On the roles of variants of Axiom of Choice in variations of Tychonoff Theorem

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Abstract

In this purely expository note, we examine the roles of Axiom of Choice and its weak variants in topology with emphasis on their connections with Tychonoff Theorem and its variations.

1 Introduction and preliminaries.

In this expository note, we examine the roles of Axiom of Choice (AC) and its weak variants in topology — in connection with Tychonoff Theorem and its variations, in particular. Most of the materials presented here can be also found e.g. in [5]. Exceptions are the second proof of Theorem 3.1 and its applications: these should be also well-known results though I could not find appropriate references (the extended version of the paper mentioned in the footnote may contain some more reference for these results.) Nevertheless, we tried hard to streamline the description. The formulation of this note is rather textbook-like. This is because

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Date: November 27, 2022 *Last update:* January 31, 2023 (15:18 JST)

MSC2020 Mathematical Subject Classification: 03E25, 03E30, 54A20, 54B10

Keywords: Tychonoff theorem, Axiom of Choice, Prime Ideal Theorem, Countable Choice

The research is supported by Kakenhi Grant-in-Aid for Scientific Research (C) 20K03717

An updated and extended version of this paper with more details and proofs is downloadable as: https://fuchino.ddo.jp/papers/RIMS2022-tychonoff-x.pdf we want to present the details so that the readers can see clearly which instance of Axiom of Choice is used/not used in which part of the proofs.

In reverse mathematics, it is proved that five weak systems of second-order arithmetic (the Big Five) are proved to be equivalent to one of many well-known mathematical theorems over the base theory RCA_0. For example, the system WKL_0 is proved to be equivalent to Heine-Borel theorem over the base theory RCA_0. This result may be interpreted as one showing the significance of the system WKL_0 but it can also be interpreted as a result showing the significance of Heine-Borel theorem in terms of the hierarchy of some kind of complexity of mathematical theorems.

Similar situations are also observed over base theories much stronger than RCA_0. The following is such an example.

Recall that a linear ordering $\sqsubseteq$ on a non-empty set X is a *well-ordering* if each non-empty subset of X has the least element with respect to $\sqsubseteq$. This is equivalent to say that we can perform (mathematical, transfinite) induction (and recursive construction) along with $\sqsubseteq$. A set X is said to be *well-orderable* if there is a well-ordering on X.

For a set X and a cardinal κ, we denote

$$[X]^\kappa := \{s \in \mathcal{P}(X) : |s| = \kappa\}.$$

$[X]^{<\kappa}$, $[X]^{\leq\kappa}$ are defined similarly.

A set X is equinumerous with another set Y if there is a bijection from X to Y.

Theorem 1.1 *The following are equivalent over* ZF*:*

- (A) *Axiom of Choice.* (A') *Every set is is well-orderable.*
- (B) (Tarski [11]) *For all infinite set X, X^2 is equinumerous with X.*
- (C) *For all infinite set X, $[X]^2$ is equinumerous with X.*
- (D) (Tychonoff Theorem, Kelley [9]) *For any index set I and any sequence $\langle X_i : i \in I\rangle$ of compact topological spaces, the product space $\prod_{i\in I} X_i$ is also compact.*

(A) $\Leftrightarrow$ (A') is classical: it is first proved by Zermelo in [14]. Nevertheless we shall give below an alternative proof of this equivalence as a part of Theorem 1.4.

(A) $\Leftrightarrow$ (B) is proved in Jech [6] as Theorem 11.7 (and Theorem 2.4). (A) $\Leftrightarrow$ (C) can be obtained by modifying this proof.

We shall give a detailed proof of (A) $\Leftrightarrow$ (D) as Theorem 4.1 below.

Theorem 4.2 which gives prominence to Prime Ideal Theorem defined below is another example of such theorems formulated in terms of the Tychonoff Theorem for Hausdorff spaces.

Some other theorems showing the relation between other weak versions of Axiom of Choice and variants of Tychonoff Theorem are considered in Section 4.

For discussions about reverse mathematical phenomenon over the full set-theory ZFC as the base theory, see e.g. [2].

In the rest of the section we review some notations, and definitions of variants of AC.

For a sequence $\langle X_i : i \in I\rangle$ of sets, we define the product of the sequence as

$$\prod_{i\in I} X_i := \{f : f : I \to \bigcup_{i\in I} X_i,\ f(i) \in X_i \text{ for all } i \in I\}.$$

Axiom of Choice is the assertion:

(AC): For any (non-empty) index set I and any sequence $\langle X_i : i \in I\rangle$ of non-empty sets, $\prod_{i\in I} X_i$ is non-empty.

If X_i's are topological spaces with $X_i = (X_i, \mathcal{O}_i)$, $\prod_{i\in I} X_i$ also denotes the product space of X_i's with the usual product topology with basic open sets of the form

$$[s] = \{f \in \prod_{i\in I} X_i : f \restriction I_0 \in \prod_{i\in I_0} s(i)\}$$

for $I_0 \in [I]^{<\aleph_0}$ and $s : I_0 \to \bigcup_{i\in I_0} \mathcal{O}_i$ with $s(i) \in \mathcal{O}_i$ for all $i \in I_0$.

We assume that a sequence $\langle a_i : i \in I\rangle$ is introduced as a function f on I such that $f(i) = a_i$ for all $i \in I$. Thus each element $f \in \prod_{i\in I} X_i$ will be also represented as $f = \langle a_i : i \in I\rangle$ where $a_i \in X_i$ for all $i \in I$.

If $X_i = X$ for all $i \in I$, $\prod_{i\in I} X_i$ is also denoted by ${}^I X$. Thus

$${}^I X = \{f : f : I \to X\}.$$

For an ordinal δ we write

$${}^{\delta>} X := \bigcup_{\alpha<\delta} {}^{\alpha} X = \{f : f : \alpha \to X \text{ for some } \alpha < \delta\}.$$

The *Prime Ideal Theorem* (PIT) is the statement that of any Boolean algebra B, there is a prime ideal on B. The *Ultrafilter Theorem* (UFT) is the statement that for any non-empty set X, and any filter $\mathcal{F}$ over X, there is an ultrafilter $\mathcal{F}^*$ over X which extends $\mathcal{F}$ (we review the definition and basic properties of filter and ultrafilter over a set, as well as ideals and prime ideals on a Boolean algebra in the next section). AC implies PIT: in many textbooks this is proved as an application of Zorn's Lemma. It is known that PIT is strictly weaker than AC over ZF (see e.g. Jech [6], Theorem 7.1). PIT and UFT are equivalent over ZFC. In Section 4 we shall see a proof of this fact as a part of Theorem 4.2. One direction is easy to see.

In the rest of the paper "(−AC)" indicates that the given statement can be proved in ZF without the Axiom of Choice. Note that this does not mean that the negation of the Axiom of Choice is assumed.

Lemma 1.2 (−AC) PIT *implies* UFT.

Proof. Assume that PIT holds. Suppose that $\mathcal{F}$ is a filter over a non-empty set X. Let $\mathcal{B} = \mathcal{P}(X)/\mathcal{F}$. (Since AC is not available here we consider $\mathcal{P}(X)/\mathcal{F}$ to be the set of all equivalence classes of elements of $\mathcal{P}(X)$ modulo $\mathcal{F}$ with the partial ordering $S \leq T$ for S, $T \in \mathcal{P}(X)/\mathcal{F}$ defined by $A \subseteq B$ modulo $\mathcal{F}$ for all $A \in S$ and $B \in T$). Let I be a prime ideal on $\mathcal{B}$ (which exists by PIT) and let F be its dual filter. Then

$$\mathcal{F}^* := \{Y \in \mathcal{P}(X) \,:\, \text{the equivalence class of } Y \text{ modulo } \mathcal{F} \text{ is in } F\}$$

is an ultrafilter over X extending $\mathcal{F}$. ❑ (Lemma 1.2)

A topological space $X = (X, \mathcal{O})$ is *compact* if for any open covering $\mathcal{U}$ of X there is a finite sub-covering $\mathcal{U}_0$ of $\mathcal{U}$. It is easy to check that the standard proof of Heine-Borel Theorem does not need Axiom of Choice [1).]

Theorem 1.3 (Heine-Borel Theorem, −AC) *Any (bounded) closed set in* $\mathbb{R}$ *(as the subspace of* $\mathbb{R}$ *with the standard topology) is compact.*

Proof. The proof given in [13] (with a slight modification) works without Axiom of Choice. ❑ (Theorem 1.3)

One of the consequences of PIT ($\Leftrightarrow$ UFT) is AC for finite sets which is defined as:

(AC(fin)): For any sequence $\langle X_i \,:\, i \in I\rangle$ of non-empty finite sets, $\prod_{i\in I} X_i \neq \emptyset$.

One way to see that AC(fin) follows from PIT is to show first the equivalence of PIT with (model-theoretic) compactness theorem and apply it. Actually it is easy to see that the compactness theorem proves that every partial ordering on

[1)] Heine-Borel Theorem in connection with Reverse Mathematics is that theorem expressible in the context of weak second-order arithmetic which is different from the same theorem in the framework of ZF. Though in our case, the proof considered in Reverse Mathematics can be translated to the general situation to see that we do not need the Axiom of Choice for the proof, there is no guarantee in general that, for a theorem formulated in the framework of Reverse Mathematics, the corresponding more general theorem in the framework of ZF do not need AC.

an arbitrary set can be extended to a total (i.e. linear) ordering, and it is also immediate to see that $\mathsf{AC}(\mathsf{fin})$ follows from this statement.

$\mathsf{AC}(\mathsf{fin})$ can be further weakened by restricting the sequences with well-ordered index set. For a cardinal κ, we define

($\mathsf{AC}_\kappa(\mathsf{fin})$): For any sequence $\langle X_\alpha : \alpha \in \kappa\rangle$ of non-empty finite sets, $\prod_{\alpha\in\kappa} X_\kappa \neq \emptyset$.

We also consider further the three types of principles DC_κ, AC_κ and $\mathsf{AC}^{\mathsf{wo}}_\kappa$: DC_κ for an infinite cardinal κ is the *Dependent Choice of length* κ:

(DC_κ): For any set S and any binary relation $R \subseteq {}^{\kappa>}S \times S$, if for any $f \in {}^{\alpha}S$ for any $\alpha < \kappa$ there is $s \in S$ such that $f \mathrel{R} s$, then there is $f \in {}^{\kappa}S$ such that $f \restriction \alpha \mathrel{R} f(\alpha)$ for all $\alpha < \kappa$.

Theorem 1.4 *The following are equivalent:* (A) AC.

(A') *Any set is well-orderable.*

(A") $\forall\kappa\, \mathsf{DC}_\kappa$.

Proof. "(A) $\Rightarrow$ (A')": Assume that AC holds and let X be an arbitrary non-empty set. We fix $f \in \prod_{U\in\mathcal{P}(X)\setminus\{\emptyset\}} U$ (which exists because of AC). Using f as a book keeping, we can try transfinitely to enumerate elements of X as a_0, a_1,... . Then there must be some $\delta \in \mathrm{On}$ such that $\langle a_\alpha : \alpha < \delta\rangle$ is an enumeration of X. The ordering $a \sqsubseteq b$ defined for $a, b \in X$ as being $a = a_\alpha$ and $b = b_\beta$ for $\alpha < \beta < \delta$ is a well ordering on X.

"(A') $\Rightarrow$ (A")": If S and R are as in the definition of DC_κ, and R^* is a well-ordering on S, then we can construct a sequence $f = \langle s_\alpha : \alpha < \kappa\rangle$ by induction on $\alpha < \kappa$, at the αth step of the construction we just choose R^*-minimal $s \in S$ such that $f_\alpha \mathrel{R} s$ where $f_\alpha = \langle s_\xi : \xi < \alpha\rangle$ is the sequence of elements of S chosen so far.

"(A") $\Rightarrow$ (A')": DC_κ implies that every non-empty set X is either well-orderable in order type $<\kappa$ or there is a 1-1 sequence of elements of X of length κ. Hence $\forall\kappa\, \mathsf{DC}_\kappa$ implies the well-orderability of an arbitrary set.

"(A') $\Rightarrow$ (A)": This is easy, for any sequence $\langle X_i : i \in I\rangle$ of non-empty sets, if R is a well-ordering of $\bigcup_{i\in I} X_i$, then we can construct $f \in \prod_{i\in I} X_i$ by choosing $f(i)$ for $i \in I$ to be the R-minimal elements of X_i. ❑ (Theorem 1.4)

(AC_κ) For any sequence $\langle X_\alpha : \alpha < \kappa\rangle$ of length κ of non-empty sets, $\prod_{\alpha<\kappa} X_\alpha \neq \emptyset$.

For a sequence $\langle X_\alpha : \alpha < \kappa\rangle$ of non-empty sets if we define $S := \bigcup_{\alpha<\kappa} X_\kappa$ and $R \subseteq {}^{\kappa>}S$ by $f \; R \; s :\Leftrightarrow$ if $\alpha < \kappa$ is the length of f then $s \in X_i$, the κ-sequence as in the definition of DC_κ is an element of $\prod_{\alpha<\kappa} X_\alpha$. Thus, for any κ, DC_κ implies AC_κ. It is known that for any κ, AC_κ does not imply DC_ω. In particular, $\forall\kappa\, \mathsf{AC}_\kappa$ is not equivalent to AC.

AC_κ can be further weakened to obtain:

($\mathsf{AC}^{\mathsf{wo}}_\kappa$) For any sequence $\langle X_\alpha : \alpha < \kappa\rangle$ of length κ of non-empty well-orderable sets, $\prod_{\alpha<\kappa} X_\alpha \neq \emptyset$.

To finish this introduction I would like to add the following remark concerning the relevance of the study of variations of the Axiom of Choice involved in mathematical theorems in "everyday" (pure) mathematics. Even if you are working in ZFC (simply assuming the full Axiom of Choice for granted), you may want to work in the set-theoretic universe in which many large large cardinals (this is not a typo) exist and/or Martin's Axiom, or even Proper Forcing Axiom or double plused version of Martin's Maximum holds. In such a universe of mathematics (i.e. set theory), there are full of inner models which do not satisfy the Axiom of Choice but diverse fragments of it: $\mathsf{L}(\mathbb{R})$ is a prominent example of such inner models — under the existence of a sufficiently large cardinal $\mathsf{L}(\mathbb{R})$ satisfies the Axiom of Determinacy and hence does not satisfy any known weakenings of AC except dependent choice (i.e. DC_ω in the notation introduced below). Mathematics in these inner models can/should be integrated to whole picture of the everyday mathematics to obtain a richer and more fertile mathematical landscape. Besides the Reverse Mathematical significance mentioned at the beginning of the section, the study of variants of AC and their topological characterizations is thought to be the first step toward this type of "inner model theory" or set-theoretic geology as Joel Hamkins put it.

A conversation with Atsushi Yamashita via twitter motivated the author to write this article. The author would like to thank Dr. Yamashita for pushing him toward this occasion.

2 Convergence of filters and ultrafilters in a topological space

Let $X = (X, \mathcal{O})$ be a topological space where the topology of X is given here by the set $\mathcal{O}$ of all open sets in X. For $p \in X$, $\mathcal{O}_p := \{O \in \mathcal{O} : p \in O\}$ denotes the open neighborhood of p.

$\mathcal{F}$ is a *filter over* X if it is a non-empty subset of $\mathcal{P}(X)$ satisfying

(2.1) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$, and

(2.2) For any A, $B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

In the following we consider only non-trivial filters, that is, we always assume additionally that

(2.3) $\emptyset \notin \mathcal{F}$.

For any non-empty X, and $\mathcal{S} \subseteq \mathcal{P}(X)$, $\mathcal{S}$ has *the finite intersection property* (the *fip*, for short) if $S_0 \cap \cdots S_{n-1} \neq \emptyset$ for any $n \in \omega$ and $S_0, ..., S_{n-1} \in \mathcal{S}$.

Lemma 2.1 ($-$AC) *For non-empty X and $\mathcal{S} \subseteq \mathcal{P}(X)$, $\mathcal{S}$ has the fip $\Leftrightarrow$ there is a filter $\mathcal{F}$ over X with $\mathcal{S} \subseteq \mathcal{F}$. Furthermore, we can uniquely specify the minimal $\mathcal{F}$ among such filters $\mathcal{F}$.*

Proof. "$\Leftarrow$": is clear by the property (2.2) (and (2.3))of the filter $\mathcal{F}$ with $\mathcal{S} \subseteq \mathcal{F}$.

"$\Rightarrow$": Assume that $\mathcal{S} \subseteq \mathcal{P}(X)$ has the fip. Then

$$\mathcal{F} := \{A \in \mathcal{P}(X) \,:\, A \supseteq S_0 \cap \cdots \cap S_n \text{ for some } n \in \omega \text{ and } S_0, ..., S_n \in \mathcal{S}\}$$

is a filter over X with $\mathcal{S} \subseteq \mathcal{F}$. It is clear that this $\mathcal{F}$ is the unique minimal filter containing $\mathcal{S}$. ❑ (Lemma 2.1)

We can generalize the notion of filter and ultrafilter in the context of Boolean algebras. For a Boolean algebra $B = \langle B, \wedge, \vee, \neg, \mathbb{0}, \mathbb{1}\rangle$, $F \subseteq B$ is said to be a filter on B if for any a, a', $b \in B$,

(2.4) $a \in F$ and $a \leq_B b$ implies $b \in F$ where $\leq_B$ denotes the partial ordering on B associated with the Boolean algebraic structure of B;

(2.5) a, $a' \in F$ implies $a \wedge a' \in F$;

(2.6) $\mathbb{0} \notin F$.

$F \subseteq B$ is an ultrafilter on B if F is a filter on B and it is maximal (with respect to $\subseteq$) among filters on B. A generalization of Lemma 2.1 proves that a filter $F \subseteq B$ is an ultrafilter, if and only if, for any $a \in B$ exactly one of $a \in F$ or $\neg a \in F$ holds. Note that we are talking about a filter "on" B. The notion of a filter over X corresponds here to the filter on the (power set algebra) Boolean algebra $\mathcal{P}(X)$.

$I \subseteq B$ is an ideal if $\neg I := \{\neg a \,:\, a \in I\}$ is a filter. Note that $\neg(\neg I) = I$. I is a prime ideal if $\neg I$ is an ultrafilter.

If X is a topological space there is an interesting interplay between filters over the set X and the topology of X.

Example 2.2 (1) *For a topological space X and $p \in X$,* $\mathcal{F}_p := \{A \in \mathcal{P}(X) : O \subseteq A$ *for some* $O \in \mathcal{O}_p\}$ *is a filter over X with* $\mathcal{O}_p \subseteq \mathcal{F}_p$.

(2) *Let* $\mathcal{F}_\infty := \{A \in \mathcal{P}(\mathbb{R}) : (a, \infty) \subseteq A$ *for some* $a \in \mathbb{R}\}$. $\mathcal{F}_\infty$ *is a filter over* $\mathbb{R}$, *and, for any filter $\mathcal{F}$ over $\mathbb{R}$ extending $\mathcal{F}_\infty$, there is no $p \in \mathbb{R}$ such that $\mathcal{O}_p \subseteq \mathcal{F}$.* □

For a topological space $X = (X, \mathcal{O})$, a filter $\mathcal{F}$ over X and $p \in X$, we say $\mathcal{F}$ *converges to p* (notation: $\mathcal{F} \to p$) if $\mathcal{O}_p \subseteq \mathcal{F}$.

The following is easy to see:

Theorem 2.3 (−AC) *A topological space $X = (X, \mathcal{O})$ is Hausdorff if and only if, for any filter $\mathcal{F}$ over X, $\mathcal{F}$ converges to at most a single point in X.* □

$p \in X$ is a *cluster point* of a filter $\mathcal{F} \subseteq \mathcal{P}(X)$, if $p \in \bigcap \overline{\mathcal{F}}$ where $\overline{\mathcal{F}} := \{\overline{F} : F \in \mathcal{F}\}$. Note that $p \in X$ is a cluster point of $\mathcal{F}$ if and only if, for any $F \in \mathcal{F}$, $O \cap F \neq \emptyset$ for all $O \in \mathcal{O}_p$.

Lemma 2.4 (−AC) *Suppose that X is a topological space and $\mathcal{F}$ is a filter over X.* (1) *The set of all cluster points of $\mathcal{F}$ is a closed subset of X.*

(2) *$\mathcal{F} \to p$ implies that p is a cluster point of $\mathcal{F}$.*

(3) *If $p \in X$ is a cluster point of a filter $\mathcal{F}$ over X then there is a filter $\mathcal{F}'$ over X extending $\mathcal{F}$ such that $\mathcal{F}' \to p$. Furthermore, such $\mathcal{F}'$ can be uniquely specified as the minimal filter with these properties.*

(4) *For an ultrafilter $\mathcal{F}$ over a topological space X, and $p \in X$, $\mathcal{F}$ converges to p if and only if p is a cluster point of $\mathcal{F}$.*

Proof. (1): This is clear since the set of all cluster points is $\bigcap \overline{\mathcal{F}}$.

(2): If $\mathcal{F} \to p$ then $\mathcal{O}_p \subseteq \mathcal{F}$. In particular $O \cap F \neq \emptyset$ for all $O \in \mathcal{O}_p$ and $F \in \mathcal{F}$.

(3): If p is a cluster point of $\mathcal{F}$, then $\mathcal{F} \cup \mathcal{O}_p$ has the fip. Thus, by Lemma 2.1, there is a filter $\mathcal{F}'$ over X which contains $\mathcal{F} \cup \mathcal{O}_p$. Clearly $\mathcal{F}' \to p$. Uniqueness of minimal such $\mathcal{F}'$ is clear from the construction of the filter (see the proof of Lemma 2.1).

(4): If $\mathcal{F}$ converges to $p \in X$ then p is a cluster point of $\mathcal{F}$ by (2). Suppose that $p \in X$ a cluster point of $\mathcal{F}$. Then $\mathcal{F} \cup \mathcal{O}_p$ has the fip. By Lemma 2.1, there is a filter $\mathcal{F}'$ over X containing this set as a subset. In particular, $\mathcal{F} \subseteq \mathcal{F}'$. Since $\mathcal{F}$ is an ultrafilter, it follows that $\mathcal{F} = \mathcal{F}'$. Hence $\mathcal{O}_p \subseteq \mathcal{F}$ i.e. $\mathcal{F}$ converges to p.

□ (Lemma 2.4)

Compactness of topological spaces can be characterized in terms of filters over them. A filter $\mathcal{F}$ over a non-empty set X is called an *ultrafilter* if it is maximal with respect to $\subseteq$ among ultrafilters over X. A filter $\mathcal{F}$ over X is an ultrafilter if and only if, for any $A \in \mathcal{P}(X)$, one of A and $X \setminus A$ is always an element of $\mathcal{F}$.

Theorem 2.5 *The following hold for any topological space X:* (1) ($-$AC) (a) *X is compact* $\Leftrightarrow$ (b) *For any family $\mathcal{B}$ of non-empty closed subset of X with the fip, $\bigcap \mathcal{B} \neq \emptyset$* $\Leftrightarrow$ (c) *Any filter $\mathcal{F}$ over X has a cluster point.*
(2) (UFT) *X is compact* $\Leftrightarrow$ *Any ultrafilter $\mathcal{F}$ over X converges to some point(s) in X.*

Proof. (1): "(a) $\Rightarrow$ (b)": Suppose that $\mathcal{B}$ is a family of non-empty closed subsets of X with the fip but (2.7): $\bigcap \mathcal{P} = \emptyset$. Let $\mathcal{Y} = \{X \setminus B \,:\, B \in \mathcal{B}\}$.

Claim 2.5.1 *$\mathcal{U}$ is an open covering of X without any finite subcover.*

$\vdash$ Elements of $\mathcal{U}$ are open by definition $\mathcal{U}$ is a covering of X by (2.7).

Suppose that $X \setminus A_0, \ldots, X \setminus A_{n-1}$ are elements of $\mathcal{U}$ where $A_0, \ldots, A_{n-1} \in \mathcal{B}$. Then

$$(X \setminus A_0) \cup \ldots \cup (X \setminus A_{n-1}) = X \setminus \overbrace{(A_0 \cap \ldots \cap A_{n-1})}^{\neq \emptyset, \text{ by the fip of } \mathcal{B}} \neq X.$$

$\dashv$ (Claim 2.5.1)

"(b) $\Rightarrow$ (c)": Clear by the definition of cluster point.

"(c) $\Rightarrow$ (a)": Suppose that X is not compact and let $\mathcal{U}$ be an open covering of X without finite subcovering. Then $\mathcal{B} := \{X \setminus O \,:\, O \in \mathcal{U}\}$ has the fip. By Lemma 2.1, there is a filter $\mathcal{F}$ over X with $\mathcal{B} \subseteq \mathcal{F}$.

Claim 2.5.2 *$\mathcal{F}$ does not have any cluster point.*

$\vdash$ For any $p \in X$, there is $O \in \mathcal{U}$ with $p \in O$. $F := X \setminus O \in \mathcal{F}$ by the choice of $\mathcal{F}$. But $O \in \mathcal{O}_p$ and $O \cap F = \emptyset$. This implies that p is not a cluster point of $\mathcal{F}$.

$\dashv$ (Claim 2.5.2)

(2), "$\Rightarrow$": Suppose that X is a compact topological space and $\mathcal{F}$ is an ultrafilter over X. By (1), $\mathcal{F}$ has a cluster point p. By Lemma 2.4, (2), $\mathcal{F}$ converges to p.

"$\Leftarrow$": Suppose that X is not compact. Let $\mathcal{U}$, $\mathcal{B}$ and $\mathcal{F}$ be as in the proof of (1), "(c) $\Rightarrow$ (a)". By UFT, there is an ultrafilter $\mathcal{F}^*$ over X with $\mathcal{F} \subseteq \mathcal{F}^*$. The proof of Claim 2.5.2 is applicable to $\mathcal{F}^*$ and shows that there is no cluster point of $\mathcal{F}^*$. By Lemma 2.4, (4), it follows that $\mathcal{F}^*$ does not converge to any point. ❑ (Theorem 2.5)

3 Proofs of Tychonoff Theorem

In this section, we examine two proofs of Tychonoff's Theorem. These proofs will be modified to obtain variations of Tychonoff Theorem (Corollary 3.2 ∼ Corollary 3.5) under various weakenings of AC.

Theorem 3.1 (Tychonoff [12], AC) *For any index set I and compact spaces X_i for $i \in I$, $Y := \prod_{i\in I} X_i$ is compact.*[2)]

The first proof. Assume that $\mathcal{F}$ is an ultrafilter over Y. By Theorem 2.5, (2) (and here we use UFT), it is enough to show that $\mathcal{F}$ converges to a point in Y.

For each $i \in I$, let

$$\mathcal{F}_i := \{U \subseteq X_i \,:\, \{f \in Y \,:\, f(i) \in U\} \in \mathcal{F}\}. \tag{3.1}$$

Then, for each $i \in I$, $\mathcal{F}_i$ is an ultrafilter over X_i. By Theorem 2.5, (2), there is $a_i \in X$ such that $\mathcal{F}_i \to a_i$. (We need AC here in general to choose the sequence $\langle a_i : i \in I\rangle$.)

We are done with the following Claim:

Claim 3.1.1 $\mathcal{F} \to \langle a_i : i \in I\rangle$.

$\vdash$ Suppose that $O \in \mathcal{O}_{\langle a_i : i\in I\rangle}$ where $\mathcal{O}$ denotes the set of all open sets in Y. We have to show that $O \in \mathcal{F}$. For this, we may assume that O is a basic open set of the product space Y of the form

$$O = \{f \in \textstyle\prod_{i\in I} X_i \,:\, f(i_0) \in O_{i_0} \text{ for all } i_0 \in I_0\}$$

for some $I_0 \in [I]^{<\aleph_0}$ and $O_{i_0} \in (\mathcal{O}_{i_0})_{a_{i_0}}$ for $i_0 \in I_0$.

By the choice of a_{i_0}, we have $O_{i_0} \in \mathcal{F}_{i_0}$ for all $i_0 \in I_0$. Thus

$$\tilde{O}_{i_0} := \{f \in \textstyle\prod_{i\in I} X_i \,:\, f(i_0) \in O_{i_0}\} \in \mathcal{F}$$

by the definition (3.1) of $\mathcal{F}_i$. It follows that $O = \bigcap_{i_0\in I} \tilde{O}_{i_0} \in \mathcal{F}$. $\dashv$ (Claim 3.1.1)

□ (Theorem 3.1)

The second proof. Suppose that $\mathcal{F}$ is a filter over $Y = \prod_{i\in I} X_i$. By Theorem 2.5, (1) it is enough to show that $\mathcal{F}$ has a cluster point.

[2)] Tychonoff formulated this theorem (indirectly)[3)] for Hausdorff spaces but his proof is applicable for spaces which are not necessarily Hausdorff.

[3)] I wrote "indirectly", since Tychonoff in [12] refers to Alexandroff and Urysohn [1] for the setting of topology. [1] cites Hausdorff's text book [4] for definition of topology in which Hausdorffness is simply one of the axioms of topological spaces.

Since we are assuming AC, we may assume that the index set I is a cardinal κ. Thus $Y = \prod_{\alpha<\kappa} X_\alpha$. Let $\mathcal{O}_Y$ be the set of open sets of the product space Y and $X_\alpha = (X_\alpha, \mathcal{O}_\alpha)$. For $\alpha < \kappa$, let $\pi_\alpha : Y \to X_\alpha;\ f \mapsto f(\alpha)$ be the projection.

By induction, we define sequences of filters $\mathcal{F}^0_\alpha \subseteq \mathcal{P}(X_\alpha)$ for $\alpha < \kappa$, and $\mathcal{F}_\alpha \subseteq \mathcal{P}(Y)$ for $\alpha \leq \kappa$ such that:

(3.2) $\mathcal{F}_0 = \mathcal{F}$;

(3.3) $\mathcal{F}^0_\alpha$ = the filter over X_α generated by $\{\pi_\alpha{}''F : F \in \mathcal{F}_\alpha\} \cup (\mathcal{O}_\alpha)_{a_\alpha}$ where a_α is a cluster point of the filter $\{\pi_\alpha{}''F : F \in \mathcal{F}_\alpha\}$ over X_α (cf. Lemma 2.4, (3) and its proof);

(3.4) $\mathcal{F}_{\alpha+1}$ = the filter generated by $\mathcal{F}_\alpha \cup \{\pi^{-1}{}''U : U \in \mathcal{F}^0_\alpha\}$; and,

(3.5) for a limit $\gamma \leq \kappa$, $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$.

$\mathcal{F}_\kappa$ is then a filter over Y extending $\mathcal{F}$.

Claim 3.1.2 *$\mathcal{F}_\kappa \to \langle a_\alpha : \alpha < \kappa\rangle$.*

$\vdash$ Suppose that O is a basic open set of Y around $\langle a_\alpha : \alpha < \kappa\rangle$. Let

$$O = [\{\langle \alpha, O_\alpha\rangle : \alpha \in I_0\}]\ (= \{f \in Y : f(\alpha) \in O_\alpha \text{ for all } \alpha \in I_0\})$$

for $I_0 \in [\kappa]^{<\aleph_0}$ and $O_\alpha \in (\mathcal{O}_\alpha)_{a_\alpha}$ for $\alpha \in I_0$.

By (3.3) and (3.4), ${\pi_\alpha}^{-1}{}''O_\alpha \in \mathcal{F}_\kappa$. Hence, $O = \bigcap_{\alpha\in I_0} {\pi_\alpha}^{-1}{}''O_\alpha \in \mathcal{F}_\kappa$.

$\dashv$ (Claim 3.1.2)

By Lemma 2.4, (2), $\langle a_\alpha : \alpha < \kappa\rangle$ is a cluster point of $\mathcal{F}_\kappa$. Hence, it is also a cluster point of $\mathcal{F}$. □ (Theorem 3.1)

All of the following are corollaries to the ideas of one of the two proofs of Theorem 3.1.

Corollary 3.2 (UFT) *For any index set I and compact Hausdorff spaces X_i for $i \in I$, $Y := \prod_{i\in I} X_i$ is compact.*

Proof. By the first proof of Theorem 3.1. Note that, by Theorem 2.3, the limit a_i of $\mathcal{F}_i$ is unique because of the Hausdorffness of X_i, and hence we can pick up the sequence $\langle a_i : i \in I\rangle$ without the help of AC. □ (Corollary 3.2)

Corollary 3.3 (DC_κ) *For sequence $\langle X_\alpha : \alpha < \kappa\rangle$ of compact spaces, the product $\prod_{\alpha<\kappa} X_\alpha$ is compact.*

Proof. By a modification of the second proof of Theorem 3.1. Suppose that $\langle X_\alpha : \alpha < \kappa \rangle$ is a sequence of compact spaces and $\mathcal{F}$ is a filter over $\prod_{\alpha<\kappa} X_\alpha$. By DC_κ we can choose the sequences $\langle a_\alpha : \alpha < \kappa \rangle$, $\langle \mathcal{F}^0_\alpha : \alpha < \kappa \rangle$, $\langle \mathcal{F}_\alpha : \alpha \leq \kappa \rangle$ as in the second proof of Theorem 3.1. Then we can show that $\mathcal{F} \to \langle a_\alpha : \alpha \in \kappa \rangle$ holds just as in the proof. □ (Corollary 3.3)

Corollary 3.4 ($\mathsf{AC}_\kappa(\mathsf{fin})$) *For any sequence $\langle X_\alpha : \alpha \in \kappa \rangle$ of finite spaces, $\prod_{\alpha\in\kappa} X_\kappa$ is compact.*

Proof. A modification of the second proof of Theorem 3.1 will do. Suppose that $\langle X_\alpha : \alpha \in \alpha \rangle$ is a sequence of finite spaces. By $\mathsf{AC}_\kappa(\mathsf{fin})$, we can find a sequence $\langle R_\alpha : \alpha \in \kappa \rangle$ such that each R_α is a well-ordering of X_α. The construction with (3.2) $\sim$ (3.5) goes through by choosing R_α-minimal a_α for each α. □ (Corollary 3.4)

In some cases we can completely avoid AC:

Corollary 3.5 ($-\mathsf{AC}$) (1) *For any compact topological spaces X, Y, the product space $X \times Y$ is compact.*

(2) *For any cardinal κ and a sequence $\langle X_\alpha : \alpha < \kappa \rangle$ such that each X_i is either a closed subset of an successor ordinal with the order topology or bounded closed subset of $\mathbb{R}$ with the order topology, then the product $\prod_{\alpha\in I} X_\alpha$ is compact. In particular, ${}^\kappa 2$, ${}^\kappa[0,1]$ are compact for any cardinal κ.*

Proof. Both (1) and (2) can be shown by the second proof of Theorem 3.1. For (2), note that X_α, $\alpha < \kappa$ as in the statement are compact by Heine-Borel Theorem (Theorem 1.3) and Lemma 3.6 below. If X_α is a closed subset of a successor ordinal, we can take a_α as the minimal element of X_α among the cluster points with respect to the canonical well-ordering. If X_α is a bounded closed subset of $\mathbb{R}$, the set of cluster points is a closed subset of the space (see Lemma 2.4, (1)) and hence we can take the minimal element among them with respect to the ordering of $\mathbb{R}$.

□ (Corollary 3.5)

Lemma 3.6 ($-\mathsf{AC}$) *An ordinal α with its order topology is compact if and only if it is a successor ordinal.* □

4 Characterizations of variants of AC in terms of Tychonoff Theorem

Theorem 4.1 (Kelley [9], see Theorem 1.1 in Section 1)
The following are equivalent over ZF*:* (A) AC.

(D) *For any index set I and any sequence $\langle X_i : i \in I\rangle$ of compact topological spaces, the product space $\prod_{i\in I} X_i$ is also compact.*

Proof. "(A) $\Rightarrow$ (D)": has been proved as Theorem 3.1.

"(D) $\Rightarrow$ (A)": Assume that (D) holds. Suppose that $\langle X_i : i \in I\rangle$ is a sequence of non-empty sets. We have to show that $\prod_{i\in I} X_i \neq \emptyset$.

Let ∞ be a set such that $\infty \notin \bigcup_{i\in I} X_i$. Let $Y_i = X_i \cup \{\infty\}$ and $\mathcal{O}_i = \{\emptyset, X_i, \{\infty\}, Y_i\}$ for $i \in I$. Then $Y_i = (Y_i, \mathcal{O}_i)$ is a compact topological space (it is compact since $\mathcal{O}_i$ is finite). Thus by the assumption of (D), $Y := \prod_{i\in I} Y_i$ is compact.

For $i \in I$, let $A_i := \{f \in Y : f(i) \in X_i\}$. Then each A_i is a closed set in Y.

Claim 4.1.1 $\mathcal{A} := \{A_i : i \in I\}$ *has the fip.*

$\vdash$ Suppose that $A_{i_0}, ..., A_{i_{k-1}} \in \mathcal{A}$. Let $g \in Y$ be defined by

$$g(i) = \begin{cases} a_i \text{ for some } a_i \in X_i, & \text{if } i = i_\ell \text{ for some } \ell < k; \\ \infty & \text{otherwise} \end{cases}$$

for $i \in I$. Note that g can be chosen without AC. We have $g \in \bigcap_{\ell<k} A_{i_\ell}$.

$\dashv$ (Claim 4.1.1)

By Theorem 2.5, (1), it follows that there is $h \in \bigcap \mathcal{A} = \prod_{i\in I} X_i$. □ (Theorem 4.1)

Theorem 4.2 (Łoś and Ryll-Nardzewski [10])
The following are equivalent over ZF*:* (E) PIT.

(F) UFT.

(G) *For any index set I and any sequence $\langle X_i : i \in I\rangle$ of compact Hausdorff topological spaces, the product space $\prod_{i\in I} X_i$ is also compact.*

Proof. "(E) $\Rightarrow$ (F)": Lemma 1.2.

"(F) $\Rightarrow$ (G)": Corollary 3.2.

"(G) $\Rightarrow$ (E)": Assume (G) and suppose that B is a Boolean algebra. Let

$$\mathcal{S} := \{B_0 : B_0 \text{ is a finite subalgebra of } B\}.$$

For $B_0 \in \mathcal{S}$, let

$$X_{B_0} := \{f : f : B_0 \to 2,\ f \text{ is a Boolean homomorphism}\} \cup \{\{\emptyset\}\}$$

be the discrete topological space. Then each X_{B_0} is compact Hausdorff. By the assumption (G), it follows that $Y := \prod_{B_0\in\mathcal{S}} X_{B_0}$ is compact.

For any distinct B_0, $B_0' \in \mathcal{S}$ with $B_0 \leq B_0'$, let

$$C_{B_0,B_0'} := \{\varphi \in Y : \varphi(B_0) \subseteq \varphi(B_0')\}.$$

The following is clear since the topology on each X_{B_0} for $B_0 \in \mathcal{S}$ is discrete and $X_{B_0} \times X_{B_0'}$ for B_0, $B_0' \in \mathcal{S}$ is finite.

Claim 4.2.1 *For any distinct* B_0, $B_0' \in \mathcal{S}$ *with* $B_0 \leq B_0'$, $C_{B_0,B_0'}$ *is closed subset of* Y. ❏

Let $\mathcal{B} := \{C_{B_0,B_0'} : B_0, B_0' \in \mathcal{S},\ B_0 \leq B_0'\}$.

Claim 4.2.2 *$\mathcal{B}$ has the fip.*

$\vdash$ Suppose that $B_{0,0}, \ldots, B_{0,k-1}, B_{0,0}', \ldots, B_{0,k-1}' \in \mathcal{S}$ and $B_{0,0} \leq B_{0,0}', \ldots, B_{0,k-1} \leq B_{0,k-1}'$. Let $B_0^* \in \mathcal{S}$ be such that $B_{0,\ell}' \leq B_0^*$ for all $\ell < k$. Since B_0^* is finite we can find a Boolean homomorphism $f : B_0^* \to 2$ (without appealing to AC for help). Let $\varphi \in Y$ be defined by

$$\varphi(B_0) := \begin{cases} f \restriction B_0, & \text{if } B_0 = B_{0,\ell} \text{ or } B_0 = B_{0,\ell}' \text{ for some } \ell < k; \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

for $B_0 \in \mathcal{S}$. Then $\varphi \in C_{B_{0,0},B_{0,0}'} \cap \ldots \cap C_{B_{0,k-1},B_{0,k-1}'}$ $\dashv$ (Claim 4.2.2)

Since Y is compact $\bigcap \mathcal{B}$ is non-empty. Let $\varphi \in \cap\mathcal{B}$. Then $\Phi = \bigcup\{\varphi(B_0) : B_0 \in \mathcal{S}\}$ is a Boolean homomorphism with $\Phi : B \to 2$, and $\Phi^{-1}{}''\{0\}$ is a prime ideal on B. ❏ (Theorem 4.2)

Theorem 4.3 *For any infinite cardinal κ, we have the implication* (H) $\Rightarrow$ (I) $\Rightarrow$ (J) $\Rightarrow$ (K) $\Rightarrow$ (L) , *where*

- (H) DC_κ.
- (I) *For any sequence $\langle X_\alpha : \alpha \in \kappa\rangle$ of compact topological spaces, the product $\prod_{\alpha\in\kappa} X_\alpha$ is compact.*
- (J) AC_κ.
- (K) *For any sequence $\langle X_\alpha : \alpha \in \kappa\rangle$ of compact topological spaces where each set X_α ($\alpha \in \kappa$) is well-orderable, the product $\prod_{\alpha\in\kappa} X_\alpha$ is compact.*
- (L) $\mathsf{AC}_\kappa^{\mathsf{WO}}$.

Proof. "(H) $\Rightarrow$ (I)": By Corollary 3.3.

"(I) $\Rightarrow$ (J)": The proof of Theorem 4.1, "(D) $\Rightarrow$ (A)" for $I = \kappa$ will do.

"(J) $\Rightarrow$ (K)": Suppose that $\langle X_\alpha : \alpha < \kappa\rangle$ is as in the statement of (K) . By AC_κ, we can choose a sequence $\langle R_\alpha : \alpha < \kappa\rangle$ such that each R_α ($\alpha < \kappa$) is a well ordering of X_α. For a filter $\mathcal{F}$ over $\prod_{\alpha<\kappa} X_\alpha$, we can construct $\langle a_\alpha : \alpha < \kappa\rangle$,

$\langle \mathcal{F}^0_\alpha : \alpha < \kappa \rangle$, $\langle \mathcal{F}_\alpha : \alpha \leq \kappa \rangle$ as in the second proof of Theorem 3.1 using this $\langle R_\alpha : \alpha < \kappa \rangle$. The same argument as in the proof shows that $\mathcal{F} \to \langle a_\alpha : \alpha \in \kappa \rangle$.

"(K) $\Rightarrow$ (L)": Again the proof of Theorem 4.1, "(D) $\Rightarrow$ (A)" for $I = \kappa$ works for $\langle X_\alpha : \alpha \in \kappa \rangle$ where each X_α is well-orderable. □ (Theorem 4.3)

It is clear that $\mathsf{AC}^{\mathsf{wo}}_\kappa$ implies $\mathsf{AC}_\kappa(\mathsf{fin})$. The latter can be characterized by a weakening of Tychonoff Theorem.

Theorem 4.4 *For an infinite cardinal κ, The following are equivalent:*

(M) $\mathsf{AC}_\kappa(\mathsf{fin})$.

(N) *For any sequence $\langle X_\alpha : \alpha \in \kappa \rangle$ of finite spaces, $\prod_{\alpha \in \kappa} X_\alpha$ is compact.*

Proof. "(M) $\Rightarrow$ (N)": By Corollary 3.3.

"(N) $\Rightarrow$ (M)": The proof of Theorem 4.1, "(D) $\Rightarrow$ (A)" for $I = \kappa$ will do. Note that, if X_i is finite, then Y_i in the proof of Theorem 4.1, "(D) $\Rightarrow$ (A)" is also finite. □ (Theorem 4.4)

References

[1] Paul Alexandroff and Paul Urysohn, Zur Theorie der topologischen Räume, Mathematische Annalen 92 (1924), 258–266.

[2] Sakaé Fuchino, A reflection principle as a reverse-mathematical fixed point over the base theory ZFC, Annals of the Japan Association for the Philosophy of Science, Vol.25, (2017), 67–77.

[3] Lorenz J. Halbeisen, Combinatorial Set Theory, With a Gentle Introduction to Forcing, Springer (2012).

[4] Felix Hausdorff, Grundzüge der Mengenlehre, Veit& Comp., Leipzig (1914).

[5] Horst Herrlich, Axiom of Choice, Lecture Notes in Mathematics, 1876, Springer (2006).

[6] Thomas Jech, The Axiom of Choice, Dover Publications, Inc., (1973/2008).

[7] Thomas Jech, Set Theory, The Third Millennium Edition, Springer (2001/2006).

[8] Akihiro Kanamori, The Higher Infinite, Springer–Verlag (1994/2003).

[9] John L. Kelley, Tychonoff's theorem implies AC, Fundamenta Mathematicae 37, (1950), 75–76.

[10] Jerzy Łoś and Czesław Ryll-Nardzewski, Effectiveness of the representation theory for Boolean algebras, Fundamenta Mathematicae 41, (1955), 49–56.

[11] Alfred Tarski, Sur quelques theorems qui equivalent a l'axiome du choix, Fundamenta Mathematicae, 5, (1924), 147 – 154.

[12] A. Tychonoff, Über die topologische Erweitrerung von Räumen, Mathematische Annalen 102 (1930), 544–561.

[13] Wikipedia, Heine-Borel Theorem, https://en.wikipedia.org/wiki/Heine-Borel_theorem.

[14] Ernst Zermelo, Beweis, daß jede Menge wohlgeordnet werden kann (Aus einem an Herrn Hilbert gerichteten Briefe), Mathematische Annalen, Vol. 59, (1904), 514–516.