = g(w)$ is a point of coincidence of f and g. However w is the only point of coincidence of f and g, so $w = f(w) = g(w)$. Moreover if $z = f(z) = g(z)$, then z is a point of coincidence of f and g, and therefore $z = w$ by uniqueness. Thus w is a unique common fixed point of f and g. □

Theorem 31. *Let (X, d, E) be an E-complete vector metric space and we assume that E is Archimedean. Let $f, g : X \to X$ be α-ψ_n mappings satisfying the following conditions;*

(i) *f is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(iii) *The range of g contains the range of f and $g(X)$ is a E-complete subspace of X.*
(iv) *f is g-continous.*
(vi) *f and g are weakly compatible.*

Then the common fixed point problem $CFP(f, g, E)$ has a solution.

Proof. Let $x_0 \in X$ be a arbitary point in X. Choose a point x_1 in X such that $\alpha(x_0, x_1) \geq 1$ and $f(x_0) = g(x_1)$. Also for x_n there exists x_{n+1} such that $f(x_n) = g(x_{n+1})$. This can be done since the range of g contains the range of f. Then

$$\begin{aligned} d(g(x_n), g(x_{n+1})) &= d(f(x_{n-1}), f(x_n)) = d(f^n(x_0), f^n(x_1)) \\ &\leq \alpha(x_1, x_0) d(f^n(x_1), f^n(x_0)) \leq \psi_n(d(g(x_1), g(x_0))) \end{aligned}$$

In this case we have

$$d(g(x_m), g(x_n)) \leq \sum_{k=n}^{m} \psi_k(d(g(x_1), g(x_0))).$$

Then there exists n_0 such taht $\sum_{k=n_0}^{\infty} \psi_k(d(g(x_0), g(x_1))) \to^o 0$, Since E is Archmedian, for any m, n with $m > n > n_0$, $d(g(x_m), g(x_n)) \to^o 0$, $d(g(x_m), g(x_n))$ is E-Cauchy in E. Hence $(g(x_n)$ is a E-Cauchy sequence in $g(X)$. Since $g(X)$ is E-complete, there exists $q \in g(X)$ such that $g(x_n) \to^{d,E} q$ as $n \to \infty$. Then there exists $p \in X$ such that $g(p) = q$. Since f is g-continous, $g(x_n) \to^{d,E} g(p)$ imples $f(x_n) \to^{d,E} f(p)$. Since

$$d(g(x_{n+1}), f(p)) = d(f(x_n), f(p)),$$

we have $g(x_n) \to^{d,E} f(p)$. The uniquness of limit in an ordered metric space E, we have $f(p) = g(p)$. From Proposition 30 and (vi), f and g have a common fixed point.

□

Corollary 32. *Let (X, d, E) be a E-complete vector metric space with E is Archimedean. Let $f, g : X \to X$ be two α-ψ mappings satisfying the same conditions in theorem 31. Then, the common fixed point problem $CFP(f, g, E)$ has a solution.*

Proof. Put $\psi_n = \psi^n$, then ψ^n satisfies the condition of Thoeorem 33 and also the rest of proof is same. □

Now we give the another versions of theorem.

Theorem 33. *Let (X, d, E) be a E-complete vector metric space and we assume that E is Archimedean. Let $f, g : X \to X$ be mutually α-ψ_n mappings satisfying the following conditions;*

(i) *f is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(iii) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\alpha(x_{n_k}, x) \geq 1 \text{ for all } k \in N.$$

(iv) *For any $t > 0$, $\psi_1(t) < t$.*
(v) *The range of g contains the range of f and $g(X)$ is a complete subspace of X.*
(vi) *f and g are weakly compatible.*

Then, the common fixed point problem $CFP(f, g, E)$ has a solution.

Proof. Following the proof of Theorem 31, we know that the sequence $\{g(x_n)\}$ defined in Theorem 31 E-converges to some $q \in X$ and there exists $p \in X$ such that $g(p) = q$. In this case by (iii) and (iv)

$$\begin{aligned} d(g(x_{n+1}), f(p)) &= d(f(x_n), f(p)) \\ &\leq \alpha(n_k, p) d(f(x_{n_k}), f(p)) \\ &\leq \psi_1(d(g(x_{n_k}), g(p))) \end{aligned}$$

Thus $g(x_{n+1}) \to^{d,E} f(p)$ as $n \to \infty$, and $f(x_n) \to^{d,E} f(p)$ as $n \to \infty$. The uniqueness of a limit in an ordered metric space implies that $f(p) = g(p)$. From Proposition 30 and (vi), f and g have a common fixed point. □

Corollary 34. *Let (X, d, E) be a E-complete vector metric space with E is Archimedean. Let $f, g : X \to X$ be two α-ψ mappings satisfying the same conditions in theorem 33. Then, the common fixed point problem $CFP(f, g, E)$ has a solution.*

Proof. Put $\psi_n = \psi^n$, then ψ^n satisfies the condition of Thoeorem 33 and also the rest of proof is same. □

In order to take an uniquness of coincidence point, we give the following condition.

We also give the following assmuption.

(i) Condition (H): For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Then we have the following theorem.

Theorem 35. *Adding to the settings of Theorem 31 or Theorem 33 we assume ψ_1 satisfies $\psi_1(t)M < t$ for all $t > 0$ and satisfies condition (H), then the coinidence point of f and g has unique common fixed point.*

Proof. Suppose that u and v are two different points of coincidence of f and g. From (H), there exists $z \in X$ such that

$$\alpha(u, z) \geq 1 \text{ and } \alpha(v, z) \geq 1. \tag{2}$$

Since f is α-admissible, from (2), we get

$$\alpha(f(u), f(z)) \geq 1 \text{ and } \alpha(f(v), f(z)) \geq 1. \tag{3}$$

Since ψ_1 is (c)-comparison, $\psi_1(t) < t$ for all $t > 0$, we have

$$\begin{aligned} d(g(u), g(z)) &= d(f(u), f(z)) \\ &\leq \alpha(u, z) d(f(u), f(z)) \leq \psi_1(d(g(u), g(z))) < d(g(u), g(z)) \end{aligned}$$

which is a contradiction. Thus $g(u) = g(v)$ From Proposition 30 and (ii), f and g have a unique common xed point. □

Remark 36. *For the condition (iv) of theorem 33, we consider the following property.*

Definition 37. *A mapping $\psi : R_+ \to R_+$ is said to be a comparison mapping if ψ satisfies:*

(a) *ψ is monotone increasing, that is, $t_1 \leq t_2$ impies $\psi(t_1) \leq \psi(t_2)$.*
(b) *$(\psi^n(t))$ converges to 0 as $n \to \infty$ for every $t \in R_+$.*

If we replace (b) by (b′)

(b′) *$\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t \in R_+$,*

then ψ is said to be a (c)-comparison mapping.

If a mapping $\psi_1 : R_+ \to R_+$ is (c)-comparison, then $\psi_1(t) < t$ for any $t > 0$ holds.

Example 4. *(*[31, Example 2.4.]*). Let $X = [0, \infty)$ and $E = (C([0, 1], R)$. Define $d : X \times X \to E$ by $d(x, y)(t) = (|x - y|)e^t$, where $e^t \in E$. Consider $f, g : X \to X$ defined by*

$$f(x) = \begin{cases} 2x - \frac{5}{3}, & \text{if } x > 1 \\ \frac{x}{3}, & \text{if } x \in [0, 1] \\ 0, & \text{if } x < 0 \end{cases}$$

$$g(x) = \begin{cases} 2x - \frac{4}{3}, & \text{if } x > 1 \\ \frac{2x}{3}, & \text{if } x \in [0, 1] \\ 0, & \text{if } x < 0 \end{cases}$$

Define

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise }. \end{cases}$$

Then

$$\alpha(x,y)d(f^n(x),f^n(y)) = \left(\frac{1}{3}\right)^n (x-y)e^t \leq \frac{1}{2}\left(\frac{1}{3}\right)^{n-1}\frac{2}{3}(x-y)e^t$$

Then

$$\psi_n(u) = \begin{cases} \frac{1}{2} \text{ if } n=1, \\ \frac{1}{2}\left(\frac{1}{3}\right)^{n-1} \text{ if } n \geq 2. \end{cases}$$

Clearly $f(X) \subseteq g(X)$ *and* $f(X) \supseteq g(X)$ *and* f *is* g*-continuous. Next let* $x, y \in R$ *with* $\alpha(x,y) = 1$. *This imples* $x, y \in [0,1]$. *In this case* $\frac{x}{3}, \frac{y}{3} \in [0,1]$ *thus* $\alpha(f(x), f(y)) \geq 1$. *Therefore* f *is* α*-admissible.*

Moreover $x_0 = \frac{1}{2}$, *then* $f(x_0) = \frac{1}{6}$. *Thus* $\alpha(x_0, f(x_0)) \geq 1$. *Therefore the conditions of Theorem 31 are satisfied.*

For more details on (c)-comparison operators, we refer to [7] and references mentioned therein.

4. APPLICATIONS

We shall study sufficient condition for the existence of common solution of the following integral equations([1, 14]) in the framework of E-metric spaces.

We consider the implicit integral equation

$$p(t, x(t)) = \int_0^1 q(t, s, x(s))ds, t, s \in [0,1], \tag{4}$$

where $x \in L^p[0,1]$, $1 < p < \infty$. Integral equations like (4) were introduced by Feckan [14] and could occur in the study of nonlinear boundary value problems of ordinary differential equations.

For $E = R$ and $X = L^1([0,1])$, its norm is defined by $\|x\| = \int_0^1 |x(t)|dt$ for any $x \in X$ and we define $d : X \times X \to R$ by

$$d(x,y) = \sup_{t\in[0,1]} |x(t) - y(t)|.$$

for any $x, y \in X$. Then d is a E-metric on X. Suppose that the following conditions holds:

(i) For all $t \in [0,1]$, $n \in N$ and $x, y \in X$, there exists $\psi : [0,1] \to R$ such that

$$\sup_{t\in[0,1]} |p(t, x(t)) - p(t, y(t))| \leq \psi\left(\sup_{t\in[0,1]} \left|\int_0^1 q(t,s,x(s)) - q(t,s,y(s))\right| ds\right),$$

$$\int_0^1 \sup_{t\in[0,1]} |q(t,s,x(s)) - q(t,s,y(s))|\, ds \leq \psi\left(\sup_{t\in[0,1]} |x(t) - y(t)|\right)$$

and ψ is (c)-comparison. For instane, take $\psi(t) = rt$, where $0 \leq r < 1$.

(ii) $p(t, x(t)) \leq \int_0^1 q(t,s,x(s))ds \leq x(t)$ for all $t \in [0,1]$.

(iii) $p(t, \int_0^1 q(t,s,x(s))ds) \leq \int_0^1 q(t,s,x(s))ds$ for all $t \in [0,1]$.

(iv) $x(t) \leq y(t)$ implies $p(t,x(t)) \leq p(t,y(t))$ for all $t \in [0,1]$.

Then, the implicit integral equation (4) has a solution in $L^1[0,1]$.

Proof. Take $f(x(t)) = p(t, x(t))$ and $g(x(t)) = \int_0^1 q(t, s, x(s))ds$. Let r_0 be a bounded positive number and M be the closed subset of $L^1[0,1]$ defined by

$$M = B_{r_0} = \{x \in L^1[0,1] \mid \|x\| \leq r_0\}\}.$$

For each $x \in M$, since $\int_0^1 q(t,s,x(s))ds \leq p(t,x(t)) \leq x(t)$ for all $t \in [0,1]$, the range of f is contained in that of g. Also since $\int_0^1 q(t,s,x(s))ds \leq r_0$, $g(M) \subset M$ and $g(M)$ is closed set, then $g(M)$ is complete metric spaces by the L^1-norm. Thus we have the condition (iv) of Theorem 31. By the condition (i), we have

$$\begin{aligned}
&d(f(x), f(y)) \\
&= \sup_{t\in[0,1]} |p\,(t, x(t))) - p(t, y(t)))| \\
&\leq \psi \left(\sup_{t\in[0,1]} \left| \int_0^1 (q(t,s,x(s) - q(t,s,y(s))ds \right| \right) \\
&\leq \psi \left(\int_0^1 \sup_{t\in[0,1]} |q(t,s,x(s)) - q(t,s,x(s)|)ds \right) \\
&\leq \psi\,(d(g(x), g(y)))
\end{aligned}$$

Also by (i), we have

$$\begin{aligned}
&d(f^2(x), f^2(y)) \\
&\sup_{t\in[0,1]} |p\,(t, p(t,x(t))) - p\,(t, p(t,y(t)))| \\
&\leq \psi \left(\left| \int_0^1 \sup_{t\in[0,1]} q\,(t,s,p(s,x(s)))\,ds - q\,(t,s,p(s,y(s)))\,ds \right| \right) \\
&\leq \psi \left(\int_0^1 \sup_{t\in[0,1]} |q\,(t,s,p(s,x(s))) - q\,(t,s,p(s,y(s)))|\,ds \right) \\
&\leq \psi \left(\psi \left(\sup_{t\in[0,1]} |p(t,x(t)) - p(t,y(t))| \right) \right) \\
&\leq \psi^2(d(g(x), g(y)))
\end{aligned}$$

Then by indution, we have

$$d(f^n(x), f^n(y)) \leq \psi^n(d(g(x, g(y))$$

Put $\psi_n = \psi^n$, then we have

$$d(f^n(x), f^n(y)) \leq \psi_n\,(d(g(x, g(y)))$$

By the definition of ψ, it satisfies (c)-comparison, ψ_n is also so.

Next let

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x \geq y, \\ 0 \text{ if } x < y. \end{cases}$$

Let $x(t) \geq y(t)$ for all $t \in [0,1]$. In this case by (iv), for any $t \in [0,1]$, $f(x(t)) \geq f(y(t))$. Thus f is α-admissible. Moreover by (ii), there exists $x_0(t)$ such that $x_0(t) - f(x_0(t)) = x_0(t) - p(t, x_0(t)) \geq 0$, thus $\alpha(x_0(t), f(x_0(t))) \geq 1$.

By by (ii) and (iii), f, g are mutually dominated. In fact

$$f(g(x(t))) = p\left(t, \int_0^1 q(t,s,x(s))ds\right) \leq \int_0^1 q(t,s,x(s))ds = g(x(t)),$$

$$g(f(x(t))) = \int_0^1 q(t,s,p(s,x(s)))ds \leq p(t,x(t)) = f(x(t)).$$

We define a sequence $\{x_n\}$ with $x_{2n+1} = f(x_{2n})$ and $x_{2n} = g(x_{2n+1})$. Then we have

$$x_1 = f(x_0) \geq gf(x_0) = g(x_1) = x_2 = g(x_1) \geq fg(x_1) = f(x_2) = x_3 \geq \ldots,$$

repeaing this arguments, we have

$$x_1 \geq x_2 \geq x_3 \geq \ldots \geq x_n \geq \ldots.$$

Then the sequence $\{x_n\}$ is decreasing and X is complete, there exists $x \in X$ and we have $x_n \to x$ as $n \to \infty$. Then for all $n \geq 1$, we have $\alpha(x_n, x_{n+1}) \geq 1$. Consider the subsequence $x_{n_k} = x_{2n}$, then we have $\alpha(x_{n_k}, x) \geq 1$. Thus the condition (iv) of Theorem 31 is satisfied.

Finally let $f(x(t)) = g(x(t))$ for all $t \in [0,1]$, then we have

$$g(f(x(t))) = \int_0^1 q(t,s,f(x(s)))ds = p(t,f(x(t))) = f(f(x(t))) = f(g(x(t))).$$

So, f and g are weakly compartible.

By using Theorem 29, the common fixed point problem $CFP(f,g,K)$ has a solution which in turn solves the integral equation (I).

□

Remark 38. *In examples, the decision of r_0 is obtained by the following assumptions, see* [3].

We assume that the function $q(s,t,x(t))$ is given by

$$q(s,t,x(t)) = \xi(s,t)h(s,y(s))$$

where $\xi : [0,1]\times[0,1] \to R$ is strongly measurable and $\int_0^1 \xi(\cdot,s)y(s))ds \in L^1[0,1]$ whenever $y \in L^1[0,1]$ and there exists a function $\theta : [0,1] \to R$ belonging to $L^\infty[0,1]$ such that $0 \leq \xi(t,s) \leq \theta(t)$ for all $t,s \in [0,1]\times[0,1]$. The function $f : [0,1]\times R \to R$ is a Caratheodory function and there exist a constant $b > 0$ and a function $a(\cdot) \in L^1[0,1]$ such that

$$|h(t,u)| \leq a(t) + b|u|$$

for all $t \in [0,1]$ and $u \in R$. Moreover, $h(t,x(t)) \geq 0$ whenever $x \in L^1_+[0,1]$, where $L^1_+[0,1] = \{x \in L^1[0,1]\} \mid x(t) \geq 0$ for all $t \in [0,1]\}$;

In this case r_0 *is defined by* $r_0 = \frac{\|\theta\|_\infty\|a\|}{1-\|b\|\|\theta\|}$. *In fact for each* $x \in M$,

$$\int_0^1 \|q(t,s,x(s))\|ds = \int_0^1 \|\xi(s,t)h(s,x(s))\|ds$$
$$\leq \theta(t)\int_0^1 (a(s)+bx(x))ds$$
$$\leq \|\theta\|_\infty(\|a\|+br_0) = r_0.$$

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