Balogh's technique for constructing spaces

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1 Introduction

Recently, the author constructed a Katětov space that is not countably paracompact [15]. The construction owes much to a technique of Z. Balogh. This technique is quite flexible. Using the technique, Balogh constructed Q-set spaces [1],[4] and Dowker spaces [2],[3],[6] and proved Morita's conjectures [5],[7]. It should also be added that the technique is based on ideas of M. E. Rudin [18]. The aim of this paper is to explain the method for constructing spaces using the technique.

Notation and definitions. Throughout this paper an ordinal is the set of smaller ordinals. Let A be a set and κ be a cardinal. By $\mathcal{P}(A)$ we denote the collection of all subsets of A. By $[A]^{<\kappa}$ and $[A]^{\kappa}$, we denote the collection of all subsets of A cardinality $< \kappa$ and $= \kappa$, respectively. By ω, ω_1, and $\mathfrak{c}$, we denote the first infinite ordinal, the first uncountable ordinal, and the first ordinal of the same cardinality as $\mathcal{P}(\omega)$, respectively. By $\mathrm{cf}(\kappa)$ we mean the first ordinal cofinal with κ.

A space X is called *Dowker* if X is normal but not countably paracompact. A regular space X is called *Q-set space* if for each $Y \subset X$, Y is a G_δ set and X is not σ-discrete. A space X is called *σ-space* if X has a σ-locally finite network. A regular space is a σ-space if and only if it has a σ-discrete network. A space X is called *weakly submetacompact* (or weakly θ-refinable) if every open cover of X has a refinement $\bigcup_{n\in\omega} \mathcal{U}_n$ such that for each $x \in X$, there is $n \in \omega$ such that $1 \leq |\{U \in \mathcal{U}_n | x \in U\}| < \omega$.

2 Bing's G

In this section, we show that Balogh's technique can be thought of as an application of Bing's G [8]. First, we present Bing's G, which is an example of a normal, not collectionwise normal space. Example 2.1 is different from his original form. We referred to [19].

Example 2.1. The set of points of X is $\omega_1 \cup G$, where $G = \{g | g : \mathcal{P}(\omega_1) \to 2\}$. Let, for each $\alpha \in \omega_1$, $h \in [\mathcal{P}(\omega_1)]^{<\omega}$, and $J \in [G]^{<\omega}$, $B(\alpha, h, J) = \{\alpha\} \cup \{g \in G | \forall Y \in$

$h(\alpha \in Y \leftrightarrow g(Y) = 1)\} \setminus J$. We consider a topology on X generated by the family $\{B(\alpha, h, J) | \alpha \in \omega_1, h \in [\mathcal{P}(\omega_1)]^{<\omega}, J \in [G]^{<\omega}\} \cup \{\{g\} | g \in G\}$.

Proposition 2.2. *X is normal.*

Proof. Let H and K be disjoint closed sets. Since every point of G is isolated, we may assume that $H \cup K \subset \omega_1$. $\bigcup_{\alpha \in H} B(\alpha, \{H\}, \emptyset)$ and $\bigcup_{\beta \in K} B(\beta, \{H\}, \emptyset)$ separate H and K. □

Since ω_1 is closed discrete in X, it follows from the following proposition that X is not collectionwise Hausdorff.

Proposition 2.3. *For each $h : \omega_1 \to [\mathcal{P}(\omega_1)]^{<\omega}$ and $J : \omega_1 \to [G]^{<\omega}$, there exist $\alpha < \beta$ such that $B(\alpha, h(\alpha), J(\alpha)) \cap B(\beta, h(\beta), J(\beta)) \neq \emptyset$.*

Proof. Let M be a countable elementary submodel containing everything relevant and take any $\beta \in \omega_1 \setminus M$. There is $\alpha \in \omega_1 \cap M$ such that $h(\beta) \cap M \subset h(\alpha)$ and for each $Y \in h(\beta) \cap M$, $\alpha \in Y$ if and only if $\beta \in Y$. Pick $g \in G \setminus (J(\alpha) \cup J(\beta))$ such that for each $Y \in h(\beta)$, $\beta \in Y$ if and only if $g(Y) = 1$ and for each $Y \in h(\alpha) \setminus h(\beta) \cap M$, $\alpha \in Y$ if and only if $g(Y) = 1$. Then, $g \in B(\alpha, h(\alpha), J(\alpha)) \cap B(\beta, h(\beta), J(\beta))$. □

Example 2.1 has the property that if α behaves the same way as β, then the neighborhoods of α and β cannot be separated. Advancing this idea, let us construct a space having the property that if α behaves the same way as β, then β is an element of the neighborhood of α. To construct such a space, we need to identify the parts of ω_1 and G. In order to do that, we equalize the sizes of them.

Example 2.4. The set of points of X is $\mathfrak{c} \cup G$, where $G = \{\langle A, B, g\rangle | A \in [\mathfrak{c}]^\omega, B \in [\mathcal{P}(A)]^\omega, g : B \to 2\}$. Let, for each $\alpha \in \mathfrak{c}$ and $h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$, $B(\alpha, h) = \{\alpha\} \cup \{\langle A, B, g\rangle \in G | \forall Y \in h[Y \cap A \in B \wedge (\alpha \in Y \leftrightarrow g(Y \cap A) = 1)]\}$. We consider a topology on X generated by the family $\{B(\alpha, h) | \alpha \in \mathfrak{c}, h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}\} \cup \{\{\langle A, B, g\rangle\} | \langle A, B, g\rangle \in G\}$. (Since this space satisfies (T_1), J does not appear in the definition).

Proposition 2.5. *For each $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$, there exist $\alpha \neq \beta$ such that $B(\alpha, h(\alpha)) \cap B(\beta, h(\beta)) \neq \emptyset$.*

Proof. Let M be a countable elementary submodel containing everything relevant. Let $A = \mathfrak{c} \cap M$ and $B = \{Y \cap M | Y \in \mathcal{P}(\mathfrak{c}) \cap M\}$. Take any $\gamma \in \mathfrak{c} \setminus M$. There are $\alpha \neq \beta \in \mathfrak{c} \cap M$ such that $h(\alpha) \cap h(\beta) = h(\gamma) \cap M$ and for each $Y \in h(\gamma) \cap M$, $\gamma \in Y$ if and only if $\alpha \in Y$ if and only if $\beta \in Y$. Since $h(\alpha) \cup h(\beta) \subset M$, we have that $Y \neq Y' \in h(\alpha) \cup h(\beta)$ implies $Y \cap A \neq Y' \cap A$. Pick $g \in G$ such that for each $Y \in h(\gamma) \cap M$, $\gamma \in Y$ if and only if $g(Y \cap A) = 1$, for each $Y \in h(\beta) \setminus h(\gamma) \cap M$, $\beta \in Y$ if and only if

$g(Y \cap A) = 1$, and for each $Y \in h(\alpha) \setminus h(\beta) \cap M$, $\alpha \in Y$ if and only if $g(Y \cap A) = 1$. Then, $\langle A, B, g\rangle \in B(\alpha, h(\alpha)) \cap B(\beta, h(\beta))$. $\square$

Now, we give a basic space for constructing various spaces.

Example 2.6. The set of points of X is $\mathfrak{c}$. Let $G = \{\langle A, B, C, g\rangle | A \in [\mathfrak{c}]^\omega,\ B, C \in [\mathcal{P}(A)]^\omega, B \cap C = \emptyset,\ g : C \to 2\}$ and fix a bijection $q : \mathfrak{c} \to G$. Let, for each $\alpha \in \mathfrak{c}$ and $h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$,

$$\begin{aligned} B(\alpha, h) = \{\beta \in \mathfrak{c} | \text{ if } q(\beta) = \langle A, B, C, g\rangle, \text{then } \forall Y \in h[&Y \cap A \in B \cup C \wedge \\ &Y \cap A \in B \to (\alpha \in Y \leftrightarrow \beta \in Y) \wedge \\ &Y \cap A \in C \to (\alpha \in Y \leftrightarrow g(Y \cap A) = 1)]\}. \end{aligned}$$

Topologize X by declaring a set U to be open if and only if for every $\alpha \in U$, there is $h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$ such that $B(\alpha, h) \subset U$.

Proposition 2.7. *X is normal.*

Proof. Let H and K be disjoint closed sets. By induction on n, we define H_n and K_n. Let $H_0 = H$ and $K_0 = K$. Pick, for each $\alpha \in X$, $h_\alpha \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$ such that $\alpha \notin H$ implies $B(\alpha, h_\alpha) \cap H = \emptyset$ and $\alpha \notin K$ implies $B(\alpha, h_\alpha) \cap K = \emptyset$. Let, for each $n \in \omega$, $H_{n+1} = \bigcup_{\alpha \in H_n} B(\alpha, h_\alpha \cup \{H_0, \cdots, H_n, K_0, \cdots, K_n\})$, $K_{n+1} = \bigcup_{\alpha \in K_n} B(\alpha, h_\alpha \cup \{H_0, \cdots, H_n, K_0, \cdots, K_n\})$. Then $\bigcup_{n\in\omega} H_n$ and $\bigcup_{n\in\omega} K_n$ separate H and K. $\square$

Proposition 2.8. *For each $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ and $n : \mathfrak{c} \to \omega$, there exist $\alpha \neq \beta$ such that $\beta \in B(\alpha, h(\alpha))$ and $n(\alpha) = n(\beta)$.*

Let $M \in N$ be countable elementary submodels containing everything relevant. Let $A = \mathfrak{c} \cap N$, $B = \{Y \cap N | Y \in \mathcal{P}(\mathfrak{c}) \cap M\}$, and $C = \{Y \cap N | Y \in \mathcal{P}(\mathfrak{c}) \cap N \setminus M\}$.

In the proof of Proposition 2.3, we can pick an element of G after taking β and α. However, since we identify $\mathfrak{c}$ with G in this space, we need to pick an element of G first. By the following lemma, roughly speaking, we have that there is a $g \in G$ such that for each $\beta \in X \setminus N$ and set in N whose elements behave the same way as β, there is α in the set such that g is an element of the neighborhood of α.

Lemma 2.9. *There exists a function g satisfying that $\langle A, B, C, g\rangle \in G$ such that whenever $v : \mathfrak{c} \to \bigcup_{a \in [\mathcal{P}(\mathfrak{c}) \setminus M]^{<\omega}} {}^a 2$ is an infinite partial function, $v \in N$, and $\{dom(v(\alpha)) | \alpha \in dom(v)\}$ are pairwise disjoint, then there exists $\alpha \in dom(v)$ such that $\{\langle Y \cap A, i\rangle | \langle Y, i\rangle \in v(\alpha)\} \subset g$.*

Proof. Let $\langle v_j\rangle_{j\in\omega}$ enumerate all functions $v \in N$ as in Lemma 2.9. By induction on j pick distinct $\{\alpha_j | j \in \omega\} \subset X \cap N$ such that $j \neq j'$ implies $\mathrm{dom}(v(\alpha_j)) \cap \mathrm{dom}(v(\alpha_{j'})) = \emptyset$.

It follows from elementarity of N that $\{\{Y \cap A | Y \in \mathrm{dom}(v_j(\alpha_j))\} | j \in \omega\}$ are pairwise disjoint and $\{Y \cap A | Y \in \mathrm{dom}(v_j(\alpha_j))\} \cap B = \emptyset$ for all $j \in \omega$. Take $g \in G$ satisfying that $\{\langle Y \cap A, i\rangle : \langle Y, i\rangle \in v_j(\alpha_j)\} \subset g$ for all $j \in \omega$. Then g is as desired. $\square$

Take $\beta \in X$ satisfying that $q(\beta) = \langle A, B, C, g\rangle$. Note that $\beta \notin N$. We obtain the following lemma by taking a set whose elements behave the same way as β.

Lemma 2.10. *There exists $E \in [X]^{\omega_1} \cap N$ such that,*
(a) *$\{h(\alpha) | \alpha \in E\}$ is a Δ-system with root $h(\beta) \cap M$,*
(b) *for each $\alpha \in E$ and $Y \in h(\beta) \cap M$, $\alpha \in Y$ if and only if $\beta \in Y$,*
(c) *for each $\alpha \in E$, $(h(\alpha) \setminus (h(\beta) \cap M)) \cap M = \emptyset$,*
(d) *for each $\alpha \in E$, $n(\alpha) = n(\beta)$.*

Proof. Let $r = h(\beta) \cap M$ and $n = n(\beta)$. Define, for each $Y \in r$, $i_Y \in 2$ by setting $i_Y = 1$ if and only if $\beta \in Y$. Let $\phi(G)$ be the statement "$\{h(\alpha)|\alpha \in G\}$ is a Δ-system with root r, for each $\alpha \in G$, $n(\alpha) = n$, and for each $\alpha \in G$ and $Y \in r$, $i_Y = 1$ if and only if $\alpha \in Y$". By Zorn's Lemma, there is a maximal F such that $\phi(F)$ holds. Since all parameters of $\phi(G)$ are in M, we may assume that $F \in M$. Suppose indirectly that F is countable; then $F \subset M$. Since $F \cup \{\beta\} \neq F$ and $\phi(F \cup \{\beta\})$ holds, it is a contradiction. Hence F is uncountable; there is $E \in [F]^{\omega_1}$ such that, for each $\alpha \in E$, $(h(\alpha) \setminus (h(\beta) \cap M)) \cap M = \emptyset$. Since h, F, and M are in N, we may assume $E \in N$. $\square$

Now, we complete our proof of Proposition 2.8. Define $v : E \to \bigcup_{a \in [\mathcal{P}(\mathfrak{c}) \setminus M]^{<\omega}} {}^a 2$ by setting $v(\alpha) = \{\langle Y, i_{\alpha,Y}\rangle | Y \in h(\alpha) \setminus (h(\beta) \cap M)\}$, where $i_{\alpha,Y} = 1$ if and only if $\alpha \in Y$. Note that $v \in N$. By Lemma 2.9, we have that there is $\alpha \in E$ such that for each $Y \in h(\alpha) \setminus (h(\beta) \cap M)$, $g(Y \cap A) = 1$ if and only if $\alpha \in Y$. Hence $\beta \in B(\alpha, h(\alpha))$.

3 Applications

Applying the basic space, we give a Dowker space, a Q-set space, and a metalindelöf, not weakly submetacompact space.

Example 3.1. The set of points of X is $\mathfrak{c} \times \omega$. Let $G = \{\langle A, B, C, g\rangle | A \in [\mathfrak{c}]^\omega,\ B, C \in [\mathcal{P}(A)]^\omega, B \cap C = \emptyset,\ g : C \to 2\}$ and fix a bijection $q : \mathfrak{c} \to G$. Let, for each $\alpha \in \mathfrak{c}$, $n \in \omega$, and $h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$,

$$\begin{aligned} B(\alpha, n+1, h) = \{\langle \beta, n\rangle \in \mathfrak{c} \times \omega |\ &\text{if } q(\beta) = \langle A, B, C, g\rangle, \text{then } \forall Y \in h[Y \cap A \in B \cup C \wedge \\ &Y \cap A \in B \to (\alpha \in Y \leftrightarrow \beta \in Y) \wedge \\ &Y \cap A \in C \to (\alpha \in Y \leftrightarrow g(Y \cap A) = 1)]\}. \end{aligned}$$

Topologize X by declaring a set U to be open if and only if for every $\langle \alpha, n+1\rangle \in U$, there is $h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$ such that $B(\alpha, n+1, h) \subset U$.

Proposition 3.2. *X is normal.*

Proof. Let H and K be disjoint closed sets. It is not difficult to prove that for each $n \in \omega$, $H \cap (\mathfrak{c} \times \{n\})$ and $H^c \cap (\mathfrak{c} \times \{n\})$ can be separated and that for each $m < n \in \omega$, $H \cap (\mathfrak{c} \times \{n\})$ and $K \cap (\mathfrak{c} \times \{m\})$ can be separated. Hence for each $n \in \omega$, $H \cap (\mathfrak{c} \times \{n\})$ and K (and H and $K \cap (\mathfrak{c} \times \{n\})$) can be separated. By the shoestring argument, we can separate H and K. $\square$

Proposition 3.3. *X is not countably paracompact.*

Proof. Suppose indirectly that X is countably paracompact. Then, $\{\mathfrak{c} \times (n+1) | n \in \omega\}$ has a one-to-one locally finite open refinement $\{U_n | n \in \omega\}$. We can take $h : \mathfrak{c} \times (\omega \setminus \{0\}) \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ satisfying that $\langle \alpha, i\rangle \in U_n$ implies $B(\alpha, i, h(\alpha, i)) \subset U_n$. Let $M \in N$ be countable elementary submodels containing everything relevant. Take $\beta \in \mathfrak{c}$ similarly in the basic space. To get a contradiction, we show that for each $n \in \omega$, there is $m > n$ such that $\langle \beta, 0\rangle \in U_m$. Fix $n \in \omega$. Since $U_n \subset \mathfrak{c} \times (n+1)$, there is $m > n$ such that $\langle \beta, n+1\rangle \in U_m$. By induction on k from $n+1$ to 0, we show that $\langle \beta, k\rangle \in U_m$. Suppose that $\langle \beta, k+1\rangle \in U_m$. By taking a set whose elements behave the same way as β including that $\langle \beta, k+1\rangle \in U_m$, we can show that there is $\alpha \neq \beta \in \mathfrak{c}$ such that $\langle \beta, k\rangle \in B(\alpha, k+1, h(\alpha, k+1))$ and $\langle \alpha, k+1\rangle \in U_m$. The proof of this is similar to the proof of Proposition 2.8. By our way of taking h, we have that $\langle \beta, k\rangle \in U_m$. $\square$

Example 3.4. The set of points of X is $\mathfrak{c}$. Let $G = \{\langle A, B, C, g\rangle | A \in [\mathfrak{c}]^\omega,\ B, C \in [\mathcal{P}(A)]^\omega, B \cap C = \emptyset,\ g : C \to 2 \times \omega\}$ and take a $q : \mathfrak{c} \to G$ satisfying that for each $\langle A, B, C, g\rangle \in G$, there is $\beta \in \mathfrak{c}$ such that $q(\beta) = \langle A, B, C, g\rangle$ and $\sup(A) < \beta$. Let, for each $Y \in \mathcal{P}(\mathfrak{c})$ and $k \in \omega$,

$$Y_k = \{\beta \in \mathfrak{c} |\text{ if } q(\beta) = \langle A, B, C, g\rangle, \text{then } Y \cap A \in C \wedge \exists i \in 2 \exists m \geq k (g(Y \cap A) = \langle i, m\rangle)\}.$$

For each $\alpha \in \mathfrak{c}, h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$, and $n \in \omega$,

$$\begin{aligned}
&B(\alpha, h, n) = \{\beta \in \mathfrak{c} |\text{ if } q(\beta) = \langle A, B, C, g\rangle, \text{then } \forall Y \in h[Y \cap A \in B \cup C \wedge \\
&Y \cap A \in B \to (\alpha \in Y \leftrightarrow \beta \in Y \wedge \forall k \in \omega(\alpha \in Y_k \leftrightarrow \beta \in Y_k)) \wedge \\
&Y \cap A \in C \to \exists m \geq n(\alpha \in Y \to g(Y \cap A) = \langle 1, m\rangle \wedge \alpha \notin Y \to g(Y \cap A) = \langle 0, m\rangle)]\}.
\end{aligned}$$

Topologize X by declaring a set U to be open if and only if for every $\alpha \in U$, there are $h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$ and $n \in \omega$ such that $B(\alpha, h, n) \subset U$.

In the same way as the proof of Proposition 2.7, we have that X is normal.

Proposition 3.5. *For all $Y \subset X$, Y is a G_δ set.*

Proof. Let, for each $n, k \in \omega$, $Y_n^0 = Y$ and $Y_n^{k+1} = \bigcup_{\alpha \in Y_n^k} B(\alpha, \{Y\}, n)$. Note that for each $n \in \omega$, $\bigcup_{k\in\omega} Y_n^k$ is open. Since $\bigcap_{n\in\omega}(Y \cup Y_n) = Y$, Proposition 3.5 follows from the following claim.

Claim 3.6. *For each $n, k \in \omega$, $Y_n^k \subset Y \cup Y_n$.*

Proof of Claim 3.6. Fix $n \in \omega$. We prove by induction on $k \in \omega$. Take any $\beta \in Y_n^{k+1}$. By induction hypothesis, there is $\alpha \in Y \cup Y_n$ such that $\beta \in B(\alpha, \{Y\}, n)$. If $q(\beta) = \langle A, B, C, g\rangle$, then $Y \cap A \in B \cup C$. And if $Y \cap A \in B$, then $\alpha \in Y$ if and only if $\beta \in Y$ and $\alpha \in Y_n$ if and only if $\beta \in Y_n$. Since $\alpha \in Y \cup Y_n$, we have $\beta \in Y \cup Y_n$. If $Y \cap A \in C$, then there is $m \geq n$ such that $\alpha \in Y$ implies $g(Y \cap A) = \langle 1, m\rangle$ and $\alpha \notin Y$ implies $g(Y \cap A) = \langle 0, m\rangle$. Hence $\beta \in Y_n$. □

Proposition 3.7. *X is not σ-discrete.*

Proof. Take any $f : X \to \omega$, $h : X \to [\mathcal{P}(X)]^{<\omega}$, and $n : X \to \omega$. We can show that there are $\alpha < \beta$ such that $f(\alpha) = f(\beta)$ and $\beta \in B(\alpha, h(\alpha), n(\alpha))$. The proof of this is similar to the proof of Proposition 2.8. Hence X cannot be σ-discrete. □

As mentioned in [1], X can be made left-separated easily. Then, by Proposition 3.9, X answers the question B16 in [16].

Question 3.8. (H. Junnila [13]) Does there exist, in ZFC, a set X and two topologies τ and π on X such that $\tau \subset \pi$, every π-open set is an F_σ set with respect to τ, the space (X, π) is metrizable but the space (X, τ) is not a σ-space?

Proposition 3.9. *X is not a σ-space.*

Proof. Suppose indirectly that X has a σ-discrete network $\bigcup_{i\in\omega} \mathcal{B}_i$. Since for each $\alpha \in X$, $[\alpha, \mathfrak{c})$ is open, there are $i_\alpha \in \omega$ and $B_\alpha \in \mathcal{B}_{i_\alpha}$ such that $\alpha \in B_\alpha \subset [\alpha, \mathfrak{c})$. We can take $h : X \to [\mathcal{P}(X)]^{<\omega}$ and $n : X \to \omega$ satisfying that for each $\alpha \in X$, $B(\alpha, h(\alpha), n(\alpha)) \cap (\bigcup \mathcal{B}_{i_\alpha} \setminus B_\alpha) = \emptyset$. We can show that there are $\alpha < \beta$ such that $\beta \in B(\alpha, h(\alpha), n(\alpha))$ and $i_\alpha = i_\beta$. Since $\alpha \notin [\beta, \mathfrak{c})$, we have $B_\alpha \neq B_\beta$. By our way of taking h and n, $B(\alpha, h(\alpha), n(\alpha)) \cap B_\beta = \emptyset$. This is a contradiction. □

Example 3.10. The set of points of X is $\mathfrak{c}$. Let $\{L_\epsilon | \epsilon \in \omega_1\}$ be a partition of $\mathfrak{c}$ such that for every $\epsilon \in \omega_1$, L_ϵ is of size $\mathfrak{c}$. Let $G = \{\langle A, B, C, g\rangle | A \in [\mathfrak{c}]^\omega,\ B, C \in [\mathcal{P}(A)]^\omega, B \cap C = \emptyset,\ g : C \to 2\}$ and take a $q : \mathfrak{c} \to G$ satisfying that for each $\langle A, B, C, g\rangle \in G$ and $\epsilon \in \omega_1$, there is $\beta \in L_\epsilon$ such that $q(\beta) = \langle A, B, C, g\rangle$ and $\sup(A) < \beta$. Let, for each $\alpha \in L_\epsilon$ and

$h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$,

$$B(\alpha,h)=\{\beta\in\bigcup_{\mu\le\epsilon}L_\mu|\text{ if }q(\beta)=\langle A,B,C,g\rangle,\text{then}\,\alpha\in A\cap\beta\wedge\forall Y\in h[Y\cap A\in B\cup C\wedge$$
$$Y\cap A\in B\to(\alpha\in Y\leftrightarrow\beta\in Y)\wedge$$
$$Y\cap A\in C\to(\alpha\in Y\leftrightarrow g(Y\cap A)=1)]\}.$$

Topologize X by declaring a set U to be open if and only if for every $\alpha \in U$, there is $h \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$ such that $B(\alpha, h) \subset U$.

Proposition 3.11. *X is metalindelöf.*

Proof. Take any open cover $\mathcal{U}$ of X. Pick, for each $\alpha \in \mathfrak{c}$, $U_\alpha \in \mathcal{U}$ such that $\alpha \in U_\alpha$. Let us take a point-countable open refinement $\{V_\alpha|\alpha \in \mathfrak{c}\}$ of $\{U_\alpha|\alpha \in \mathfrak{c}\}$. Fix $\alpha \in \mathfrak{c}$. Take, for each $\beta \in U_\alpha$, $h_\beta \in [\mathcal{P}(\mathfrak{c})]^{<\omega}$ such that $B(\beta, h_\beta) \subset U_\alpha$. Let $V^0 = \{\alpha\}$, for each $n \in \omega$, $V^{n+1} = \bigcup_{\beta\in V^n} B(\beta, h_\beta)$, and $V_\alpha = \bigcup_{n\in\omega} V^n$. To show that $\{V_\alpha|\alpha \in \mathfrak{c}\}$ is point-countable, we define a set $D_\beta \subset \mathfrak{c}$, inductively, by setting $D_\beta = \{\beta\} \cup \bigcup_{\alpha\in A_\beta\cap\beta} D_\alpha$. Here, A_β is the first term of $q(\beta)$. Note that for each $\beta \in \mathfrak{c}$, D_β is countable. By the construction of V_α, we have that $\beta \in V_\alpha$ implies $\alpha \in D_\beta$. Hence $\{V_\alpha|\alpha \in \mathfrak{c}\}$ is point-countable. □

Proposition 3.12. *X is not weakly submetacompact.*

Proof. Suppose indirectly that X is weakly submetacompact. Then, $\{\bigcup_{\mu\le\epsilon} L_\mu|\epsilon \in \omega_1\}$ has an open refinement $\bigcup_{n\in\omega} \mathcal{U}_n$ satisfying that for each $\alpha \in \mathfrak{c}$, there is $n_\alpha \in \omega$ such that $1 \le |\{U \in \mathcal{U}_{n_\alpha}|\alpha \in U\}| < \omega$. We can take $h : \mathfrak{c} \to [\mathcal{P}(\mathfrak{c})]^{<\omega}$ satisfying that for each $\alpha \in \mathfrak{c}$, $B(\alpha, h(\alpha)) \subset \bigcap\{U \in \mathcal{U}_{n_\alpha}|\alpha \in U\}$. Let $M \in N$ be countable elementary submodels containing everything relevant. Take $\beta \in L_{\omega_1\cap M}$ with $\sup(\mathfrak{c} \cap N) < \beta$ similarly in the basic space. We obtain the following lemma by taking a set whose elements behave the same way as β. The proof of Lemma 3.13 is similar to the proof of Lemma 2.10.

Lemma 3.13. *There exists $E \in [\mathfrak{c}]^{\omega_1} \cap N$ such that*
(a) *$\{h(\alpha)|\alpha \in E\}$ is a Δ-system with root $h(\beta) \cap M$,*
(b) *for each $\alpha \in E$ and $Y \in h(\beta) \cap M$, $\alpha \in Y$ if and only if $\beta \in Y$,*
(c) *for each $\alpha \in E$, $(h(\alpha) \setminus (h(\beta) \cap M)) \cap M = \emptyset$,*
(d) *$\{\epsilon \in \omega_1|E \cap L_\epsilon \ne \emptyset\}$ is unbounded in ω_1,*
(e) *for each $\alpha \in E$, $n_\alpha = n_\beta$.*

Inductively, let us take a set $\{\alpha_i|i \in \omega\} \subset E \cap N$ satisfying that for each $i \in \omega$, $\beta \in B(\alpha_i, h(\alpha_i))$ and for each $j > i$ and $U \in \mathcal{U}_{n_\beta}$, $\alpha_i \in U$ implies $\alpha_j \notin U$. Suppose we have taken $\{\alpha_k|k < i\} \subset E \cap N$. Since $\bigcup_{n\in\omega} \mathcal{U}_n \prec \{\bigcup_{\mu\le\epsilon} L_\mu|\epsilon \in \omega_1\}$, we have that there is $\delta \in \omega_1 \cap N$ such that for each $k < i$ and $U \in \mathcal{U}_{n_\beta}$, $\alpha_k \in U$ implies $U \subset \bigcup_{\mu\le\delta} L_\mu$. Define

$v : E \cap \bigcup_{\mu>\delta} L_\mu \to \bigcup_{a\in[\mathcal{P}(\mathfrak{c})\setminus M]^{<\omega}} {}^a2$ by setting $v(\alpha) = \{\langle Y, i_{\alpha,Y}\rangle | Y \in h(\alpha) \setminus (h(\beta) \cap M)\}$, where $i_{\alpha,Y} = 1$ if and only if $\alpha \in Y$. Note that $v \in N$. By Lemma 2.9, we have that there is $\alpha_i \in E \cap \bigcup_{\mu>\delta} L_\mu \cap N$ such that $\beta \in B(\alpha_i, h(\alpha_i))$. Since $\alpha_i \in \bigcup_{\mu>\delta} L_\mu$, we have that for each $k < i$ and $U \in \mathcal{U}_{n_\beta}$, $\alpha_k \in U$ implies $\alpha_i \notin U$. Hence α_i is as desired. By our way of taking h, we have that for each $i \in \omega$ and $U \in \mathcal{U}_{n_\beta}$, $\alpha_i \in U$ implies $\beta \in U$. Hence $\{U \in \mathcal{U}_{n_\beta} | \beta \in U\}$ is infinite. This is a contradiction. $\square$

Example 3.10 can be made collectionwise normal (see §4) and perfectly normal (see Example 3.4). To our knowledge, the first discovery of a metalindelöf, not weakly submetacompact space in ZFC is due to G. Gruenhage in [11].

4 Complete neighborhoods

Applying the basic space, let us construct a collectionwise normal space. First, we consider the following example.

Example 4.1. The set of points of X is $\mathfrak{c}$. Let

$$I = \{\langle F_\rho\rangle_{\rho\in\mathfrak{c}} \in {}^{\mathfrak{c}}\mathcal{P}(\mathfrak{c}) | \{F_\rho\}_{\rho\in\mathfrak{c}} : \text{pairwise disjoint} \wedge (\rho \neq \sigma \wedge F_\rho, F_\sigma \neq \emptyset) \to F_\rho \neq F_\sigma\}.$$

Let, for each $\mathcal{F} = \langle F_\rho\rangle_{\rho\in\mathfrak{c}} \in I$ and $A \in [\mathfrak{c}]^\omega$, $\mathcal{F} \restriction A = \langle F_\rho \cap A\rangle_{\rho\in A}$ and $I \restriction A = \{\mathcal{F} \restriction A | \mathcal{F} \in I\}$. Let $G = \{\langle A, B, C, g\rangle | A \in [\mathfrak{c}]^\omega,\ B, C \in [I \restriction A]^\omega, B \cap C = \emptyset,\ g : C \to \mathfrak{c} \cup \{-1\}\}$ and fix a bijection $q : \mathfrak{c} \to G$. Let, for each $\alpha \in \mathfrak{c}$ and $h \in [I]^{<\omega}$,

$$\begin{aligned} B(\alpha, h) = \{\beta \in \mathfrak{c} | \text{ if } q(\beta) = \langle A, B, C, g\rangle, \text{then } \forall \mathcal{F} = \langle F_\rho\rangle_{\rho\in\mathfrak{c}} \in h[\mathcal{F} \restriction A \in B \cup C \wedge \\ \mathcal{F} \restriction A \in B \to (\alpha \notin \bigcup \mathcal{F} \leftrightarrow \beta \notin \bigcup \mathcal{F} \wedge \alpha \in F_\rho \leftrightarrow \beta \in F_\rho) \wedge \\ \mathcal{F} \restriction A \in C \to (\alpha \notin \bigcup \mathcal{F} \leftrightarrow g(\mathcal{F} \restriction A) = -1 \wedge \alpha \in F_\rho \leftrightarrow g(\mathcal{F} \restriction A) = \rho)]\}. \end{aligned}$$

Topologize X by declaring a set U to be open if and only if for every $\alpha \in U$, there is $h \in [I]^{<\omega}$ such that $B(\alpha, h) \subset U$.

Example 4.1 is collectionwise normal. However, the proof of Proposition 2.8 for this space does not work. The reason why is that if $\mathcal{F} = \langle F_\rho\rangle_{\rho\in\mathfrak{c}} \in M, \beta \in F_\rho$, and $\rho \notin M$, then α and β are not elements of the same F_ρ. By using Balogh's great idea, complete neighborhoods, we can have that if $\mathcal{F} \in M$ and $\beta \in F_\rho$, then $\rho \in M$, so α and β are elements of the same F_ρ.

Example 4.2. The set of points of X is $\mathfrak{c}$. Let $\langle \mathcal{F}_\xi\rangle_{\xi\in 2^{\mathfrak{c}}}$ list all terms of I mentioning each $2^{\mathfrak{c}}$ times. Let, for each $\xi \in 2^{\mathfrak{c}}$, $\mathcal{F}_\xi = \langle F^\xi_\rho\rangle_{\rho\in\mathfrak{c}}$. Inductively, we define $H \subset 2^{\mathfrak{c}}$ and, for every $\xi \in 2^{\mathfrak{c}}$, a topology τ_ξ on X. Suppose $\xi \in 2^{\mathfrak{c}}$, and for every $\eta < \xi$, we have decided whether $\eta \in H$. The topology τ_ξ is defined by declaring a set U to be open if and only if for every $\alpha \in U$, there are $h \in [H \cap \xi]^{<\omega}$ and $J \in [X]^{<\omega}$ such that $B(\alpha, h) \setminus J \subset U$. This $B(\alpha, h)$ is defined in the same way as Example 4.1. Suppose $\{F^\xi_\rho | \rho \in \mathfrak{c}\}$ is a discrete family of closed sets in τ_ξ, and there is no $\eta \in H \cap \xi$ such that $\{F^\eta_\rho | \rho \in \mathfrak{c}\} = \{F^\xi_\rho | \rho \in \mathfrak{c}\}$. Then, let $\xi \in H$. The topology τ is defined by declaring a set U to be open if and only if for every $\alpha \in U$, there are $h \in [H]^{<\omega}$ and $J \in [X]^{<\omega}$ such that $B(\alpha, h) \setminus J \subset U$.

Proposition 4.3. *τ is collectionwise normal.*

Proof. Let $\mathcal{F}$ be a discrete family of closed sets in τ. By the construction of τ, it follows from $|X| = \mathfrak{c} < \mathrm{cf}(2^\mathfrak{c})$ that $\tau = \bigcup_{\xi \in 2^\mathfrak{c}} \tau_\xi$. Hence there is $\xi \in 2^\mathfrak{c}$ such that $\mathcal{F}$ is a discrete family of closed sets in τ_ξ. Since each term of I is listed in $\langle \mathcal{F}_\xi \rangle_{\xi \in 2^\mathfrak{c}}$ $2^\mathfrak{c}$ times, there is $\xi \in 2^\mathfrak{c}$ such that $\{F^\xi_\rho | \rho \in \mathfrak{c}\} = \mathcal{F}$ and $\{F^\xi_\rho | \rho \in \mathfrak{c}\}$ is a discrete family of closed sets in τ_ξ. Let us take the smallest such ξ, then $\xi \in H$. For each $\alpha \in \mathfrak{c}$, let us take $h_\alpha \in H$ and $J_\alpha \in [X]^{<\omega}$ satisfying that for each $\rho \in \mathfrak{c}$, $\alpha \notin F^\xi_\rho$ implies $(B(\alpha, h_\alpha) \setminus J_\alpha) \cap F^\xi_\rho = \emptyset$. Let $\mathcal{F}_1 = \{F^\xi_\rho \cup \bigcup_{\alpha \in F^\xi_\rho} (B(\alpha, h_\alpha \cup \{\xi\}) \setminus J_\alpha) | \rho \in \mathfrak{c}\}$. Then $\mathcal{F}_1$ is a discrete family of closed sets. (Let, for each $\beta \in X$, $U^\beta_0 = \{\beta\}$, $U^\beta_{i+1} = \bigcup_{\gamma \in U^\beta_i} (B(\gamma, h_\gamma \cup \{\xi\}) \setminus J_\gamma) (i \in \omega)$, and $U^\beta = \bigcup_{i \in \omega} U^\beta_i$. Then, $\{U^\beta | \beta \in X\}$ witnesses the fact that $\mathcal{F}_1$ is a discrete family of closed sets). Hence there is $\xi_1 \in H$ such that $\{F^{\xi_1}_\rho | \rho \in \mathfrak{c}\} = \mathcal{F}_1$. For each $i > 1$, we can define $\mathcal{F}_i$ and take ξ_i, similarly. Let, for each $\rho \in \mathfrak{c}$, $U_{\rho,0} = F^\xi_\rho$, $U_{\rho,i+1} = U_{\rho,i} \cup \bigcup_{\alpha \in U_{\rho,i}} (B(\alpha, h_\alpha \cup \{\xi, \xi_1, \cdots \xi_i\}) \setminus J_\alpha) (i \in \omega)$, and $U_\rho = \bigcup_{i \in \omega} U_{\rho,i}$. Then $\{U_\rho | \rho \in \mathfrak{c}\}$ separate $\mathcal{F}$. □

Let, for each $\xi \in H$,

$$O^\xi_\beta = \begin{cases} X \setminus \bigcup \mathcal{F}_\xi, & \text{if } \beta \notin \bigcup \mathcal{F}_\xi, \\ (X \setminus \bigcup \mathcal{F}_\xi) \cup F^\xi_\rho, & \text{if } \beta \in F^\xi_\rho. \end{cases}$$

Definition 4.4. Let us call a pair $\langle h, J \rangle \in [H]^{<\omega} \times [X]^{<\omega}$ *complete at β* if for each $\xi \in h$, $B(\beta, h \cap \xi) \setminus J \subset O^\xi_\beta$.

Lemma 4.5. *If $\langle h, J \rangle \in [H]^{<\omega} \times [X]^{<\omega}$ is not complete at β, then there exist $h' \supset h$ and $J' \supset J$ such that $\langle h', J' \rangle$ is complete at β.*

Proof. let $\xi_{h,J}$ be the largest $\xi \in h$ with $B(\beta, h \cap \xi) \setminus J \not\subset O^\xi_\beta$. Since $2^\mathfrak{c}$ is well-founded, it suffices to prove that there exist $h' \supset h$ and $J' \supset J$ such that $\langle h', J' \rangle$ is complete at β or $\xi_{h',J'} < \xi_{h,J}$. Let $\eta = \xi_{h,J}$. Since O^η_β is a τ_η-open neighborhood of β, there exist $\hat{h} \in [H \cap \eta]^{<\omega}$ and $\hat{J} \in [X \setminus \{\beta\}]^{<\omega}$ such that $B(\beta, \hat{h} \cap \xi) \setminus \hat{J} \subset O^\eta_\beta$. Then $h' = h \cup \hat{h}$ and $J' = J \cup \hat{J}$ are as desired. □

Proposition 4.6. *For all $h : X \to [H]^{<\omega}$, $J : X \to [X]^{<\omega}$, and $n : X \to \omega$, there exist $\alpha \neq \beta$ such that $\beta \in B(\alpha, h(\alpha)) \setminus J(\alpha)$ and $n(\alpha) = n(\beta)$.*

Proof. We may assume that for all $\beta \in X$, $\langle h(\beta), J(\beta) \rangle$ is complete at β. Let $M \in N$

be countable elementary submodels containing everything relevant. Let $A = \mathfrak{c} \cap N$, $B = \{\mathcal{F}_\xi \upharpoonright A | \xi \in H \cap M\}$, and $C = \{\mathcal{F}_\xi \upharpoonright A | \xi \in H \cap N \setminus M\}$. The proof of the following lemma is similar to the proof of Lemma 2.9.

Lemma 4.7. *There exists a function g satisfying that $\langle A, B, C, g\rangle \in G$ such that whenever $v : \mathfrak{c} \to \bigcup_{a \in [H\setminus M]^{<\omega}} {}^a(\mathfrak{c} \cup \{-1\})$ is an infinite partial function, $v \in N$, and $\{dom(v(\alpha)) | \alpha \in dom(v)\}$ are pairwise disjoint, then there exists $\alpha \in dom(v)$ such that $\{\langle \mathcal{F}_\xi \upharpoonright A, \rho\rangle | \langle \xi, \rho\rangle \in v(\alpha)\} \subset g$.*

Take $\beta \in X$ satisfying that $q(\beta) = \langle A, B, C, g\rangle$. We obtain the following lemma by taking a set whose elements behave the same way as β. The proof of this is similar to the proof of Lemma 2.10.

Lemma 4.8. *There is $E \in [X]^{\omega_1} \cap N$ such that*
(a) *$\{h(\alpha) : \alpha \in E\}$ is a Δ-system with root $h(\beta) \cap M$;*
(b) *for all $\alpha \in E$, $(h(\alpha) \setminus (h(\beta) \cap M)) \cap M = \emptyset$;*
(c) *for all $\alpha \in E$ and $\xi \in h(\beta) \cap M$, $\alpha \in \bigcup \mathcal{F}_\xi$ if and only if $\beta \in \bigcup \mathcal{F}_\xi$;*
(d) *for all $\alpha \in E$, $n(\alpha) = n(\beta)$.*

Let, for each $\xi \in H$ and $\alpha \in X$, $\rho_{\xi,\alpha} = \rho$ if $\alpha \in F^\xi_\rho$ and $\rho_{\xi,\alpha} = -1$ if $\alpha \notin \bigcup \mathcal{F}_\xi$. Define $v : E \to \bigcup_{a \in [H\setminus M]^{<\omega}} {}^a(\mathfrak{c} \cup \{-1\})$ by setting $v(\alpha) = \{\langle \xi, \rho_{\xi,\alpha}\rangle | \xi \in h(\alpha) \setminus (h(\beta) \cap M)\}$. Note that $v \in N$. By Lemma 4.7, there exists $\alpha \in \mathrm{dom}(v)$ such that $\{\langle \mathcal{F}_\xi \upharpoonright A, \rho_{\xi,\alpha}\rangle | \langle \xi, \rho_{\xi,\alpha}\rangle \in v(\alpha)\} \subset g$. Let us show that $\beta \in B(\alpha, h(\alpha)) \setminus J(\alpha)$ by induction on ξ. Since $J, \alpha \in N$, we have $J(\alpha) \subset N$. Hence $\beta \in B(\alpha, \emptyset) \setminus J(\alpha)$. Suppose $\xi \in h(\alpha)$ and we have proved $\beta \in B(\alpha, h(\alpha) \cap \xi) \setminus J(\alpha)$.

Case 1. Suppose $\xi \in h(\alpha) \cap M$. Note that $\mathcal{F}_\xi \upharpoonright A \in B$. If $\alpha \notin \bigcup \mathcal{F}_\xi$, then by (c), $\beta \notin \bigcup \mathcal{F}_\xi$. Hence $\beta \in B(\alpha, \{\xi\})$. If there is $\rho \in \mathfrak{c}$ such that $\alpha \in F^\xi_\rho$, then by induction hypothesis and completeness, $\beta \in B(\alpha, h(\alpha) \cap \xi) \setminus J(\alpha) \subset O^\xi_\alpha$. Hence, by (c), $\beta \in F^\xi_\rho$, so $\beta \in B(\alpha, \{\xi\})$.

Case 2. Suppose $\xi \in h(\alpha) \setminus (h(\beta) \cap M)$. Note that $\mathcal{F}_\xi \upharpoonright A \in \mathrm{dom}(g)(= C)$. Then by the definition of $\rho_{\xi,\alpha}$, if $\alpha \in F^\xi_\rho$, then $\rho_{\xi,\alpha} = \rho$ and if $\alpha \notin \bigcup \mathcal{F}_\xi$, then $\rho_{\xi,\alpha} = -1$. Hence $\beta \in B(\alpha, \{\xi\})$. $\square$

5 CH constructions

In this section, we show that Balogh's technique can be thought of as an application of the HFC construction. First, we present two examples. The Kunen line [14] is an example of a first countable S-space (hereditarily separable, not Lindelöf space). The HFC [12] is an example of an L-space (hereditarily Lindelöf, not separable space). These spaces are constructed under CH. Example 5.1 is a simpler version of the Kunen line (it is not first countable).

Example 5.1. (CH) The set of points of X is $\mathbb{R}(=\{x_\alpha|\alpha\in\omega_1\})$. Let $\langle M_\alpha\rangle_{\alpha\in\omega_1}$ be a continuous chain of countable elementary submodels. At α-th stage, let $\{A_n|n\in\omega\}$ be a list of $\{A|x_\alpha\in \mathrm{Cl}_{\mathbb{R}}(A), A\in[\{x_\beta|\beta<\alpha\}]^\omega\cap M_\alpha\}$ mentioning each infinite times. Take, for each $n\in\omega$, $y_n^\alpha\in A_n\cap B(x_\alpha,\frac{1}{n})$. Topologize X by declaring a set U to be open if and only if for every $x_\alpha\in U$, there is $J\in[\mathbb{R}]^{<\omega}$ such that $\{y_n^\alpha|n\in\omega\}\setminus J\subset U$.

Since for each $\alpha\in\omega_1$, $\{x_\beta|\beta\leq\alpha\}$ is open, X is not Lindelöf. Note that X refines the usual topology on $\mathbb{R}$. Since X has a base of $\mathbb{R}$-closed sets, X is Tychonoff. Since for each $A\in[\mathbb{R}]^\omega$, $\mathrm{Cl}_{\mathbb{R}}(A)\setminus\mathrm{Cl}(A)$ is countable, X is hereditarily separable.

Example 5.2. (CH) We define $X=\{f_\alpha|\alpha\in\omega_1\}\subset{}^{\omega_1}2$. Let $\langle M_\alpha\rangle_{\alpha\in\omega_1}$ be a continuous chain of countable elementary submodels. At α-th stage, let $\{u_n|n\in\omega\}$ be a list of $\{u:\omega\to\bigcup_{a\in[\omega_1]^{<\omega}}{}^a2|u\in M_\alpha, \{\mathrm{dom}(u(i))|i\in\omega\}$ are pairwise disjoint$\}$. Take, for each $n\in\omega$, $i_n\in\omega$ such that $\{\mathrm{dom}(u_n(i_n))|n\in\omega\}$ are pairwise disjoint. Define f_α so $f_\alpha\supset u_n(i_n)$ for all $n\in\omega$, $f_\alpha(\omega_1\cap M_\alpha)=1$, and $f_\alpha(\beta)=0$ for all $\beta>\omega_1\cap M_\alpha$.

Since for each $\alpha\in\omega_1$, $\{f_\beta|\beta\geq\alpha\}$ is open, X is not separable.

Proposition 5.3. *For all $Y\in[\omega_1]^{\omega_1}$ and neighborhood assignments $h:Y\to\bigcup_{a\in[\omega_1]^{<\omega}}{}^a2$, there exists $\gamma\in\omega_1$ such that $\{f_\beta|\beta\in Y\}\subset\bigcup_{\alpha\in Y\cap\gamma}[h(\alpha)]$.*

Proof. Let M be a countable elementary submodel containing everything relevant. Then $\gamma=\omega_1\cap M$ is as desired. Take any $\beta\in Y\setminus M$. By taking a set whose elements behave the same way as β, we have that there is $\{\alpha_n|n\in\omega\}\in[Y]^\omega\cap M$ such that $\{\mathrm{dom}(h(\alpha_n))|n\in\omega\}$ is a Δ-system with root $\mathrm{dom}(h(\beta))\cap M$ and that for each $n\in\omega$ and $\epsilon\in\mathrm{dom}(h(\beta))\cap M$, $h(\alpha_n)(\epsilon)=h(\beta)(\epsilon)$. Define $u:\omega\to\bigcup_{a\in[\omega_1]^{<\omega}}{}^a2$ by setting $u(n)=h(\alpha_n)\restriction\mathrm{dom}(h(\alpha_n))\setminus(\mathrm{dom}(h(\beta))\cap M)$. Since $u\in M$, we have $u\in M_{\omega_1\cap M}\subset M_\beta$. By the construction of f_β, there is $n\in\omega$ such that $f_\beta\in[h(\alpha_n)]$. $\square$

These two constructions are basically the same except that the objects of the diagonal arguments are different. We can apply these constructions in ZFC by listing countable subsets of the objects length $\mathfrak{c}$. Van Douwen's space [9] is an application of the Kunen line in ZFC. We give a simpler version.

Example 5.4. The set of points of X is $\mathbb{R}(=\{x_\alpha|\alpha\in\mathfrak{c}\})$. Let $\langle\mathcal{A}_\alpha\rangle_{\alpha\in\mathfrak{c}}$ be a list of $[[\mathbb{R}]^\omega]^\omega$ satisfying that for each uncountable $\mathbb{R}$-closed set K and $\mathcal{A}\in[[\mathbb{R}]^\omega]^\omega$, there are uncountable $\alpha\in\mathfrak{c}$ such that $x_\alpha\in K$, $\mathcal{A}_\alpha=\mathcal{A}$, and $\sup(\{\beta\in\mathfrak{c}|x_\beta\in\bigcup\mathcal{A}_\alpha\})<\alpha$. At α-th stage, let $\{A_n|n\in\omega\}$ be a list of $\{A|x_\alpha\in\mathrm{Cl}_{\mathbb{R}}(A), A\in[\{x_\beta|\beta<\alpha\}]^\omega\cap\mathcal{A}_\alpha\}$

mentioning each infinite times. Take, for each $n \in \omega$, $y_n^\alpha \in A_n \cap B(x_\alpha, \frac{1}{n})$. Topologize X by declaring a set U to be open if and only if for every $x_\alpha \in U$, there is $J \in [\mathbb{R}]^{<\omega}$ such that $\{y_n^\alpha | n \in \omega\} \setminus J \subset U$.

It follows that X is normal and countably paracompact from the following proposition. See [9].

Proposition 5.5. *If $\langle F_n : n \in \omega \rangle$ is any sequence of closed subsets in X such that $|\bigcap_{n\in\omega} F_n| \leq \omega$, then $|\bigcap_{n\in\omega} \mathrm{Cl}_{\mathbb{R}}(F_n)| \leq \omega$.*

Proof. Suppose indirectly that $\bigcap_{n\in\omega} \mathrm{Cl}_{\mathbb{R}}(F_n)$ is uncountable. Let M be a countable elementary submodel containing everything relevant. Since $\mathbb{R}$ is hereditarily separable, for each $n \in \omega$, there is $D_n \in [F_n]^\omega \cap M$ such that $F_n \subset \mathrm{Cl}_{\mathbb{R}}(D_n)$. By our way of listing, there are uncountable $\alpha \in \mathfrak{c}$ such that $x_\alpha \in \bigcap_{n\in\omega} \mathrm{Cl}_{\mathbb{R}}(F_n)$, $\mathcal{A}_\alpha = [\mathbb{R}]^\omega \cap M$, and $\sup(\{\beta | x_\beta \in \bigcup \mathcal{A}_\alpha\}) < \alpha$. By the construction, for such α, $x_\alpha \in \bigcap_{n\in\omega} \mathrm{Cl}(D_n) \subset \bigcap_{n\in\omega} F_n$. This is a contradiction. $\square$

In the same way as constructing van Douwen's space from the Kunen line, we can construct $\{f_\alpha | \alpha \in \mathfrak{c}\} \subset {}^{\mathfrak{c}}2$ from the HFC satisfying that for all neighborhood assignments $h : \mathfrak{c} \to \bigcup_{a\in[\mathfrak{c}]^{<\omega}} {}^a2$, there are $\alpha \neq \beta$ behaving in the same way such that $\beta \in [h(\alpha)]$. By using elementarity with respect to '$\neq$', we obtain the following example.

Example 5.6. We define $\{f_\alpha | \alpha \in \mathfrak{c}\} \subset {}^{\mathcal{P}(\mathfrak{c})}2$. Let, for each $A \in [\mathfrak{c}]^\omega$, $\mathcal{U}_A = \{u | u : \omega \to \bigcup_{a\in[\mathcal{P}(A)]^{<\omega}} {}^a2, \{\mathrm{dom}(u(i)) | i \in \omega\}$: pairwise disjoint$\}$. Let $\langle A_\alpha, \mathcal{V}_\alpha \rangle_{\alpha\in\mathfrak{c}}$ be a list of $\{\langle A, \mathcal{V} \rangle | A \in [\mathfrak{c}]^\omega, \mathcal{V} \in [\mathcal{U}_A]^\omega\}$. At α-th stage, let $\{u_n | n \in \omega\}$ be a list of $\mathcal{V}_\alpha$. Take, for each $n \in \omega$, $i_n \in \omega$ such that $\{\mathrm{dom}(u_n(i_n)) | n \in \omega\}$ are pairwise disjoint. Define, for all $Y \in \mathcal{P}(\mathfrak{c})$, if $Y \cap A_\alpha \in \mathrm{dom}(u_n(i_n))$ for some n, then $f_\alpha(Y) = u_n(i_n)(Y \cap A_\alpha)$.

Proposition 5.7. *For all neighborhood assignments $h : \mathfrak{c} \to \bigcup_{a\in[\mathcal{P}(\mathfrak{c})]^{<\omega}} {}^a2$ and $m : \mathfrak{c} \to \omega$, there exist $\alpha \neq \beta$ such that $f_\beta \in [h(\alpha)]$ and $m(\alpha) = m(\beta)$.*

Proof. Let M be a countable elementary submodel containing everything relevant. Let $A = [\mathfrak{c}]^\omega \cap M$ and $\mathcal{V} = \{u \in \mathcal{U}_A | \exists v \in M (v : \omega \to \bigcup_{a\in[\mathcal{P}(\mathfrak{c})]^{<\omega}} {}^a2 \wedge \forall n \in \omega (u(n) = \{\langle Y \cap A, v(n)(Y) \rangle | Y \in \mathrm{dom}(v(n))\}))\}$. By elementarity of M, we have that $\mathcal{V} \in [\mathcal{U}_A]^\omega$. Pick $\beta \in \mathfrak{c}$ such that $\langle A_\beta, \mathcal{V}_\beta \rangle = \langle A, \mathcal{V} \rangle$. By taking a set whose elements behave the same way as β, we have that there is $\{\alpha_n | n \in \omega\} \in [\mathfrak{c}]^\omega \cap M$ such that $\{\mathrm{dom}(h(\alpha_n)) | n \in \omega\}$ is a Δ-system with root $\mathrm{dom}(h(\beta)) \cap M$, for each $n \in \omega$, $m(\alpha_n) = m(\beta)$, and for each $n \in \omega$ and $Y \in \mathrm{dom}(h(\beta)) \cap M$, $h(\alpha_n)(Y) = h(\beta)(Y)$. Define $v : \omega \to \bigcup_{a\in[\mathcal{P}(\mathfrak{c})]^{<\omega}} {}^a2$ by setting $v(k) = h(\alpha_n) \restriction \mathrm{dom}(h(\alpha_n)) \setminus (\mathrm{dom}(h(\beta)) \cap M)$ and $u : \omega \to \bigcup_{a\in[\mathcal{P}(A)]^{<\omega}} {}^a2$ by setting $u(n) = \{\langle Y \cap A_\beta, v(n)(Y) \rangle | Y \in \mathrm{dom}(v(n))\}$. Since $v \in M$, we have $u \in \mathcal{V}_\beta$. By the construction of f_β, there is $n \in \omega$ such that $f_\beta \in [h(\alpha_n)]$. $\square$

Next, we present the HFD [12], which is an example of an S-space.

Example 5.8. (CH) We define $X = \{f_\alpha | \alpha \in \omega_1\} \subset {}^{\omega_1}2$. Let $\langle M_\alpha \rangle_{\alpha\in\omega_1}$ be a continuous chain of countable elementary submodels. At α-th stage, let $\{\langle A_n, i_n\rangle | n \in \omega\}$ be a list of $\{\langle A, i\rangle \in [\alpha]^\omega \times 2 | A \in M_\alpha\}$ mentioning each infinite times. Take, for each $n \in \omega$, $\gamma_n \in A_n$, with all γ_n distinct. Define $f_{\gamma_n}(\alpha) = i_n$ for all $n \in \omega$, $f_\alpha(\alpha) = 1$, and $f_\beta(\alpha) = 0$ for all $\beta > \alpha$.

Since for each $\alpha \in \omega_1$, $\{f_\beta | \beta \leq \alpha\}$ is open, X is not Lindelöf.

Proposition 5.9. *X is hereditarily separable.*

Proof. Take any $Y \in [X]^{\omega_1}$. Let M be a countable elementary submodel containing everything relevant. Take any $f_\beta \in Y \setminus M$ and $\sigma \in \bigcup_{a\in[\omega_1]^{<\omega}} {}^a2$ as a basic neighborhood of f_β. Let $\mathrm{dom}(\sigma) = \{\gamma_0, \cdots, \gamma_n, \delta_0, \cdots, \delta_m\}$ ($\gamma_0 < \cdots < \gamma_n < \omega_1 \cap M \leq \delta_0 < \cdots < \delta_m$). By taking a set whose elements behave the same way as β, we have that there is $A \in [Y]^\omega \cap M$ such that for all $f_\alpha \in A$, $f_\alpha(\gamma_0) = \sigma(\gamma_0), \cdots, f_\alpha(\gamma_n) = \sigma(\gamma_n)$. Note that $A \in M_{\omega_1\cap M} \subset M_{\delta_0}$. By the construction at δ_0-th stage, there is $A_0 \in [A]^\omega$ such that for all $f_\alpha \in A_0$, $f_\alpha(\delta_0) = \sigma(\delta_0)$. By elementarity of M_{δ_0+1}, we may assume that $A_0 \in M_{\delta_0+1}$. Hence $A_0 \in M_{\delta_1}$. Repeating this argument m times, we get A_m. Since $A_m \subset [\sigma]$, we have $f_\beta \in \mathrm{Cl}(Y \cap M)$. $\square$

The chain is essentially used in the HFD construction. Hence we cannot apply the HFD construction in ZFC in the same way as the Kunen line and the HFC.

Question 5.10. Can we apply the HFD construction in ZFC?

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