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ACTIONS OF AUTOMORPHISM GROUPS OF FREE GROUPS ON SPACES OF JACOBI DIAGRAMS. II

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Abstract The automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n acts on a space $A_d(n)$ of Jacobi diagrams of degree d on n oriented arcs. We study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$ by using two actions on the associated graded vector space of $A_d(n)$: an action of the general linear group $\mathrm{GL}(n,\mathbb{Z})$ and an action of the graded Lie algebra $\mathrm{gr}(\mathrm{IA}(n))$ of the IA-automorphism group $\mathrm{IA}(n)$ of F_n associated with its lower central series. We extend the action of $\mathrm{gr}(\mathrm{IA}(n))$ to an action of the associated graded Lie algebra of the Andreadakis filtration of the endomorphism monoid of F_n. By using this action, we study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$. We obtain an indecomposable decomposition of $A_d(n)$ as $\mathrm{Aut}(F_n)$-modules for $n \geq 2d$. Moreover, we obtain the radical filtration of $A_d(n)$ for $n \geq 2d$ and the socle of $A_3(n)$.

Key words and phrases: Jacobi diagrams, automorphism groups of free groups, general linear groups, IA-automorphism groups of free groups, Andreadakis filtration

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Contents

1. Introduction

Jacobi diagrams are uni-trivalent graphs, which graphically encode the algebraic structures of Lie algebras and their representations. Jacobi diagrams were introduced for the *Kontsevich integral*, which is a universal finite type link invariant and unifies all quantum link invariants [2, 18, 15, 19]. The associated graded vector space of finite type link invariants is isomorphic to the space of *weight systems*, which is the dual to the space of Jacobi diagrams.

Let $\Bbbk$ be a field of characteristic 0. We study the $\Bbbk$-vector space $A(n)$ of Jacobi diagrams on n-component oriented arcs, which is the target space of the Kontsevich integral for *string links* [8, 3] or *bottom tangles* [9]. We consider the degree d part $A_d(n)$ of $A(n)$, where the degree of a Jacobi diagram is determined by half the number of its vertices. The space $A_d(n)$ encodes the universal enveloping algebra $U(\mathfrak{g})$ of any finite-dimensional semisimple Lie algebra $\mathfrak{g}$. More precisely, the weight system maps $A_d(n)$ to the $\mathfrak{g}$-invariant part of $U(\mathfrak{g})^{\otimes n}$.

We consider a filtration for $A_d(n)$ defined by the number of trivalent vertices. The associated graded vector space of $A_d(n)$ is identified via the PBW (Poincaré–Birkhoff–Witt) map [2, 3] with a graded vector space $B_d(n)$ of *open Jacobi diagrams* of degree d that are colored by elements of an n-dimensional $\Bbbk$-vector space. For a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, the weight system maps $B_d(n)$ to the $\mathfrak{g}$-invariant part of the tensor product $\mathfrak{S}(\mathfrak{g})^{\otimes n}$ of the symmetric algebra $\mathfrak{S}(\mathfrak{g})$ of $\mathfrak{g}$.

In a previous paper [16], we proved that the vector spaces $A_d(n)$ define a functor $A_d : \mathbf{F}^{\mathrm{op}} \to \mathbf{fVect}$ from the opposite category $\mathbf{F}^{\mathrm{op}}$ of the category $\mathbf{F}$ of finitely generated free groups to the category $\mathbf{fVect}$ of filtered vector spaces. By functoriality on $\mathbf{F}^{\mathrm{op}}$, $A_d(n)$ inherits an action of the *automorphism group* $\mathrm{Aut}(F_n)$ and of the *endomorphism monoid* $\mathrm{End}(F_n)$ of the free group F_n of rank n. We proved in [16] that the action of $\mathrm{Aut}(F_n)$ on $A_d(n)$ induces an action of the *outer automorphism group* $\mathrm{Out}(F_n)$ of F_n on $A_d(n)$ and we observed that the $\mathrm{Aut}(F_n)$-action on $A_d(n)$ induces two actions on $B_d(n)$: an action of the *general linear group* $\mathrm{GL}(n;\mathbb{Z})$ and an action of the graded Lie algebra $\mathrm{gr}(\mathrm{IA}(n))$ of the *IA-automorphism group* $\mathrm{IA}(n)$ of F_n associated with the lower central series. We used these two actions on $B_d(n)$ to study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$ for $d=2$. However, it is rather difficult to compute the $\mathrm{gr}(\mathrm{IA}(n))$-action on $B_d(n)$ directly for general d.

The aim of the present paper is to study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$ for general d and especially $d=3$ in detail. We consider the *Andreadakis filtration* $\mathcal{E}_*(n)$ of the endomorphism monoid $\mathrm{End}(F_n)$ of F_n. We extend the action of the graded Lie algebra $\mathrm{gr}(\mathrm{IA}(n))$ to an action of the associated graded Lie algebra $\mathrm{gr}(\mathcal{E}_*(n))$ of the Andreadakis filtration. On the other hand, we construct a graphical version of the $\mathrm{gr}(\mathcal{E}_*(n))$-action on $B_d(n)$. By using this graphical action, we study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$. We obtain an indecomposable decomposition of $A_d(n)$ as $\mathrm{Aut}(F_n)$-modules for $n \geq 2d$. Moreover, we obtain the radical filtration of $A_d(n)$ for $n \geq 2d$ and the socle of $A_3(n)$.

1.1. Andreadakis filtration of $\mathrm{End}(F_n)$

Let $\Gamma_r := \Gamma_r(F_n)$ denote the r-th term of the lower central series of the free group F_n. Let $\mathcal{L}_r(n) := \Gamma_r/\Gamma_{r+1}$ for $r \geq 1$, and set $H := \mathcal{L}_1(n)$. Note that $\mathcal{L}_r(n)$ is the degree r part of the free Lie algebra $\mathcal{L}_*(n)$ on H.

Let $\mathrm{IA}(n)$ denote the IA-automorphism group of F_n, which is the kernel of the canonical homomorphism $\mathrm{Aut}(F_n) \to \mathrm{GL}(n;\mathbb{Z})$.

The *Andreadakis filtration* $\mathcal{A}_*(n)$ of $\mathrm{Aut}(F_n)$ [1, 22]

$$\mathrm{Aut}(F_n) = \mathcal{A}_0(n) \supset \mathcal{A}_1(n) = \mathrm{IA}(n) \supset \mathcal{A}_2(n) \supset \cdots$$

is defined by

$$\mathcal{A}_r(n) = \ker(\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/\Gamma_{r+1})).$$

For $r \geq 1$, we have an injective homomorphism

$$\tau_r : \mathrm{gr}^r(\mathcal{A}_*(n)) \hookrightarrow \mathrm{Hom}(H, \mathcal{L}_{r+1}(n)),$$

which is called the *Johnson homomorphism*. By Andreadakis [1] and Kawazumi [17], we have $\mathrm{gr}^1(\mathrm{IA}(n)) \cong \mathrm{gr}^1(\mathcal{A}_*(n)) \cong \mathrm{Hom}(H, \mathcal{L}_2(n))$.

We construct the Andreadakis filtration $\mathcal{E}_*(n)$ of $\mathrm{End}(F_n)$ in a similar way by

$$\mathcal{E}_r(n) = \ker(\mathrm{End}(F_n) \to \mathrm{End}(F_n/\Gamma_{r+1})).$$

We define an equivalence relation on the monoid $\mathcal{E}_r(n)$ and consider the quotient group $\mathrm{gr}^r(\mathcal{E}_*(n))$, which includes $\mathrm{gr}^r(\mathcal{A}_*(n))$ (see Section 3.3). We also construct the Johnson homomorphism

$$\tilde{\tau}_r : \mathrm{gr}^r(\mathcal{E}_*(n)) \xrightarrow{\cong} \mathrm{Hom}(H, \mathcal{L}_{r+1}(n))$$

of $\mathrm{End}(F_n)$, which turns out to be an abelian group isomorphism (see Proposition 3.8).

The target group $\mathrm{Hom}(H, \mathcal{L}_{r+1}(n)) \cong H^* \otimes \mathcal{L}_{r+1}(n)$ of the Johnson homomorphism is identified with the degree r part $\mathrm{Der}_r(\mathcal{L}_*(n))$ of the derivation Lie algebra $\mathrm{Der}(\mathcal{L}_*(n))$ of the free Lie algebra $\mathcal{L}_*(n)$ and with the tree module $T_r(n)$, which we define in Section 3.2. From the above, we have abelian group isomorphisms

$$\mathrm{gr}^r(\mathcal{E}_*(n)) \cong H^* \otimes \mathcal{L}_{r+1}(n) \cong \mathrm{Der}_r(\mathcal{L}_*(n)) \cong T_r(n).$$

Thus, we have

$$\mathrm{gr}^1(\mathrm{IA}(n)) \cong \mathrm{gr}^1(\mathcal{E}_*(n)) \cong H^* \otimes \mathcal{L}_2(n) \cong \mathrm{Der}_1(\mathcal{L}_*(n)) \cong T_1(n).$$

Moreover, we have isomorphisms of graded Lie algebras

$$\mathrm{gr}(\mathcal{E}_*(n)) = \bigoplus_{r\geq 1} \mathrm{gr}^r(\mathcal{E}_*(n)) \cong \mathrm{Der}(\mathcal{L}_*(n)) \cong \bigoplus_{r \geq 1} T_r(n) \tag{1.1}$$

(see Section 3.5). In what follows, we identify these three graded Lie algebras.

1.2. Actions of the derivation Lie algebra on $B_d(n)$

Let $A_d(n)$ be the $\Bbbk$-vector space spanned by Jacobi diagrams of degree d on n oriented arcs. We consider a filtration for $A_d(n)$

$$A_d(n) = A_{d,0}(n) \supset A_{d,1}(n) \supset A_{d,2}(n) \supset \cdots,$$

where $A_{d,k}(n)$ is the subspace of $A_d(n)$ spanned by Jacobi diagrams with at least k trivalent vertices. By restricting the functor $A_d : \mathbf{F}^{\mathrm{op}} \to \mathbf{fVect}$ that we defined in [16] to the endomorphisms, we obtain an action of $\mathrm{End}(F_n)$ on $A_d(n)$. (See Section 2.3 and Section 4.)

Let V_n be an n-dimensional $\Bbbk$-vector space, which will be identified with the first cohomology of a handlebody of genus n. The associated graded vector space of $A_d(n)$ is isomorphic via the PBW map [3] to a graded vector space $B_d(n) = \bigoplus_{k\geq 0} B_{d,k}(n)$ of V_n-colored open Jacobi diagrams of degree d, where $B_{d,k}(n)$ is the subspace of $B_d(n)$ spanned by open Jacobi diagrams with exactly k trivalent vertices.

We defined in [16] a $\mathrm{gr}(\mathrm{IA}(n))$-action on $B_d(n)$ by using the bracket map

$$[\cdot,\cdot] : B_{d,k}(n) \otimes_{\mathbb{Z}} \mathrm{gr}^r(\mathrm{IA}(n)) \to B_{d,k+r}(n).$$

We extend the $\mathrm{gr}(\mathrm{IA}(n))$-action to an action of $\mathrm{gr}(\mathcal{E}_*(n))$ on $B_d(n)$.

We define a $\Bbbk$-linear map

$$[\cdot,\cdot] : B_{d,k}(n) \otimes_{\mathbb{Z}} \mathrm{gr}^r(\mathcal{E}_*(n)) \to B_{d,k+r}(n)$$

by using the following theorem.

Theorem 1.1 (see Theorem 4.1). *For any $r \geq 1$, we have*

$$[A_{d,k}(n), \mathcal{E}_r(n)] \subset A_{d,k+r}(n).$$

To prove this theorem, we introduce a category $\mathbf{A}^L$, which includes as full subcategories the category $\mathbf{A}$ of Jacobi diagrams in handlebodies and the category isomorphic to the PROP for Casimir Lie algebras [13]. (See Section 4 and Appendix A).

By using the bracket maps, we obtain $\Bbbk$-linear maps

$$\tilde{\beta}^r_{d,k} : \mathrm{gr}^r(\mathcal{E}_*(n)) \to \mathrm{Hom}(B_{d,k}(n), B_{d,k+r}(n)),$$

which form an action of the graded Lie algebra $\mathrm{gr}(\mathcal{E}_*(n))$ on the graded vector space $B_d(n)$.

We also define a $\Bbbk$-linear map

$$c : B_{d,k}(n) \otimes_{\mathbb{Z}} T_r(n) \to B_{d,k+r}(n),$$

which is an analogue of the contraction map for a vector space and its dual vector space (see Section 5). By using the map c, we obtain $\Bbbk$-linear maps

$$\gamma^r_{d,k} : T_r(n) \to \mathrm{Hom}(B_{d,k}(n), B_{d,k+r}(n)),$$

which form an action of the graded Lie algebra $\bigoplus_{r\geq 1} T_r(n)$ on the graded vector space $B_d(n)$.

Via the isomorphisms (1.1), these two actions of the derivation Lie algebra $\mathrm{Der}(\mathcal{L}_*(n))$ on $B_d(n)$ coincide up to sign. (See Theorem 6.1.)

By using the linear map c for computation, we obtain the surjectivity of the bracket map.

Proposition 1.2 (see Proposition 7.8). *For $n \geq 2d-k$, the bracket map*

$$[\cdot,\cdot] : B_{d,k}(n) \otimes_{\mathbb{Z}} \mathrm{gr}^1(\mathrm{IA}(n)) \to B_{d,k+1}(n)$$

is surjective.

1.3. The $\mathrm{GL}(n;\mathbb{Z})$-module structure of $B_d(n)$

The $\mathrm{GL}(n;\mathbb{Z})$-action on $B_d(n)$ that is induced by the $\mathrm{Aut}(F_n)$-action on $A_d(n)$ naturally extends to a polynomial $\mathrm{GL}(V_n)$-action on $B_d(n)$ [16]. Therefore, the $\mathrm{GL}(V_n)$-module $B_d(n)$ can be decomposed into the direct sum of images of the Schur functors. In general, however, it remains open to obtain an irreducible decomposition of $B_d(n)$ as $\mathrm{GL}(V_n)$-modules. We can reduce this problem to the connected parts $B^c_{d,k}(n) \subset B_{d,k}(n)$ (see Theorem 7.2).

For a partition $\lambda \vdash N$, let V_λ denote the image of V_n under the Schur functor $\mathbb{S}_\lambda$. By using the results by Bar-Natan [4], we have isomorphisms of $\mathrm{GL}(V_n)$-modules

$$B_3(n) = B_{3,0}(n) \oplus \cdots \oplus B_{3,4}(n),$$

where

$$\begin{aligned}
B_{3,0}(n) &\cong V_{(6)} \oplus V_{(4,2)} \oplus V_{(2^3)},\\
B_{3,1}(n) &\cong V_{(3,1^2)} \oplus V_{(2,1^3)},\\
B_{3,2}(n) &\cong V_{(4)} \oplus V_{(3,1)} \oplus (V_{(2^2)})^{\oplus 2},\\
B_{3,3}(n) &= B^c_{3,3} \cong V_{(1^3)},\\
B_{3,4}(n) &= B^c_{3,4} \cong V_{(2)}
\end{aligned}$$

(see Proposition 7.6 for the cases $d = 3,4,5$).

In general degrees, we obtain irreducible decompositions of $B_{d,k}(n)$ as $\mathrm{GL}(V_n)$-modules for $k = 0,1$.

Proposition 1.3 (see Proposition 7.7). *For any $d \geq 1$, we have*

$$B_{d,0}(n) \cong \bigoplus_{\lambda \vdash d} V_{2\lambda},$$

where $2\lambda = (2\lambda_1, \cdots, 2\lambda_r) \vdash 2d$ for $\lambda = (\lambda_1, \cdots, \lambda_r) \vdash d$. For any $d \geq 2$, we have

$$B_{d,1}(n) \cong \bigoplus_{\lambda \vdash 2d-1 \text{ with exactly 3 odd parts}} V_\lambda.$$

1.4. The $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$

We consider the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$ and give an indecomposable decomposition of $A_d(n)$. We have

$$A_0(n) = \Bbbk \quad (n \geq 0), \quad A_d(0) = 0 \quad (d \geq 1)$$

and we studied the cases where $d = 1,2$ in [16]. Thus, we mainly consider the cases where $d \geq 3, n \geq 1$.

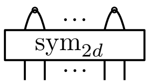
For $X \in A_d(2d)$, let

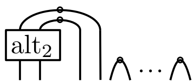
$$A_d X : \mathbf{F}^{\mathrm{op}} \to \mathbf{fVect}$$

denote the subfunctor of A_d generated by X. That is, for any $n \in \mathbb{N}$, $A_d X(n)$ is the $\mathrm{Aut}(F_n)$-submodule of $A_d(n)$ defined by

$$A_d X(n) := \mathrm{Span}_{\Bbbk}\{A_d(f)(X) \mid f \in \mathbf{F}^{\mathrm{op}}(2d,n)\}.$$

Set

$P =$ 

, $Q =$ 

$\in A_d(2d)$.

Then, we have the following direct decomposition of $A_d(n)$ as $\mathrm{Aut}(F_n)$-modules, which is indecomposable for $n \geq 2d$.

Theorem 1.4 (see Theorems 8.2, 8.9). *We have $A_d(n) = A_d P(n) \oplus A_d Q(n)$ for any $d, n \geq 1$. This direct decomposition is indecomposable for $n \geq 2d$.*

In degree 1, we have $A_1 Q(n) = 0$ and $A_1(n) \cong \mathrm{Sym}^2(V_n)$ is simple for $n \geq 1$. In [16], we obtained that the direct decomposition of $A_2(n)$ is indecomposable for $n \geq 3$ (see Theorem 6.9 of [16]). We improve Theorem 1.4 for $d = 3,4$ (see Theorems 8.12 and 8.17).

In general degree d, we obtain the radical of $A_{d,k}(n)$ for any $k \geq 0$ if $n \geq 2d$.

Theorem 1.5 (see Theorem 8.6). *Let $n \geq 2d$. The filtration of $A_d(n)$ by the number of trivalent vertices coincides with the radical filtration of $A_d(n)$.*

In degree 3, we obtain the socle of $A_3(n)$ as well (see Proposition 8.15).

1.5. Direct decomposition of the functor A_d

Lastly, we give an indecomposable decomposition of the functor A_d.

By Theorem 1.4, we obtain an indecomposable decomposition of the functor A_d.

Theorem 1.6 (see Theorem 10.1). *We have an indecomposable decomposition*

$$A_d = A_d P \oplus A_d Q \tag{1.2}$$

in the functor category $\mathbf{fVect}^{\mathbf{F}^{\mathrm{op}}}$.

In degree 1, we have $A_1 Q = 0$ and $A_1 = A_1 P$. In [16], we obtained the direct decomposition (1.2) of the functor A_2 and proved that equation (1.2) is indecomposable (see Proposition 6.5 and Theorem 6.14 of [16]).

1.6. Organization of the paper

In Section 2, we recall the category $\mathbf{A}$ of Jacobi diagrams in handlebodies, N-series and graded Lie algebras, contents of the previous paper [16], Hopf algebras and Lie algebras in a linear symmetric strict monoidal category. In Section 3, we construct the Andreadakis filtration and the Johnson homomorphism of $\mathrm{End}(F_n)$. In Section 4, we construct an action of the derivation Lie algebra $\mathrm{Der}(\mathcal{L}_*(n))$ on $B_d(n)$, which is defined by the bracket map. In preparation for the definition of the bracket map, we construct an extended category $\mathbf{A}^L$ of the category $\mathbf{A}$, which includes a Lie algebra structure. In Section 5, we define a contraction map, which forms another action of $\mathrm{Der}(\mathcal{L}_*(n))$ on $B_d(n)$. In Section 6, we prove that two actions of $\mathrm{Der}(\mathcal{L}_*(n))$ on $B_d(n)$ defined in Sections 4 and 5 coincide up to sign. In Section 7, we compute the $\mathrm{GL}(n;\mathbb{Z})$-module structure of $B_d(n)$. In Section 8, we study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$ by using the $\mathrm{GL}(n;\mathbb{Z})$-module structure of $B_d(n)$ and the action of $\mathrm{Der}(\mathcal{L}_*(n))$ on $B_d(n)$. In Section 10, we give an indecomposable decomposition of the functor A_d. In Appendix A, we study an expected presentation of the category $\mathbf{A}^L$.

2. Preliminaries

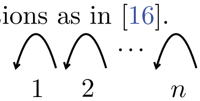
In this section, we recall the contents of the previous paper [16] and definitions of the category $\mathbf{A}$ of Jacobi diagrams in handlebodies, Hopf algebras and Lie algebras in a symmetric strict monoidal category and an action of an N-series on a filtered vector space and that of a graded Lie algebra on a graded vector space.

In what follows, we work over a fixed field $\Bbbk$ of characteristic 0. For a vector space V and an abelian group G, we just write $V \otimes G$ instead of $V \otimes_{\mathbb{Z}} G$. For vector spaces V and W, we also write $V \otimes W$ instead of $V \otimes_{\Bbbk} W$.

For $n \geq 0$, let $[n] := \{1, \cdots, n\}$.

2.1. The category A of Jacobi diagrams in handlebodies

Here, we briefly review the category $\mathbf{A}$ of Jacobi diagrams in handlebodies defined in [11]. We use the same notations as in [16].

For $n \geq 0$, let $X_n =$  be the oriented 1-manifold consisting of n arc components.

Let $I = [-1,1]$. For $n \geq 0$, let $U_n \subset \mathbb{R}^3$ denote the handlebody of genus n that is obtained from the cube I^3 by attaching n handles on the top square $I^2 \times \{1\}$ as depicted in Figure 1. We call $l := I \times \{0\} \times \{-1\}$ the *bottom line* of U_n and $l' := I \times \{0\} \times \{1\}$ the *upper line* of U_n. We call $S := I^2 \times \{-1\}$ the *bottom square* of U_n.

For $i \in [n]$, let x_i be a loop which goes through only the i-th handle of the handlebody U_n just once, and let x_i denote its homotopy class as well. In what follows, for loops γ_1 and γ_2 with base points on l, let $\gamma_2\gamma_1$ denote the loop that goes through γ_1 first and then goes through γ_2. That is, we write a product of elements of the fundamental group of U_n in the opposite order to the usual one. Let $H = H_1(U_n;\mathbb{Z})$, and let $\bar{x}_i \in H$ be the

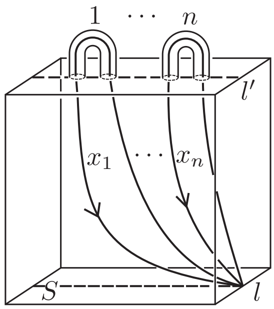


Figure 1. The handlebody U_n.

homology class of x_i. We have $H = \bigoplus_{i=1}^{n} \mathbb{Z}\bar{x}_i$ and $\pi_1(U_n) = \langle x_1, \cdots, x_n \rangle$. Let

$$V_n = H^1(U_n;\Bbbk) = \mathrm{Hom}(H,\Bbbk),$$

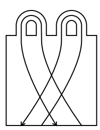
and let $\{v_1, \cdots, v_n\}$ be the dual basis of $\{\bar{x}_1, \cdots, \bar{x}_n\}$.

The objects in $\mathbf{A}$ are nonnegative integers.

For $m,n \geq 0$, the hom-set $\mathbf{A}(m,n)$ is the $\Bbbk$-vector space spanned by (m,n)-*Jacobi diagrams* modulo the STU relation. An (m,n)-Jacobi diagram is a Jacobi diagram on X_n mapped into U_m in such a way that the endpoints of X_n are uniformly distributed on the bottom line l of U_m (see [11, 16] for further details). We usually depict (m,n)-Jacobi diagrams by drawing their images under the orthogonal projection of $\mathbb{R}^3$ onto $\mathbb{R} \times \{0\} \times \mathbb{R}$.

The *degree* of an (m,n)-Jacobi diagram is the degree of its Jacobi diagram. Let $\mathbf{A}_d(m,n) \subset \mathbf{A}(m,n)$ be the subspace spanned by (m,n)-Jacobi diagrams of degree d. We have $\mathbf{A}(m,n) = \bigoplus_{d \geq 0} \mathbf{A}_d(m,n)$.

The category $\mathbf{A}$ has a structure of a linear symmetric strict monoidal category. The tensor product on objects is addition. The monoidal unit is 0. The tensor product on morphisms is juxtaposition followed by horizontal rescaling and relabelling of indices. The symmetry is determined by

$P_{1,1} =$  $: 2 \to 2.$

2.2. N-series and graded Lie algebras

Here, we briefly review the definition of an action of an N-series on a filtered vector space and the induced action of the graded Lie algebra on the graded vector space (see [16] for details).

An *N-series* $K_* = (K_n)_{n \geq 1}$ of a group K is a descending series

$$K = K_1 \supset K_2 \supset \cdots$$

such that $[K_n, K_m] \subset K_{n+m}$ for all $n,m \geq 1$.

A *morphism* $f: G_* \to K_*$ between N-series is a group homomorphism $f: G_1 \to K_1$ such that we have $f(G_n) \subset K_n$ for all $n \geq 1$.

For a filtered vector space W_*, set

$$\mathrm{Aut}_n(W_*) := \{\phi \in \mathrm{Aut}_{\mathbf{fVect}}(W_*) \mid [\phi, w] \in W_{k+n} \text{ for all } w \in W_k, k \geq 0\} \quad (n \geq 1),$$

where $[\phi, w] := \phi(w) - w$ for $w \in W_k$. We can easily check that $\mathrm{Aut}_*(W_*) := (\mathrm{Aut}_n(W_*))_{n \geq 1}$ is an N-series.

Definition 2.1. (Action of N-series on filtered vector spaces) Let K_* be an N-series and W_* be a filtered vector space. An *action* of K_* on W_* is a morphism $f: K_* \to \mathrm{Aut}_*(W_*)$ between N-series.

For an N-series K_*, we have a graded Lie algebra $\mathrm{gr}(K_*) = \bigoplus_{n \geq 1} K_n/K_{n+1}$, where the Lie bracket is defined by the commutator.

For a graded vector space $W = \bigoplus_{k \geq 0} W_k$, set

$$\mathrm{End}_n(W) := \{\phi \in \mathrm{End}(W) \mid \phi(W_k) \subset W_{k+n} \text{ for } k \geq 0\} \quad (n \geq 1).$$

We can check that $\mathrm{End}_+(W) = \bigoplus_{n \geq 1} \mathrm{End}_n(W)$ is a graded Lie algebra, where the Lie bracket is defined by

$$[f, g] := f \circ g - g \circ f \quad \text{for} \quad f \in \mathrm{End}_k(W), g \in \mathrm{End}_l(W) \ (k, l \geq 1).$$

Definition 2.2. (Action of graded Lie algebras on graded vector spaces) Let $L_+ = \bigoplus_{n \geq 1} L_n$ be a graded Lie algebra and $W = \bigoplus_{k \geq 0} W_k$ be a graded vector space. An *action* of L_+ on W is a morphism $f: L_+ \to \mathrm{End}_+(W)$ between graded Lie algebras.

Proposition 2.3. *An action of an N-series K_* on a filtered vector space W_* induces an action of the graded Lie algebra $\mathrm{gr}(K_*)$ on the graded vector space $\mathrm{gr}(W_*)$, which is a morphism*

$$\rho_+ : \bigoplus_{n \geq 1} \mathrm{gr}^n(K_*) \to \bigoplus_{n \geq 1} \mathrm{End}_n(\mathrm{gr}(W_*))$$

defined by $\rho_+(gK_{n+1})([v]_{W_{k+1}}) = [[g, v]]_{W_{k+n+1}}$ for $gK_{n+1} \in \mathrm{gr}^n(K_*)$, $[v]_{W_{k+1}} \in \mathrm{gr}^k(W_*)$.

The proof can be seen in Proposition 5.14 of [16].

2.3. Contents of the previous paper

Here, we briefly review the notations and contents of the previous paper [16]. Let $\mathrm{Aut}(F_n)$ denote the automorphism group of the free group F_n of rank n and $\mathrm{GL}(n;\mathbb{Z})$ the general linear group of degree n. Let $\mathrm{IA}(n)$ denote the IA-automorphism group of F_n, that is the kernel of the canonical surjection

$$\mathrm{Aut}(F_n) \to \mathrm{Aut}(H_1(F_n;\mathbb{Z})) \cong \mathrm{GL}(n;\mathbb{Z}).$$

Let $\Gamma_*(\mathrm{IA}(n)) = (\Gamma_r(\mathrm{IA}(n)))_{r \geq 1}$ denote the lower central series of $\mathrm{IA}(n)$, and $\mathrm{gr}(\mathrm{IA}(n)) = \bigoplus_{r \geq 1} \mathrm{gr}^r(\mathrm{IA}(n))$ the associated graded Lie algebra, where $\mathrm{gr}^r(\mathrm{IA}(n)) = \Gamma_r(\mathrm{IA}(n))/\Gamma_{r+1}(\mathrm{IA}(n))$.

Let $A_d(n) = \mathbf{A}_d(0,n)$ denote the $\Bbbk$-vector space of Jacobi diagrams of degree d on X_n. We consider a filtration for $A_d(n)$

$$A_d(n) = A_{d,0}(n) \supset A_{d,1}(n) \supset \cdots \supset A_{d,2d-2}(n) \supset A_{d,2d-1}(n) = 0$$

such that $A_{d,k}(n) \subset A_d(n)$ is the subspace spanned by Jacobi diagrams with at least k trivalent vertices. Hence, $A_d(n)$ is a filtered vector space.

Let $\mathbf{F}$ denote the category of finitely generated free groups and **fVect** the category of filtered vector spaces over $\Bbbk$.

We have a $\Bbbk$-vector space isomorphism

$$Z : \Bbbk\mathbf{F}^{\mathrm{op}}(m,n) \xrightarrow{\cong} \mathbf{A}_0(m,n)$$

from the hom-set $\Bbbk\mathbf{F}^{\mathrm{op}}(m,n)$ of the $\Bbbk$-linearization of the opposite category of $\mathbf{F}$ to the degree 0 part of the hom-set $\mathbf{A}(m,n)$ [11]. We define a functor

$$A_d : \mathbf{F}^{\mathrm{op}} \to \mathbf{fVect}$$

by $A_d(n) = \mathbf{A}_d(0,n)$ for an object $n \in \mathbb{N}$ and $A_d(f) = Z(f)_*$ for a morphism $f \in \mathbf{F}^{\mathrm{op}}(m,n)$, where $Z(f)_*$ denotes the post-composition with $Z(f)$. The functor A_d is a polynomial functor of degree $2d$ in the sense of [12, 20] (see Remark 3.1 of [16]). By restricting this functor to the automorphism group, we obtain an action of the opposite group $\mathrm{Aut}(F_n)^{\mathrm{op}}$ of $\mathrm{Aut}(F_n)$ on $A_d(n)$ for each $n \geq 0$. We consider this action as a right action of $\mathrm{Aut}(F_n)$ on $A_d(n)$. The $\mathrm{Aut}(F_n)$-action on $A_d(n)$ induces an action on $A_d(n)$ of the outer automorphism group $\mathrm{Out}(F_n)$ of F_n (see Theorem 5.1 in [16]).

On the other hand, the associated graded vector space $\mathrm{gr}(A_d(n))$ of $A_d(n)$ is identified via the PBW map [2, 3]

$$\theta_{d,n} : \mathrm{gr}(A_d(n)) \xrightarrow{\cong} B_d(n) \tag{2.1}$$

with the graded $\Bbbk$-vector space $B_d(n) = \bigoplus_{k\geq 0} B_{d,k}(n) = \bigoplus_{k=0}^{2d-2} B_{d,k}(n)$ of V_n-colored open Jacobi diagrams of degree d, where the grading is determined by the number of trivalent vertices. Note that we have $\theta_{d,n} = \bigoplus_k \theta_{d,n,k}$, where

$$\theta_{d,n,k} : \mathrm{gr}^k(A_d(n)) \xrightarrow{\cong} B_{d,k}(n).$$

Let **FAb** denote the category of finitely generated free abelian groups and **gVect** the category of graded vector spaces over $\Bbbk$.

We define a functor

$$B_d : \mathbf{FAb}^{\mathrm{op}} \to \mathbf{gVect}$$

by sending an object $n \in \mathbb{N}$ to the graded vector space $B_d(n)$ and a morphism $f \in \mathbf{FAb}^{\mathrm{op}}(m,n) = \mathrm{Mat}(m,n;\mathbb{Z})$ to $B_d(f)$, which is a right action on each coloring, where we consider an element of V_n as a $(1 \times n)$-matrix. By restricting this functor to the automorphism group, we obtain an action of the opposite group $\mathrm{GL}(n;\mathbb{Z})^{\mathrm{op}}$ of $\mathrm{GL}(n;\mathbb{Z})$ on $B_d(n)$ for each $n \geq 0$. We consider this action as a right action of $\mathrm{GL}(n;\mathbb{Z})$ on $B_d(n)$. Note that the $\mathrm{GL}(n;\mathbb{Z})$-action on $B_d(n)$ naturally extends to a $\mathrm{GL}(V_n)$-action on $B_d(n)$.

Proposition 2.4 (see Proposition 3.2 of [16]). *For $d \geq 0$, the PBW maps equation (2.1) give a natural isomorphism*

$$\theta_d : \mathrm{gr} \circ A_d \overset{\cong}{\Rightarrow} B_d \circ \mathrm{ab}^{\mathrm{op}},$$

where $\mathrm{ab}^{\mathrm{op}}$ denotes the opposite functor of the abelianization functor and gr denote the functor that sends a filtered vector space to its associated graded vector space.

By this proposition, it turns out that the $\mathrm{Aut}(F_n)$-action on $A_d(n)$, which is an action of an extended N-series on a filtered vector space, induces two actions on $B_d(n)$, which form an action of an extended graded Lie algebra on a graded vector space (see Theorem 5.15 of [16] and [10] for extended N-series and extended graded Lie algebras). One of them is the $\mathrm{GL}(n;\mathbb{Z})$-action, and the other of them is an action of the graded Lie algebra $\mathrm{gr}(\mathrm{IA}(n))$ on the graded vector space $B_d(n)$, which consists of $\mathrm{GL}(n;\mathbb{Z})$-module homomorphisms

$$[\cdot,\cdot] : B_{d,k}(n) \otimes \mathrm{gr}^r(\mathrm{IA}(n)) \to B_{d,k+r}(n) \tag{2.2}$$

for $k \geq 0, r \geq 1$ (see Proposition 5.10 and Theorem 5.15 of [16]). By using these two actions on $B_d(n)$, we obtained an indecomposable decomposition of $A_2(n)$ as $\mathrm{Aut}(F_n)$-modules (see Theorem 6.9 of [16]).

2.4. Hopf algebra in a symmetric strict monoidal category

We review the definition of a Hopf algebra in a symmetric strict monoidal category. Let $\mathcal{C} = (\mathcal{C}, \otimes, I, P)$ be a symmetric strict monoidal category. A *Hopf algebra* in $\mathcal{C}$ is an object H in $\mathcal{C}$ equipped with morphisms

$$\mu : H \otimes H \to H, \quad \eta : I \to H, \quad \Delta : H \to H \otimes H, \quad \epsilon : H \to I, \quad S : H \to H,$$

called the *multiplication, unit, comultiplication, counit* and *antipode*, respectively, satisfying

(1) $\mu(\mu \otimes \mathrm{id}_H) = \mu(\mathrm{id}_H \otimes \mu), \quad \mu(\eta \otimes \mathrm{id}_H) = \mathrm{id}_H = \mu(\mathrm{id}_H \otimes \eta)$,
(2) $(\Delta \otimes \mathrm{id}_H)\Delta = (\mathrm{id}_H \otimes \Delta)\Delta, \quad (\epsilon \otimes \mathrm{id}_H)\Delta = \mathrm{id}_H = (\mathrm{id}_H \otimes \epsilon)\Delta$,
(3) $\epsilon\eta = \mathrm{id}_I, \quad \epsilon\mu = \epsilon \otimes \epsilon, \quad \Delta\eta = \eta \otimes \eta$,
(4) $\Delta\mu = (\mu \otimes \mu)(\mathrm{id}_H \otimes P_{H,H} \otimes \mathrm{id}_H)(\Delta \otimes \Delta)$,
(5) $\mu(\mathrm{id}_H \otimes S)\Delta = \mu(S \otimes \mathrm{id}_H)\Delta = \eta\epsilon$.

A Hopf algebra H is said to be *cocommutative* if $P_{H,H}\Delta = \Delta$.
Define $\mu_n : H^{\otimes n} \otimes H^{\otimes n} \to H^{\otimes n}$ and $\Delta_m : H^{\otimes m} \to H^{\otimes m} \otimes H^{\otimes m}$ inductively by

$$\mu_0 = \mathrm{id}_I, \quad \mu_{n+1} = (\mu_n \otimes \mu)(\mathrm{id}_{H^{\otimes n}} \otimes P_{H,H^{\otimes n}} \otimes \mathrm{id}_H)$$

for $n \geq 0$ and by

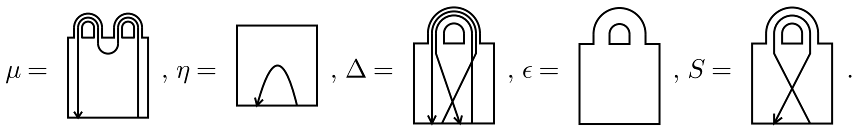
$$\Delta_0 = \mathrm{id}_I, \quad \Delta_{m+1} = (\mathrm{id}_{H^{\otimes m}} \otimes P_{H^{\otimes m},H} \otimes \mathrm{id}_H)(\Delta_m \otimes \Delta)$$

for $m \geq 0$.

For morphisms $f,\ f' : H^{\otimes m} \to H^{\otimes n}$, $m,n \geq 0$, the *convolution* $f * f'$ of f and f' is defined by

$$f * f' := \mu_n (f \otimes f') \Delta_m.$$

The category $\mathbf{A}$ has a cocommutative Hopf algebra with the object 1, where



2.5. Lie algebra in a linear symmetric strict monoidal category

We review the definition of a Lie algebra in a linear symmetric strict monoidal category. Let $\mathcal{C} = (\mathcal{C}, \otimes, I, P)$ be a linear symmetric strict monoidal category. A *Lie algebra* in $\mathcal{C}$ is an object L in $\mathcal{C}$ equipped with a morphism

$$[\cdot,\cdot] : L \otimes L \to L$$

satisfying

(1) $[\cdot,\cdot](\mathrm{id}_{L \otimes L} + P_{L,L}) = 0$,

(2) $[\cdot,\cdot](\mathrm{id}_L \otimes [\cdot,\cdot])(\mathrm{id}_{L^{\otimes 3}} + \sigma + \sigma^2) = 0$, where $\sigma = (1,2,3) : L^{\otimes 3} \to L^{\otimes 3}$.

3. Andreadakis filtration $\mathcal{E}_*(n)$ of $\mathrm{End}(F_n)$

We briefly review the Andreadakis filtration and the Johnson homomorphism of $\mathrm{Aut}(F_n)$. See [22] for further details. Then we consider its extension to the endomorphism monoid $\mathrm{End}(F_n)$ of F_n.

3.1. Andreadakis filtration $\mathcal{A}_*(n)$ of $\mathrm{Aut}(F_n)$

In what follows, we consider the left action of $\mathrm{Aut}(F_n)$ on F_n. Let $\Gamma_r := \Gamma_r(F_n)$ denote the r-th term of the lower central series of the free group F_n of rank n. Let $\mathcal{L}_r(n) := \Gamma_r/\Gamma_{r+1}$ for $r \geq 1$. Note that $H = \mathcal{L}_1(n)$ and that $\mathcal{L}_r(n)$ is the degree r part of the free Lie algebra $\mathcal{L}_*(n)$ on H.

For $r \geq 0$, the left action of $\mathrm{Aut}(F_n)$ on each nilpotent quotient F_n/Γ_{r+1} induces a group homomorphism

$$\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/\Gamma_{r+1}).$$

Set

$$\mathcal{A}_r(n) := \ker(\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n/\Gamma_{r+1})) \lhd \mathrm{Aut}(F_n).$$

Then we have a filtration, which is called the *Andreadakis filtration* of $\mathrm{Aut}(F_n)$:

$$\mathrm{Aut}(F_n) = \mathcal{A}_0(n) \supset \mathcal{A}_1(n) = \mathrm{IA}(n) \supset \mathcal{A}_2(n) \supset \cdots.$$

For $r \geq 1$, the *Johnson homomorphism*

$$\tau_r : \mathrm{gr}^r(\mathcal{A}_*(n)) \hookrightarrow \mathrm{Hom}(H,\mathcal{L}_{r+1}(n))$$

is the injective homomorphism induced by the group homomorphism

$$\tau'_r : \mathcal{A}_r(n) \to \mathrm{Hom}(H,\mathcal{L}_{r+1}(n))$$

defined by

$$\tau'_r(f)(x\Gamma_2) := f(x)x^{-1}\Gamma_{r+2} \quad \text{for } f \in \mathcal{A}_r(n), x \in F_n.$$

3.2. The target group of the Johnson homomorphism

The target group $\mathrm{Hom}(H,\mathcal{L}_{r+1}(n)) \cong H^* \otimes \mathcal{L}_{r+1}(n)$ of the Johnson homomorphism is identified with the degree r part $\mathrm{Der}_r(\mathcal{L}_*(n))$ of the derivation Lie algebra $\mathrm{Der}(\mathcal{L}_*(n))$ of the free Lie algebra $\mathcal{L}_*(n)$ and with the tree module $T_r(n)$ via abelian group isomorphisms

$$H^* \otimes \mathcal{L}_{r+1}(n) \cong \mathrm{Der}_r(\mathcal{L}_*(n)) \cong T_r(n). \tag{3.1}$$

Here, we briefly review the derivation Lie algebra and the tree module. (See [22] for details.)

A *derivation* f of $\mathcal{L}_*(n)$ is a $\mathbb{Z}$-linear map $f : \mathcal{L}_*(n) \to \mathcal{L}_*(n)$ such that $f([a,b]) = [f(a),b] + [a,f(b)]$ for any $a,b \in \mathcal{L}_*(n)$. The derivation Lie algebra $\mathrm{Der}(\mathcal{L}_*(n))$ of the Lie algebra $\mathcal{L}_*(n)$ is the set of all derivations of $\mathcal{L}_*(n)$. The degree r part $\mathrm{Der}_r(\mathcal{L}_*(n))$ of the derivation Lie algebra is defined to be

$$\mathrm{Der}_r(\mathcal{L}_*(n)) = \{f \in \mathrm{Der}(\mathcal{L}_*(n)) \mid f(a) \in \mathcal{L}_{r+1}(n) \text{ for any } a \in H\}.$$

Then we have $\mathrm{Der}(\mathcal{L}_*(n)) = \bigoplus_{r \geq 0} \mathrm{Der}_r(\mathcal{L}_*(n))$ and abelian group isomorphisms

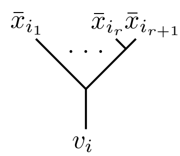
$$\mathrm{Der}_r(\mathcal{L}_*(n)) \cong \mathrm{Hom}(H,\mathcal{L}_{r+1}(n)) \cong H^* \otimes \mathcal{L}_{r+1}(n).$$

We call a connected Jacobi diagram with no cycle a *trivalent tree*. For $r \geq 0$, a trivalent tree is called a *rooted trivalent tree of degree* r if it has one univalent vertex (called the *root*) that is colored by an element of H^* and $r+1$ univalent vertices (called *leaves*) that are colored by elements of H. Let $T_r(n)$ denote the $\mathbb{Z}$-module spanned by rooted trivalent trees of degree r modulo the AS, IHX and multilinearity relations. We have an abelian group isomorphism

$$\Phi : H^* \otimes \mathcal{L}_{r+1}(n) \xrightarrow{\cong} T_r(n)$$

defined by

$$\Phi(v_i \otimes [\bar{x}_{i_1}, \cdots, [\bar{x}_{i_r}, \bar{x}_{i_{r+1}}] \cdots]) =$$



for $v_i \in H^*$, $[\bar{x}_{i_1}, \cdots, [\bar{x}_{i_r}, \bar{x}_{i_{r+1}}] \cdots] \in \mathcal{L}_{r+1}(n)$.

3.3. Andreadakis filtration $\mathcal{E}_*(n)$ of $\mathrm{End}(F_n)$

We extend the above construction to the endomorphism monoid $\mathrm{End}(F_n)$ of F_n. For $r \geq 0$, consider the canonical map

$$\rho_r : \mathrm{End}(F_n) \to \mathrm{End}(F_n/\Gamma_{r+1})$$

and set $\mathcal{E}_r(n) := \ker(\rho_r)$. Then we have a filtration of monoids

$$\mathrm{End}(F_n) = \mathcal{E}_0(n) \supset \mathcal{E}_1(n) \supset \cdots,$$

and we call $\mathcal{E}_*(n) = (\mathcal{E}_r(n))_{r\geq 0}$ the *Andreadakis filtration* of $\mathrm{End}(F_n)$.

For $f \in \mathrm{End}(F_n)$ and $x,y \in F_n$, set

$$[f,x] := f(x)x^{-1}, \quad {}^{y}x = yxy^{-1},$$

and for a subset $T \subset F_n$, set

$$[f,T] = \{[f,x] \in F_n \mid x \in T\}.$$

We can easily check the following lemma.

Lemma 3.1.

$$f \in \mathcal{E}_r(n) \quad \Leftrightarrow \quad [f,F_n] \subset \Gamma_{r+1} \quad \Leftrightarrow \quad [f,x_i] \in \Gamma_{r+1} \ (\textit{for any } i \in [n]).$$

For subsets $S \subset \mathrm{End}(F_n)$ and $T \subset F_n$, let $[S,T]$ denote the subgroup of F_n generated by the elements $[f,x]$ for $f \in S, x \in T$.

Lemma 3.2. *We have*

$$[\mathcal{E}_r(n),\Gamma_k] \subset \Gamma_{k+r}$$

for $r \geq 0, k \geq 1$.

Proof. It is well known that $[\mathcal{A}_r(n),\Gamma_k] \subset \Gamma_{k+r}$ by Andreadakis [1]. The same proof can be applied to $\mathcal{E}_r(n)$. We use induction on k. When $k = 1$, we have $[\mathcal{E}_r(n),F_n] \subset \Gamma_{r+1}$ by the definition of $\mathcal{E}_r(n)$. Suppose that $[\mathcal{E}_r(n),\Gamma_{k-1}] \subset \Gamma_{k-1+r}$. We will show that $[\mathcal{E}_r(n),\Gamma_k] \subset \Gamma_{k+r}$. Let $f \in \mathcal{E}_r(n)$. Recall that Γ_k is generated by the commutator $[x,y]$ with $x \in \Gamma_{k-1}, y \in F_n$. We can check that for $x \in \Gamma_{k-1}, y \in F_n$, we have

$$[f,[x,y]] = {}^{[f,y]}([[f,y]^{-1},f(x)] \cdot [[f,x],{}^{x}y] \cdot [[x,y],[f,y]^{-1}]) \in \Gamma_{k+r}.$$

For $z,w \in \Gamma_k$, we have

$$[f,zw] = [f,z] \cdot {}^{z}[f,w] \equiv [f,z][f,w] \pmod{\Gamma_{k+r+1}},$$

and by letting $w = z^{-1}$, we have

$$[f,z^{-1}] \equiv [f,z]^{-1} \pmod{\Gamma_{k+r+1}}.$$

Therefore, we have $[f,z] \in \Gamma_{k+r}$ for any $z \in \Gamma_k$. □

Define a map

$$\sigma : \mathrm{End}(F_n) \to \mathrm{End}(F_n)$$

by $\sigma(f)=\tilde{f}$ for $f\in \mathrm{End}(F_n)$, where

$$\tilde{f}(x_i)=[f,x_i]^{-1}x_i=x_if(x_i)^{-1}x_i$$

for $i\in[n]$.

Lemma 3.3. *We have*

$$\sigma^2=\mathrm{id}_{\mathrm{End}(F_n)} \tag{3.2}$$

$$f\in\mathcal{E}_r(n) \quad\Rightarrow\quad \sigma(f)\in\mathcal{E}_r(n) \tag{3.3}$$

$$f\in\mathcal{E}_r(n) \quad\Rightarrow\quad f\sigma(f),\sigma(f)f\in\mathcal{E}_{2r}(n). \tag{3.4}$$

Proof. We have equation (3.2) since for any $f\in\mathrm{End}(F_n)$ and $i\in[n]$, we have

$$\sigma^2(f)(x_i)=x_i\tilde{f}(x_i)^{-1}x_i=x_ix_i^{-1}f(x_i)x_i^{-1}x_i=f(x_i).$$

We have equation (3.3) since, for any $f\in\mathcal{E}_r(n)$ and $i\in[n]$, we have

$$[\tilde{f},x_i]=[f,x_i]^{-1}\in\Gamma_{r+1}.$$

We prove equation (3.4). Let $f\in\mathcal{E}_r(n)$. We have

$$[f\tilde{f},x_i]=f([\tilde{f},x_i])[f,x_i]=f([f,x_i]^{-1})[f,x_i]=[f,[f,x_i]^{-1}]\in\Gamma_{2r+1}$$

for any $i\in[n]$. Thus, we have

$$f\tilde{f}\in\mathcal{E}_{2r}(n). \tag{3.5}$$

By equation (3.3), we have $\tilde{f}\in\mathcal{E}_r(n)$, and by equations (3.2) and (3.5),

$$\tilde{f}f=\tilde{f}\tilde{\tilde{f}}\in\mathcal{E}_{2r}(n).$$

□

For $N\geq r\geq 0$, we define an equivalence relation $\sim_N$ on the monoid $\mathcal{E}_r(n)$ by

$$f\sim_N g \quad\overset{\text{def}}{\Leftrightarrow}\quad [f,x]\equiv[g,x]\ (\mathrm{mod}\ \Gamma_{N+1}) \quad \text{for any } x\in F_n$$

for $f,g\in\mathcal{E}_r(n)$. Thus, we have

$$f\sim_N \mathrm{id}_{F_n} \quad\Leftrightarrow\quad [f,x]\in\Gamma_{N+1} \quad\text{for any } x\in F_n \quad\Leftrightarrow\quad f\in\mathcal{E}_N(n).$$

Lemma 3.4. *Let $r\geq 1$. For $f\in\mathcal{E}_r(n)$, define f^R_N and f^L_N for $N\geq r+1$ inductively by*

$$f^R_N=\begin{cases}\tilde{f} & (N=r+1)\\ f^R_{N-1}\widetilde{ff^R_{N-1}} & (N\geq r+2),\end{cases}$$

$$f^L_N=\begin{cases}\tilde{f} & (N=r+1)\\ \widetilde{f^L_{N-1}f}f^L_{N-1} & (N\geq r+2).\end{cases}$$

Then we have

$$f_N^R \in \mathcal{E}_r(n), \quad f f_N^R \in \mathcal{E}_N(n), \quad f_N^R \sim_{N-1} f_{N-1}^R,$$
$$f_N^L \in \mathcal{E}_r(n), \quad f_N^L f \in \mathcal{E}_N(n), \quad f_N^L \sim_{N-1} f_{N-1}^L.$$

Proof. We use induction on $N \geq r+1$. When $N = r+1$, by Lemma 3.3, we have $\tilde{f} \in \mathcal{E}_r(n)$ and $f\tilde{f} \in \mathcal{E}_{2r}(n) \subset \mathcal{E}_{r+1}(n)$. Suppose that $f_{N-1}^R \in \mathcal{E}_r(n)$ satisfies $f f_{N-1}^R \in \mathcal{E}_{N-1}(n)$. By Lemma 3.3, we have $\widetilde{f f_{N-1}^R} \in \mathcal{E}_{N-1}(n)$ and $f f_{N-1}^R \widetilde{f f_{N-1}^R} \in \mathcal{E}_{2N-2}(n) \subset \mathcal{E}_N(n)$. Then we have $f_N^R = f_{N-1}^R \widetilde{f f_{N-1}^R} \in \mathcal{E}_r(n)$ and $f f_N^R \in \mathcal{E}_N(n)$. Since $\widetilde{f f_{N-1}^R} \in \mathcal{E}_{N-1}(n)$, we have $f_N^R \sim_{N-1} f_{N-1}^R$. The case for f_N^L is similar. □

Proposition 3.5. *For $N \geq 1$, we have a filtration of groups*

$$\mathcal{E}_1(n)/\sim_N \supset \mathcal{E}_2(n)/\sim_N \supset \cdots \supset \mathcal{E}_{N-1}(n)/\sim_N \supset \mathcal{E}_N(n)/\sim_N = 1.$$

Moreover, this is an N-series.

Proof. Firstly, we show that $\mathcal{E}_r(n)/\sim_N$ is a group for each $r \geq 1$. For $f, f', g \in \mathcal{E}_r(n)$ such that $f \sim_N f'$, we can easily check that $fg \sim_N f'g$ and $gf \sim_N gf'$. Thus, the composition makes the set $\mathcal{E}_r(n)/\sim_N$ a monoid. For $[f] \in \mathcal{E}_r(n)/\sim_N$, by Lemma 3.4, it follows that $[f][f_N^R] = [f_N^L][f] = 1 \in \mathcal{E}_r(n)/\sim_N$. Since $\mathcal{E}_r(n)/\sim_N$ is a monoid, we have $[f_N^R] = [f_N^L]$, and this is the inverse of $[f]$. Therefore, $\mathcal{E}_r(n)/\sim_N$ is a group for each $r \geq 1$.

Since $\mathcal{E}_r(n) \supset \mathcal{E}_{r+1}(n)$, we have $\mathcal{E}_r(n)/\sim_N \supset \mathcal{E}_{r+1}(n)/\sim_N$. Secondly, we show that the descending series is an N-series. It suffices to show that, for $f \in \mathcal{E}_r(n), g \in \mathcal{E}_s(n)$, we have

$$[[f],[g]] = [f][g][f]^{-1}[g]^{-1} = [fgf_N^R g_N^R] \in \mathcal{E}_{r+s}(n)/\sim_N .$$

Note that, by Lemma 3.4, we can take $f_N^R, g_N^R \in \mathcal{E}_r(n)$ such that $ff_N^R, gg_N^R \in \mathcal{E}_N(n) \cap \mathcal{E}_{r+s}(n)$. By commutator calculus, for $x \in F_n$, we have

$$[fg,x] = [f,[g,x]]\,[g,x]\,[f,x] \equiv [g,x]\,[f,x] \pmod{\Gamma_{r+s+1}},$$

$$[g,[g_N^R,x]\,[f_N^R,x]] = [g,[g_N^R,x]]^{[g_N^R,x]}[g,[f_N^R,x]] \equiv [g,[g_N^R,x]] \pmod{\Gamma_{r+s+1}}.$$

Similarly, we have

$$[f_N^R g_N^R,x] \equiv [g_N^R,x]\,[f_N^R,x] \pmod{\Gamma_{r+s+1}},$$

$$[f,[g_N^R,x]\,[f_N^R,x]] \equiv [f,[f_N^R,x]] \pmod{\Gamma_{r+s+1}}.$$

Thus, we have

$$\begin{aligned}[fg,[f_N^R g_N^R,x]] &\equiv [g,[f_N^R g_N^R,x]]\,[f,[f_N^R g_N^R,x]]\\ &\equiv [g,[g_N^R,x]\,[f_N^R,x]]\,[f,[g_N^R,x]\,[f_N^R,x]]\\ &\equiv [g,[g_N^R,x]]\,[f,[f_N^R,x]] \pmod{\Gamma_{r+s+1}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}[fgf_N^Rg_N^R,x] &= [fg,[f_N^Rg_N^R,x]]\,[f_N^Rg_N^R,x]\,[fg,x]\\ &\equiv [g,[g_N^R,x]]\,[f,[f_N^R,x]]\,[g_N^R,x]\,[f_N^R,x]\,[g,x]\,[f,x]\\ &\equiv [g,[g_N^R,x]]\,[g_N^R,x]\,[g,x]\,[f,[f_N^R,x]]\,[f_N^R,x]\,[f,x]\\ &= [gg_N^R,x]\,[ff_N^R,x]\\ &\equiv 1 \pmod{\Gamma_{r+s+1}},\end{aligned}$$

and the proof is complete. □

For $N \geq r \geq 1$, we have a canonical projection

$$p_{N+1} : \mathcal{E}_r(n)/\sim_{N+1} \twoheadrightarrow \mathcal{E}_r(n)/\sim_N .$$

Let $\hat{\mathcal{E}}_r(n)$ denote the projective limit $\varprojlim_N(\mathcal{E}_r(n)/\sim_N)$ and

$$\pi_N : \hat{\mathcal{E}}_r(n) \twoheadrightarrow \mathcal{E}_r(n)/\sim_N$$

denote the projection. By Proposition 3.5, we have a descending series of groups

$$\hat{\mathcal{E}}_1(n) \supset \hat{\mathcal{E}}_2(n) \supset \cdots$$

satisfying

$$\bigcap_{r\geq 1} \hat{\mathcal{E}}_r(n) = \{\mathrm{id}\}.$$

Proposition 3.6. *The descending series $\hat{\mathcal{E}}_*(n) := (\hat{\mathcal{E}}_r(n))_{r\geq 1}$ is an N-series.*

Proof. By Proposition 3.5, we have $[\mathcal{E}_r(n)/\sim_N, \mathcal{E}_s(n)/\sim_N] \subset \mathcal{E}_{r+s}(n)/\sim_N$ for each $N > r,s$. By taking the projective limits, we have $[\hat{\mathcal{E}}_r(n), \hat{\mathcal{E}}_s(n)] \subset \hat{\mathcal{E}}_{r+s}(n)$. □

We have a graded Lie algebra $\mathrm{gr}(\hat{\mathcal{E}}_*(n))$ associated to the N-series $\hat{\mathcal{E}}_*(n)$. Let $\mathrm{gr}^r(\mathcal{E}_*(n)) := \mathcal{E}_r(n)/\sim_{r+1}$ for $r \geq 1$ and $\mathrm{gr}(\mathcal{E}_*(n)) := \bigoplus_{r\geq 1} \mathrm{gr}^r(\mathcal{E}_*(n))$.

Proposition 3.7. *We have a group isomorphism*

$$\bar{\pi}_{r+1} : \mathrm{gr}^r(\hat{\mathcal{E}}_*(n)) \xrightarrow{\cong} \mathrm{gr}^r(\mathcal{E}_*(n))$$

induced by the projection $\pi_{r+1} : \hat{\mathcal{E}}_r(n) \to \mathrm{gr}^r(\mathcal{E}_*(n))$. Therefore, $\mathrm{gr}(\mathcal{E}_*(n))$ is a graded Lie algebra.

Proof. The projection π_{r+1} induces $\bar{\pi}_{r+1}$ since, for $f \in \hat{\mathcal{E}}_{r+1}(n)$, we have $\pi_{r+1}(f) \in \mathcal{E}_{r+1}(n)/\sim_{r+1} = 1$.

We will check that $\bar{\pi}_{r+1}$ is surjective. For any $f \in \mathcal{E}_r(n)$, let $\Phi(f) \in \hat{\mathcal{E}}_r(n)$ satisfy $\pi_N(\Phi(f)) = [f] \in \mathcal{E}_r(n)/\sim_N$ for each $N > r$. We have $\bar{\pi}_{r+1}([\Phi(f)]) = \pi_{r+1}(\Phi(f)) = [f] \in \mathcal{E}_r(n)/\sim_{r+1}$. Therefore, $\bar{\pi}_{r+1}$ is surjective.

Finally, we show that $\bar{\pi}_{r+1}$ is injective. Let $f \in \hat{\mathcal{E}}_r(n)$ satisfy $\bar{\pi}_{r+1}([f]) = 1 \in \mathcal{E}_r(n)/\sim_{r+1}$ and $\pi_N(f) = [f_N] \in \mathcal{E}_r(n)/\sim_N$ for $f_N \in \mathcal{E}_r(n)$. Then, we have $f_{r+1} \in \mathcal{E}_{r+1}(n)$ and $f_N \sim_{r+1} f_{r+1}$ for any $N > r$. Therefore, we have $\pi_N(f) = [f_N] \in \mathcal{E}_{r+1}(n)/\sim_N$ for each $N > r$ and thus $[f] = 1 \in \mathrm{gr}^r(\hat{\mathcal{E}}_*(n))$. The proof is complete. □

3.4. Johnson homomorphism of $\mathrm{End}(F_n)$

For $r \geq 1$, by using Lemma 3.2, we can define a monoid homomorphism

$$\tilde{\tau}'_r : \mathcal{E}_r(n) \to \mathrm{Hom}(H, \mathcal{L}_{r+1}(n))$$

by $\tilde{\tau}'_r(f)(x\Gamma_2) := [f,x]\Gamma_{r+2}$ for $f \in \mathcal{E}_r(n), x \in F_n$. It is easily checked that the monoid homomorphism $\tilde{\tau}'_r$ induces an injective group homomorphism

$$\tilde{\tau}_r : \mathrm{gr}^r(\mathcal{E}_*(n)) \hookrightarrow \mathrm{Hom}(H, \mathcal{L}_{r+1}(n)).$$

We call it the *r-th Johnson homomorphism* of $\mathrm{End}(F_n)$.

Proposition 3.8. *The map $\tilde{\tau}_r : \mathrm{gr}^r(\mathcal{E}_*(n)) \hookrightarrow \mathrm{Hom}(H, \mathcal{L}_{r+1}(n))$ is an abelian group isomorphism.*

Proof. It suffices to show that $\tilde{\tau}_r$ is surjective. For any $\varphi \in \mathrm{Hom}(H, \mathcal{L}_{r+1}(n))$, we fix a representative of $\varphi(x_i\Gamma_2) \in \mathcal{L}_{r+1}(n)$ and write it $\varphi(x_i) \in \Gamma_{r+1}$, for $i \in [n]$. Define $\psi \in \mathrm{End}(F_n)$ by

$$\psi(x_i) = \varphi(x_i)x_i \text{ for } i \in [n].$$

It turns out that $[\psi,x]\Gamma_{r+2} = \varphi(x\Gamma_2) \in \mathcal{L}_{r+1}(n)$ for any $x \in F_n$ by induction on the word length of $x \in F_n$. Therefore, we have $\tilde{\tau}_r(\psi) = \varphi$, and thus the map $\tilde{\tau}_r$ is surjective. □

Then we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{gr}^r(\mathcal{A}_*(n)) & & \\ {\scriptstyle \text{inclusion}} \downarrow & \searrow^{\tau_r} & \\ \mathrm{gr}^r(\mathcal{E}_*(n)) & \xrightarrow[\tilde{\tau}_r]{\cong} & \mathrm{Hom}(H, \mathcal{L}_{r+1}(n)). \end{array}$$

Remark 3.9. It is well known that the Andreadakis filtration $\mathcal{A}_*(n)$ of $\mathrm{Aut}(F_n)$ includes the lower central series of $\mathrm{IA}(n)$:

$$\Gamma_r(\mathrm{IA}(n)) \subset \mathcal{A}_r(n).$$

We have $\mathcal{A}_1(n) = \mathrm{IA}(n)$ by definition. Andreadakis [1] conjectured that

$$\mathcal{A}_r(n) = \Gamma_r(\mathrm{IA}(n)) \tag{3.6}$$

for all $r \geq 2, n \geq 2$. Andreadakis [1] ($n = 3$) and Kawazumi [17] (for any n) showed that equation (3.6) holds for $r = 2$. Moreover, Andreadakis [1] showed that the first Johnson homomorphism τ_1 of $\mathrm{Aut}(F_n)$ is an isomorphism. Therefore, we have abelian group isomorphisms

$$\mathrm{gr}^1(\mathrm{IA}(n)) \cong \mathrm{Hom}(H, \mathcal{L}_2(n)) \cong \mathrm{gr}^1(\mathcal{E}_*(n)). \tag{3.7}$$

Recently, Satoh [23] showed that equation (3.6) holds for $r = 3$. On the other hand, Bartholdi [5] showed that

$$(\mathcal{A}_5(3)/\Gamma_5(\mathrm{IA}(3))) \otimes \mathbb{Q} \cong \mathbb{Q}^{\oplus 3},$$

which is a counterexample of the Andreadakis conjecture. Now, the Andreadakis conjecture remains open for $n \gg r$.

3.5. The derivation Lie algebra

By equation (3.1) and Proposition 3.8, we have abelian group isomorphisms

$$\mathrm{gr}^r(\mathcal{E}_*(n)) \cong H^* \otimes \mathcal{L}_{r+1}(n) \cong \mathrm{Der}_r(\mathcal{L}_*(n)) \cong T_r(n).$$

We write $\tilde{\tau}_r : \mathrm{gr}^r(\mathcal{E}_*(n)) \xrightarrow{\cong} \mathrm{Der}_r(\mathcal{L}_*(n))$ as well.

Proposition 3.10. *The abelian group isomorphism*

$$\tilde{\tau} = \bigoplus_{r \geq 1} \tilde{\tau}_r : \mathrm{gr}(\mathcal{E}_*(n)) \xrightarrow{\cong} \mathrm{Der}(\mathcal{L}_*(n))$$

is an isomorphism of graded Lie algebras.

Proof. We only need to check that the Lie bracket of $\mathrm{gr}(\mathcal{E}_*(n))$ is sent to the Lie bracket of $\mathrm{Der}(\mathcal{L}_*(n))$. For $f \in \hat{\mathcal{E}}_r(n), g \in \hat{\mathcal{E}}_s(n)$ and $x \in F_n$, we have

$$\begin{aligned}[\tilde{\tau}_r([f]), \tilde{\tau}_s([g])](x\Gamma_2) &= \tilde{\tau}_r([f])\tilde{\tau}_s([g])(x\Gamma_2) - \tilde{\tau}_s([g])\tilde{\tau}_r([f])(x\Gamma_2)\\ &= [f,[g,x]]\,[g,[f,x]]^{-1} = [[f,g],x] \in \mathcal{L}_{r+s+1}(n).\end{aligned}$$

On the other hand, we have

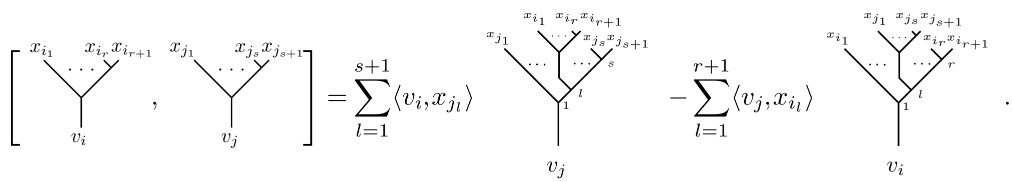
$$\tilde{\tau}_{r+s}([[f,g]])(x\Gamma_2) = [[f,g],x] \in \mathcal{L}_{r+s+1}(n).$$

Therefore, $\tilde{\tau}$ is an isomorphism of graded Lie algebras. □

Remark 3.11. The tree module $\bigoplus_{r \geq 1} T_r(n)$ also has a graded Lie algebra structure which is induced by the Lie algebra structure of $\mathrm{Der}(\mathcal{L}_*(n))$. The Lie bracket

$$[\cdot,\cdot] : T_r(n) \times T_s(n) \to T_{r+s}(n)$$

is defined by the difference between two linear sums obtained by contracting the root of one of the trees and the leaves of the other tree



4. Action of $\mathrm{gr}(\mathcal{E}_*(n))$ on $B_d(n)$

We defined the bracket maps (2.2) in [16]. In this section, we extend them to linear maps

$$[\cdot,\cdot] : B_{d,k}(n) \otimes \mathrm{gr}^r(\mathcal{E}_*(n)) \to B_{d,k+r}(n).$$

In Section 4.1, we state Theorem 4.1, which we use to obtain the extended bracket map. In Section 4.2, we extend the category $\mathbf{A}$ to a category $\mathbf{A}^L$, which includes a Lie algebra structure besides the Hopf algebra structure in $\mathbf{A}$. In Section 4.3, we observe some relations for morphisms of $\mathbf{A}^L$. By using these relations, we prove Theorem 4.1 in Section 4.4.

4.1. Bracket map $[\cdot,\cdot]: B_{d,k}(n)\otimes \mathrm{gr}^r(\mathcal{E}_*(n)) \to B_{d,k+r}(n)$

We have a right $\mathrm{End}(F_n)$-action on $A_d(n)$ by letting

$$u\cdot g := A_d(g)(u)$$

for $u\in A_d(n), g\in \mathrm{End}(F_n)$. We define

$$[\cdot,\cdot]: A_d(n)\times \mathrm{End}(F_n)\to A_d(n) \tag{4.1}$$

by $[u,g] := u\cdot g - u$ for $u\in A_d(n), g\in \mathrm{End}(F_n)$, which we call the *bracket map.*

Theorem 4.1. *The N-series $\hat{\mathcal{E}}_*(n)$ acts on the right on the filtered vector space $A_d(n)$. That is, we have*

$$[A_{d,k}(n),\mathcal{E}_r(n)]\subset A_{d,k+r}(n)$$

for any $r\geq 1$.

Note that we have $[A_{d,k}(n),\Gamma_r(\mathrm{IA}(n))]\subset A_{d,k+r}(n)$ (see Lemma 5.7 in [16]). We will prove Theorem 4.1 in Section 4.4.

By using Theorem 4.1, we can extend the bracket map

$$[\cdot,\cdot]: B_{d,k}(n)\otimes \mathrm{gr}^r(\mathrm{IA}(n))\to B_{d,k+r}(n)$$

to $\mathrm{gr}^r(\mathcal{E}_*(n))$.

Corollary 4.2. *Let $r\geq 1$. The bracket map (4.1) induces a $\Bbbk$-linear map*

$$[\cdot,\cdot]: B_{d,k}(n)\otimes \mathrm{gr}^r(\mathcal{E}_*(n))\to B_{d,k+r}(n).$$

We can also extend the $\mathrm{GL}(n;\mathbb{Z})$-module map

$$\beta^r_{d,k}: \mathrm{gr}^r(\mathrm{IA}(n))\to \mathrm{Hom}(B_{d,k}(n),B_{d,k+r}(n))$$

defined by $\beta^r_{d,k}(g)(u)=[u,g]$ for $g\in \mathrm{gr}^r(\mathrm{IA}(n)), u\in B_{d,k}(n)$ to a group homomorphism

$$\tilde{\beta}^r_{d,k}: \mathrm{gr}^r(\mathcal{E}_*(n))\to \mathrm{Hom}(B_{d,k}(n),B_{d,k+r}(n)),$$

which $\beta^r_{d,k}$ factors through. That is, we have $\beta^r_{d,k}=\tilde{\beta}^r_{d,k}i$, where the map $i:\mathrm{gr}^r(\mathrm{IA}(n))\to \mathrm{gr}^r(\mathcal{E}_*(n))$ is induced by the inclusion map $\Gamma_r(\mathrm{IA}(n))\hookrightarrow \mathcal{E}_r(n)$.

Remark 4.3. The right action of the N-series $\hat{\mathcal{E}}_*(n)$ on $A_d(n)$ induces an action of the graded Lie algebra $\mathrm{gr}(\mathcal{E}_*(n))$ on the graded vector space $B_d(n)$:

$$\mathrm{gr}(\mathcal{E}_*(n))\xrightarrow{\cong}\mathrm{gr}(\hat{\mathcal{E}}_*(n))\to \bigoplus_{r\geq 1}\mathrm{End}_r(B_d(n)),$$

which is given by the group homomorphisms $\tilde{\beta}^r_{d,k}$. This induced action can be regarded as an action of the derivation Lie algebra $\mathrm{Der}(\mathcal{L}_*(n))$ on the graded vector space $B_d(n)$ by the identification in Section 3.5.

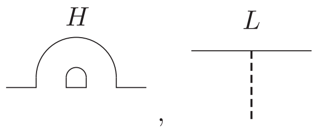


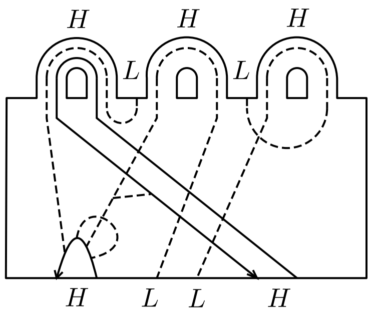
Figure 2. Source of a morphism

4.2. The category $\mathbf{A}^L$ of extended Jacobi diagrams in handlebodies

The category $\mathbf{A}$ has a cocommutative Hopf algebra with the underlying object 1, which we recalled in Section 2.4. Moreover, the morphisms of the category $\mathbf{A}$ have Jacobi diagrams, and the STU relations correspond to relations of Lie algebras. In a proof of Theorem 4.1, we use graphical computations which deal with the Hopf algebra structure and the Lie algebra structure. For this purpose, we extend the category $\mathbf{A}$ to another category $\mathbf{A}^L$ which includes the Hopf algebra structure and the Lie algebra structure. In Appendix A, we give an expected presentation of the category $\mathbf{A}^L$.

Construct the category $\mathbf{A}^L$ as follows. The set of objects of $\mathbf{A}^L$ is the free monoid generated by two objects H and L, where multiplication is denoted by $\otimes$. The category $\mathbf{A}^L$ includes the category $\mathbf{A}$ as a full subcategory with the free monoid generated by H as the set of objects. (On the other hand, the full subcategory with the free monoid generated by L is isomorphic to a category in [13]. See Remark A.4.) In the category $\mathbf{A}^L$, we consider diagrams that are obtained from Jacobi diagrams in handlebodies by attaching univalent vertices of the Jacobi diagrams to the bottom line l and the upper line l'.

Example 4.4. Here is a morphism in $\mathbf{A}^L(H \otimes L \otimes H \otimes L \otimes H, H \otimes L^{\otimes 2} \otimes H)$:



As depicted in Figure 2, the objects H and L in the source of a morphism of $\mathbf{A}^L$ correspond to a handle of the handlebody and a univalent vertex attached to the upper line l', respectively.

As depicted in Figure 3, the objects H and L in the target of a morphism of $\mathbf{A}^L$ correspond to an arc component mapped into the handlebody and a univalent vertex attached to the bottom line l, respectively.

In the category $\mathbf{A}^L$, the object H is considered as a Hopf algebra and L is considered as a Lie algebra. See Section 4.3 and Appendix A.

To define morphisms of the category $\mathbf{A}^L$ precisely, we give the following definition.

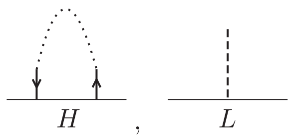


Figure 3. Target of a morphism

Definition 4.5. For a finite set T, an (X_m,T)-*diagram* is a quadruple (D,V,f,g), where

- D is a vertex-oriented uni-trivalent graph such that each connected component has at least one univalent vertex,
- V is a subset of $\partial D = \{\text{univalent vertices of } D\}$,
- f is an embedding of V into the interior of X_m,
- g is a bijection from T to $\partial D \setminus V$.

Note that an $(X_m,\emptyset)$-diagram is a Jacobi diagram on X_m.

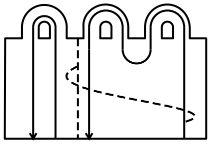
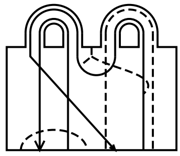
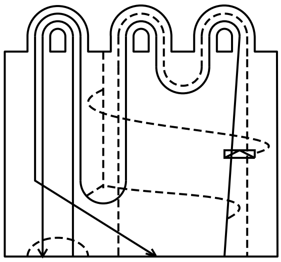
For an object $w = H^{\otimes m_1} \otimes L^{\otimes n_1} \otimes \cdots \otimes H^{\otimes m_r} \otimes L^{\otimes n_r} \in \mathbf{A}^L$, let $m := \sum_{i=1}^r m_i$ and $n := \sum_{i=1}^r n_i$. For $p \geq 0$, let $[p]^+ := \{1^+,\cdots,p^+\}$ and $[p]^- := \{1^-,\cdots,p^-\}$ be two copies of $[p]$.

Definition 4.6. For objects $w = H^{\otimes m_1} \otimes L^{\otimes n_1} \otimes \cdots \otimes H^{\otimes m_r} \otimes L^{\otimes n_r} \in \mathbf{A}^L$ and $w' = H^{\otimes m'1} \otimes L^{\otimes n'1} \otimes \cdots \otimes H^{\otimes m's} \otimes L^{\otimes n's} \in \mathbf{A}^L$, a (w,w')-*diagram* consists of

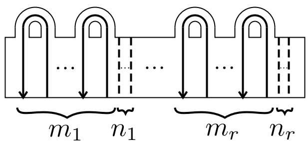
- an $(X_{m'},[n]^+ \sqcup [n']^-)$-diagram (D,V,f,g) such that each connected component of D has at least one univalent vertex in $V \cup g([n']^-)$
- a map $\varphi : X_{m'} \cup D \to U_m$ such that
  1. the pair (the empty set $\emptyset$, the restriction $\varphi|_{X_{m'}}$) is an (m,m')-Jacobi diagram; that is, φ maps X'_m into U_m in such a way that endpoints of X'_m are arranged in the bottom line l from left to right,
  2. $g([n]^+)$ is mapped into l' so that the corresponding object in $\mathbf{A}^L$ with respect to Figure 2 will be w when we look at the top line l' from left to right,
  3. $g([n']^-)$ is mapped into l so that the corresponding object in $\mathbf{A}^L$ with respect to Figure 3 will be w' when we look at the bottom line l from left to right.

We identify two (w,w')-diagrams if they are homotopic in U_m relative to the endpoints of $X'_m \cup D$. In what follows, we simply write D for a (w,w')-diagram. For objects w and w', the hom-set $\mathbf{A}^L(w,w')$ is the $\Bbbk$-vector space spanned by (w,w')-diagrams modulo the STU, AS and IHX relations.

The composition of $\mathbf{A}^L$ is defined in a similar way to that of the category $\mathbf{A}$. We can define a square diagram for an (w,w')-diagram similarly. Let D be a diagram in $\mathbf{A}^L(w,w')$ and D' a diagram in $\mathbf{A}^L(w',w'')$. Deform D' to have only the parallel copies of the handle cores in each handle. Then the composition $D' \circ D$ is a diagram obtained by stacking the cabling of D on top of the square presentation of D'.

Example 4.7. For $D =$  and $D' =$ , the composition $D' \circ D$ is , where the box notation represents a linear sum of Jacobi diagrams. (See [11] and [16] for the definition of the box notation.)

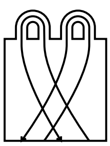
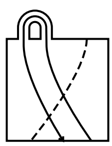
The identity morphism $\mathrm{id}_{H^{\otimes m_1} \otimes L^{\otimes n_1} \otimes \cdots \otimes H^{\otimes m_r} \otimes L^{\otimes n_r}}$ is the following diagram:

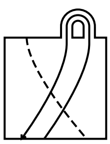
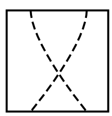


.

We can naturally extend the linear symmetric strict monoidal structure of $\mathbf{A}$ to the category $\mathbf{A}^L$, where the tensor product is defined to be the juxtaposition of the handlebodies.

Note that the symmetries in $\mathbf{A}^L$ are determined by

$P_{H,H} =$  $: H \otimes H \to H \otimes H$, $P_{H,L} =$  $: H \otimes L \to L \otimes H$,

$P_{L,H} =$  $: L \otimes H \to H \otimes L$, $P_{L,L} =$  $: L \otimes L \to L \otimes L$.

The degree of a (w,w')-diagram is defined by

$$\frac{1}{2}\#\{\text{vertices}\} - \#\{\text{univalent vertices attached to the upper line } l'\}.$$

Let $\mathbf{A}^L_d(w,w') \subset \mathbf{A}^L(w,w')$ be the subspace spanned by (w,w')-diagrams of degree d. We have $\mathbf{A}^L(w,w') = \bigoplus_{d \geq 0} \mathbf{A}^L_d(w,w')$. Since we have

$$\mathbf{A}^L_{d'}(w',w'') \circ \mathbf{A}^L_d(w,w') \subset \mathbf{A}^L_{d+d'}(w,w'')$$

and

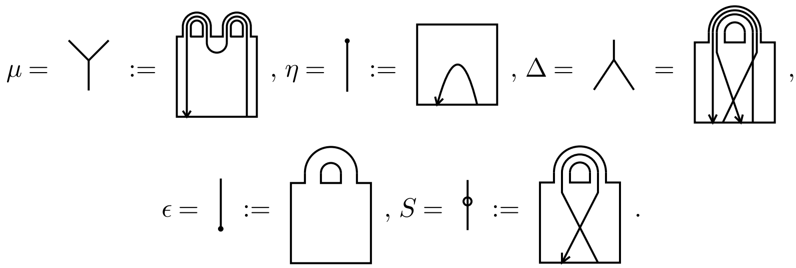
$$\mathbf{A}^L_{d'}(w,w') \otimes \mathbf{A}^L_d(z,z') \subset \mathbf{A}^L_{d+d'}(w \otimes z, w' \otimes z')$$

for any $w,w',w'',z,z' \in \mathbf{A}^L$, this grading is an $\mathbb{N}$-grading on $\mathbf{A}^L$. Note that we have $\mathbf{A}_d(m,n) = \mathbf{A}^L_d(H^{\otimes m}, H^{\otimes n})$ for $m,n \geq 0$.

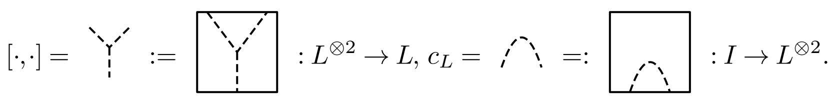
4.3. Relations for morphisms in $\mathbf{A}^L$

Here, we observe some relations for morphisms of $\mathbf{A}^L$, which we use in the proof of Theorem 4.1.

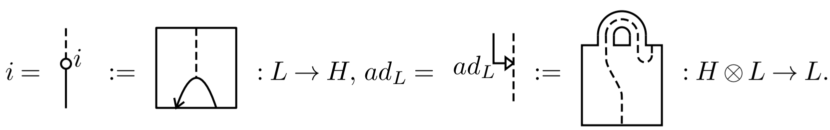
The cocommutative Hopf algebra $(H,\mu,\eta,\Delta,\epsilon,S)$ in $\mathbf{A}$ naturally induces a cocommutative Hopf algebra in $\mathbf{A}^L$ such that



Additionally, the triple $(L,[\cdot,\cdot],c_L)$ is a Lie algebra with a symmetric invariant 2-tensor in $\mathbf{A}^L$ (see Appendix A.2), where

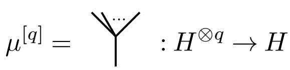


Moreover, $\mathbf{A}^L$ has two morphisms

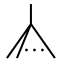


The degree of the morphism c_L is 1 and that of the others of the above morphisms is 0.

The iterated multiplications



and the iterated comultiplications

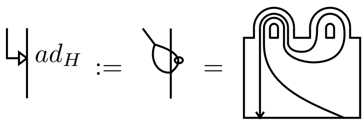
$\Delta^{[q]} =$  $: H \to H^{\otimes q}$

for $q \geq 0$ are inductively defined by

$$\mu^{[0]} = \eta, \quad \mu^{[1]} = \mathrm{id}_H, \quad \mu^{[q+1]} = \mu(\mu^{[q]} \otimes \mathrm{id}_H) \quad (q \geq 1),$$

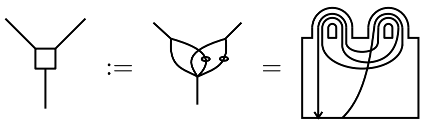
$$\Delta^{[0]} = \epsilon, \quad \Delta^{[1]} = \mathrm{id}_H, \quad \Delta^{[q+1]} = (\Delta^{[q]} \otimes \mathrm{id}_H)\Delta \quad (q \geq 1).$$

Let

$ad_H =$ 

,

which denotes the *adjoint action*, and

$comm =$  , (4.2)

which denotes the *commutator*.

Lemma 4.8. *We have*

(1) $S \circ i = -i$
(2) $\Delta \circ i = i \otimes \eta + \eta \otimes i$
(3) $\epsilon \circ i = 0$
(4) $ad_H(i \otimes i) = -i \circ [\cdot, \cdot]$.

Proof. They can be checked by diagrammatic computation. □

Let $\mathfrak{g}$ be a Lie algebra and $U = U(\mathfrak{g})$ be the universal enveloping algebra. We have a filtration $F_*(U)$ of U induced by the usual filtration of the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$. Since U has a cocommutative Hopf algebra structure, we can define the commutator operator

$$comm : U^{\otimes 2} \to U$$

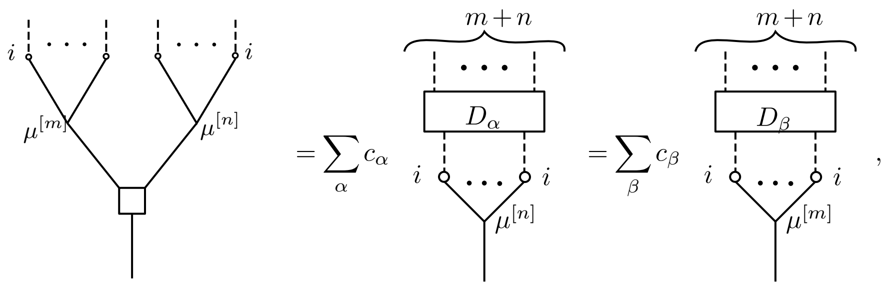
in a similar way as equation (4.2). For $x_1,\cdots,x_m,y_1,\cdots,y_n \in \mathfrak{g}$, we have

$$comm(x_1\cdots x_m, y_1\cdots y_n) \in F_{\min(m,n)}(U).$$

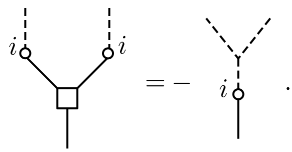
The following lemma is a diagrammatic version of this fact.

Lemma 4.9.

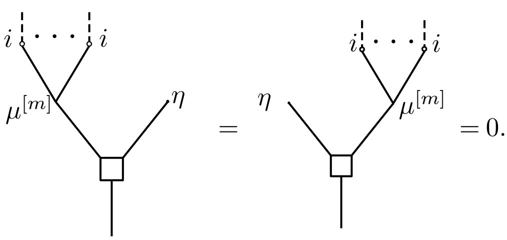
(1) *Let $m,n \geq 1$. We have*



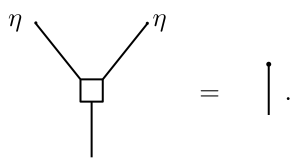
where $c_\alpha, c_\beta \in \mathbb{Z}$, and where D_α (resp. D_β) is a union of trees with m (resp. n) trivalent vertices. Moreover, for $m=n=1$, we have



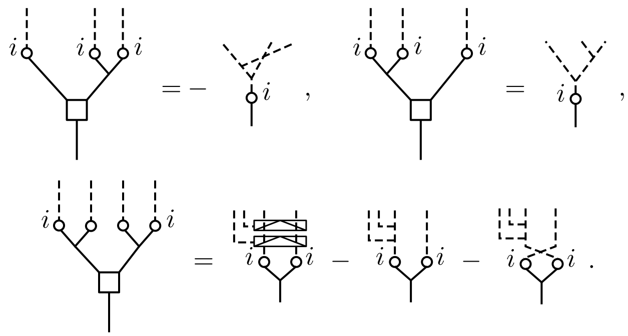
(2) *Let $m \geq 1$. We have*

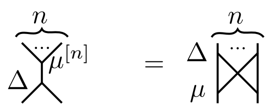


(3) *We have*

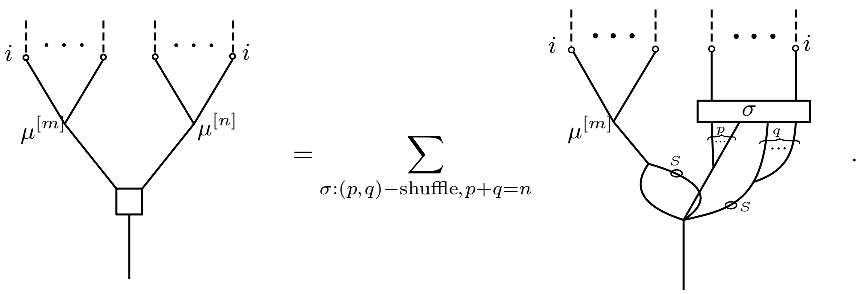


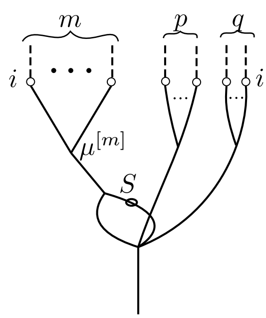
For example, we have

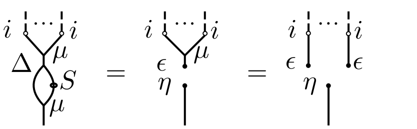


Proof of Lemma 4.9. By using Lemma 4.8 (2) and 

, we have

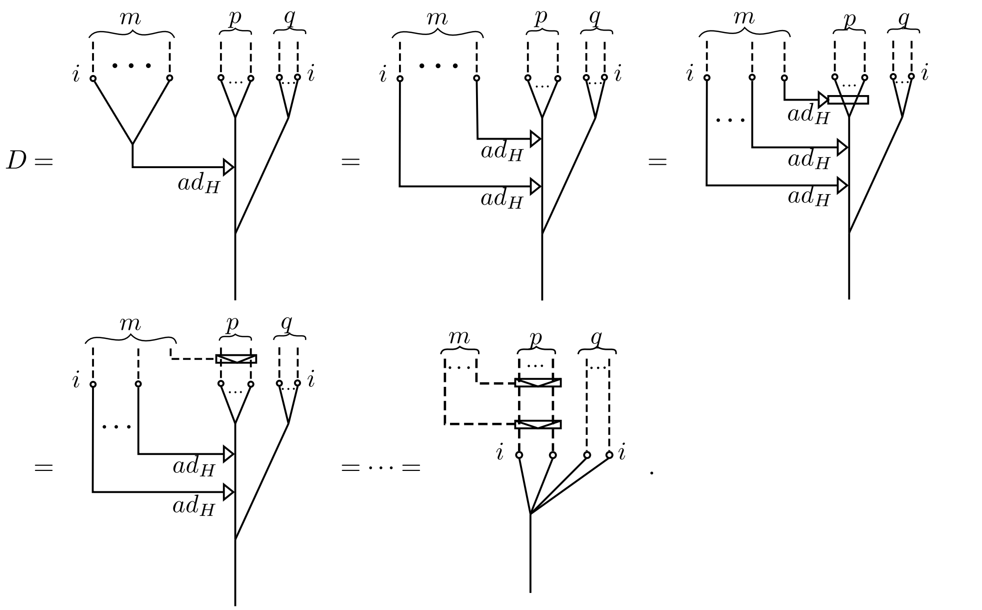


By Lemma 4.8 (1), it suffices to consider $D =$ 

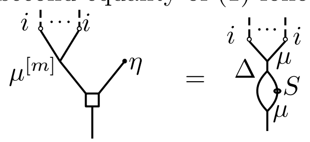
. By Lemma 4.8 (3), we have 

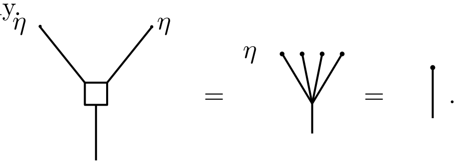
$= 0$. Thus, when $p = 0$, we have $D = 0$.

When $p \geq 1$, by Lemma 4.8 (4), we have



Note that the last term is a $\mathbb{Z}$-linear sum of unions of tree diagrams with m trivalent vertices. Therefore, the first equality of (1) follows. If $m = n = 1$, then the equality follows from the case where $m = p = 1, q = 0$. The second equality of (1) follows similarly.

The first equality of (2) follows from  $= 0$. The second equality follows similarly.

We have (3) because  . ∎

4.4. Proof of Theorem 4.1

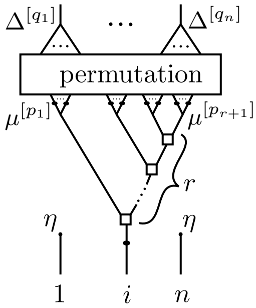
In this subsection, we prove Theorem 4.1.

For any $y_1, \cdots, y_r \in F_n$, we call $[y_1, \cdots, [y_{r-1}, y_r]] \in \Gamma_r$ an *r-fold commutator.*

For $i \in [n]$, define $d_i \in \mathrm{End}(F_n) = \mathbf{F}^{\mathrm{op}}(n,n)$ by

$$d_i(x_i) = [y_1, \cdots, [y_r, y_{r+1}]]^{\epsilon}, \quad d_i(x_j) = 1 \quad (j \neq i)$$

for $y_1, \cdots, y_{r+1} \in F_n, \epsilon \in \{\pm 1\}$, which we call an *$(r+1)$-fold commutator at i.* Via the isomorphism $\Bbbk\mathbf{F}^{\mathrm{op}}(n,n) \cong \mathbf{A}_0(n,n)$, we identify $d_i \in \mathbf{F}^{\mathrm{op}}(n,n)$ with a morphism of the following form



$\in \mathbf{A}_0(n,n),$

which we also call an $(r+1)$-*fold commutator at* i, where each (vertical line with a dot) depicts S or id_H, and $q_k, p_l \geq 0$ satisfy $\sum_{k=1}^{n} q_k = \sum_{l=1}^{r+1} p_l$.

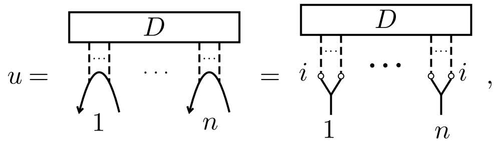
Claim 1. *An element* $g \in \mathcal{E}_r(n)$ *can be written as a convolution product*

$$g = d_{1,1} * \cdots * d_{1,l_1} * \cdots * d_{n,1} * \cdots * d_{n,l_n} * \mathrm{id}_{H^{\otimes n}},$$

where $d_{i,j}$ *is an* $(r+1)$-*fold commutator at* i *for* $i \in [n]$ $(l_i \geq 0, 1 \leq j \leq l_i)$.

Proof. Let $g \in \mathcal{E}_r(n)$. Since Γ_{r+1} is generated by $(r+1)$-fold commutators, $g(x_i)x_i^{-1}$ is a product of $(r+1)$-fold commutators or their inverses for any $i \in [n]$. Thus, we can decompose g into a convolution product of $(r+1)$-fold commutators and $\mathrm{id}_{H^{\otimes n}}$. □

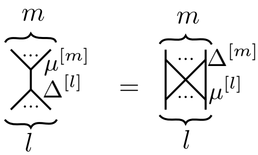
Proof of Theorem 4.1. We show that $[A_{d,k}(n), \mathcal{E}_r(n)] \subset A_{d,k+r}(n)$. We can write an element of $A_{d,k}(n) \subset \mathbf{A}^L(I, H^{\otimes n})$ as a linear sum of the following diagrams:

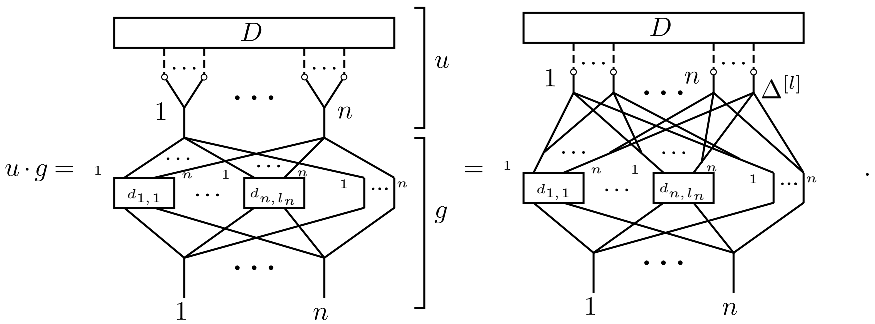


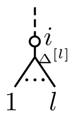
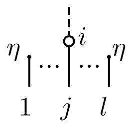
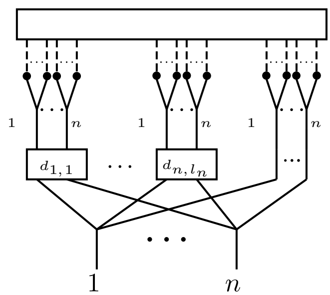
where D is a Jacobi diagram with at least k trivalent vertices. Let $g \in \mathcal{E}_r(n)$. By Claim 1, we can write g as a convolution product

$$g = d_{1,1} * \cdots * d_{1,l_1} * \cdots * d_{n,1} * \cdots * d_{n,l_n} * \mathrm{id}_{H^{\otimes n}},$$

where $d_{i,j} \in \mathbf{A}_0(n,n)$ is an $(r+1)$-fold commutator at i. Let $l = 1 + \sum_{i=1}^{n} l_i$.

By using  , we have



Here, each  is once connected to all of the diagrams $d_{1,1},\cdots,d_{n,l_n}$ and $\mathrm{id}_{H^{\otimes n}}$. Since we have  $=\sum_{j=1}^{l}$  by Lemma 4.8 (2), the element $u\cdot g$ is a linear sum of diagrams of shape  , where  denotes  or . If all  that are connected to $\mathrm{id}_{H^{\otimes n}}$ are  , then it is easily checked that the corresponding summand is just u by using Lemma 4.9 (3). Otherwise, at least one of  that are connected to diagrams $d_{1,1},\cdots,d_{n,l_n}$ are  . By using Lemma 4.9, it follows that each summand is a linear sum of diagrams with at least $k+r$ trivalent vertices. Therefore, we have $[u,g]=u\cdot g-u\in A_{d,k+r}(n)$. □

5. Contraction map

Recall that $H=\mathcal{L}_1(n)=\bigoplus_{i=1}^n\mathbb{Z}\bar{x}_i$ and $H^*=\bigoplus_{i=1}^n\mathbb{Z}v_i$. In what follows, we identify $H^*\otimes\mathcal{L}_{r+1}(n)$ with $T_r(n)$ as we remarked in Section 3.2.

5.1. Preliminaries to computation

Let $N\geq 1$. We briefly review the construction of the irreducible representations of the symmetric group $\mathfrak{S}_N$. See Fulton–Harris [6] and Sagan [21] for basic facts of representation theory of $\mathfrak{S}_N$. Let $\lambda=(\lambda_1,\cdots,\lambda_l)$ be a partition of N, and write $\lambda\vdash N$. A *Young diagram* of λ consists of λ_i boxes in the i-th row for $i\in[l]$ such that the rows of boxes are lined up on the left. A λ*-tableau* is a numbering of the boxes by the integers in $[N]$. We call a λ-tableau *standard* if the numbering increases in each row and in each column. The *canonical* λ-tableau is a standard tableau whose numbering starts from the first row from left to right and then the second row from left to right and so on.

Let t_0 be the canonical λ-tableau. Define R_{t_0} (resp. C_{t_0}) to be the subgroup of $\mathfrak{S}_N$ that preserves each row (resp. column) of t_0. We define

$$a_\lambda:=\sum_{\sigma\in R_{t_0}}\sigma,\quad b_\lambda:=\sum_{\sigma\in C_{t_0}}\mathrm{sgn}(\sigma)\sigma\in\Bbbk\mathfrak{S}_N.$$

For each $\lambda\vdash N$, the *Young symmetrizer* c_λ is defined by

$$c_\lambda=b_\lambda a_\lambda\in\Bbbk\mathfrak{S}_N. \tag{5.1}$$

The Specht module S^λ, which is an irreducible representation of $\mathfrak{S}_N$ corresponding to λ, can be constructed as

$$S^\lambda=\Bbbk\mathfrak{S}_N\cdot c_\lambda.$$

Lemma 5.1. *We have the following decomposition of $\Bbbk\mathfrak{S}_N$-bimodules*

$$\Bbbk\mathfrak{S}_N=\bigoplus_{\lambda\vdash N}\Bbbk\mathfrak{S}_N\cdot c_\lambda\cdot\Bbbk\mathfrak{S}_N.$$

Proof. This follows from basic facts of representation theory. The reader is referred to [6] and [21]. □

For $N',N''\geq 0$, let $N=N'+N''$. For $\mu\vdash N',\nu\vdash N''$, let $S^\mu\diamond S^\nu$ denote the representation of $\mathfrak{S}_N$ induced from the tensor product representation $S^\mu\boxtimes S^\nu$ of $\mathfrak{S}_{N'}\times\mathfrak{S}_{N''}$ by the inclusion of $\mathfrak{S}_{N'}\times\mathfrak{S}_{N''}$ in $\mathfrak{S}_N$. By the Littlewood–Richardson rule, we have

$$S^\mu\diamond S^\nu=\bigoplus_{\lambda\vdash N}(S^\lambda)^{LR^\lambda_{\mu,\nu}},$$

where $LR^\lambda_{\mu,\nu}$ denotes the Littlewood–Richardson coefficient. We have the following lemma by using basic facts of representation theory of $\mathfrak{S}_N$.

Lemma 5.2. *Let $N=N'+N''$ for $N',N''\geq 0$. Let $\lambda\vdash N,\mu\vdash N',\nu\vdash N''$, respectively. We have*

$$\dim_\Bbbk((c_\mu\diamond c_\nu)\cdot\Bbbk\mathfrak{S}_N\cdot c_\lambda)=LR^\lambda_{\mu,\nu}.$$

In particular, if the Littlewood–Richardson coefficient $LR^{\lambda}_{\mu,\nu} = 0$*, then we have*

$$(c_\mu \diamond c_\nu) \cdot \Bbbk\mathfrak{S}_N \cdot c_\lambda = 0.$$

5.2. Contraction map

We have an isomorphism of $\mathrm{GL}(V_n)$-modules

$$B_{d,k}(n) \cong V_n^{\otimes 2d-k} \otimes_{\Bbbk\mathfrak{S}_{2d-k}} D_{d,k}, \tag{5.2}$$

where $D_{d,k}$ is the $\Bbbk$-vector space spanned by $[2d-k]$-colored open Jacobi diagrams of degree d such that the map $\{\text{univalent vertices of } D\} \to [2d-k]$ that gives the coloring of D is a bijection. Thus, any element of $B_{d,k}(n)$ can be written in the form

$$u(w_1, \cdots, w_{2d-k}) := (w_1 \otimes \cdots \otimes w_{2d-k}) \otimes u$$

for $u \in D_{d,k}$ and $w_1, \cdots, w_{2d-k} \in V_n$.

For $\lambda \vdash 2d-k$, let $B_{d,k}(n)_\lambda$ be the *isotypic component* of $B_{d,k}(n)$ corresponding to λ; that is,

$$B_{d,k}(n)_\lambda \cong V_n^{\otimes 2d-k} \otimes_{\Bbbk\mathfrak{S}_{2d-k}} \Bbbk\mathfrak{S}_{2d-k} c_\lambda D_{d,k}.$$

We have $B_{d,k}(n) = \bigoplus_{\lambda \vdash 2d-k} B_{d,k}(n)_\lambda$.

We define a contraction map

$$c : B_{d,k}(n) \otimes T_r(n) \to B_{d,k+r}(n),$$

which is an analogue of the contraction map defined in Appendix B of [6].

Let $p \geq q$. For $I = (i_1, \cdots, i_q)$ such that $i_1, \cdots, i_q$ are distinct elements of $[p]$, define a contraction map

$$c^I : V_n^{\otimes p} \otimes (V_n^*)^{\otimes q} \to V_n^{\otimes (p-q)}$$

by

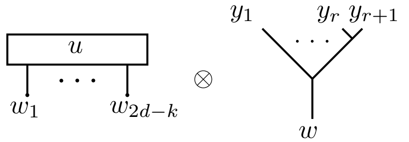
$$c^I((w_1 \otimes \cdots \otimes w_p) \otimes (y_1 \otimes \cdots \otimes y_q)) = \left(\prod_{j=1}^{q} \langle w_{i_j}, y_j \rangle \right) w_1 \otimes \cdots \hat{w}_{i_1} \cdots \hat{w}_{i_q} \cdots \otimes w_p,$$

where $\hat{w}_{i_1} \cdots \hat{w}_{i_q}$ denotes the omission of $w_{i_1}, \cdots, w_{i_q}$ and where $\langle -,- \rangle : V_n \otimes V_n^* \to \Bbbk$ denotes the dual pairing. (See [6] for details.)

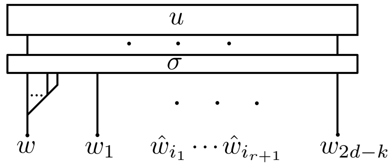
We next consider a diagrammatic version of the above contraction map c^I. Let $2d-k \geq r+1$. For $I = (i_1, \cdots, i_{r+1}) \in [2d-k]^{r+1}$ such that $i_1, \cdots, i_{r+1}$ are distinct, we define a linear map

$$c^I : B_{d,k}(n) \otimes T_r(n) \to B_{d,k+r}(n)$$

by contracting colorings of a Jacobi diagram and leaves of a rooted trivalent tree; that is,

$c^I\Bigg($ 

$\Bigg)$

$$= \left(\prod_{j=1}^{r+1} \langle w_{i_j}, y_j \rangle\right)$$ 

,

where $\sigma^{-1} = \begin{pmatrix} 1 & \cdots & r+1 & r+2 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & 2d-k \\ i_1 & \cdots & i_{r+1} & 1 & \cdots \ \hat{\hat{i}}_1 \ \cdots \ \hat{\hat{i}}_{r+1} \ \cdots & 2d-k \end{pmatrix}$. We define a *contraction map*

$$c : B_{d,k}(n) \otimes T_r(n) \to B_{d,k+r}(n)$$

by $c = \sum_{I=(i_1,\cdots,i_{r+1})\in[2d-k]^{r+1}:\text{distinct}} c^I$. By using the contraction map c, we define a map

$$\gamma^r_{d,k} : T_r(n) \to \mathrm{Hom}(B_{d,k}(n), B_{d,k+r}(n))$$

by $\gamma^r_{d,k}(g)(u') := c(u' \otimes g)$ for $g \in T_r(n), u' = u(w_1, \cdots, w_{2d-k}) \in B_{d,k}(n)$.

5.3. Vanishing conditions for the contraction map

Here, we observe that the contraction map vanishes under certain specific conditions.

For $r \geq 0$, a trivalent tree is called a *based trivalent tree of degree* r if it has one distinguished univalent vertex with no coloring (called a *base*) and $r+1$ univalent vertices (called *leaves*) that are colored by distinct elements of $[r+1]$. (Note that a based trivalent tree is different from a rooted trivalent tree.) Let L_r denote the $\mathbb{Z}$-module spanned by based trivalent trees of degree r modulo the AS and IHX relations. The symmetric group $\mathfrak{S}_{r+1}$ acts on the $\mathbb{Z}$-module L_r by the action on colorings of based trivalent trees. Then we have

$$\mathcal{L}_{r+1}(n) \cong H^{\otimes(r+1)} \otimes_{\mathbb{Z}\mathfrak{S}_{r+1}} L_r.$$

On the other hand, $\mathcal{L}_{r+1}(n)$ has a $\mathrm{GL}(n;\mathbb{Z})$-module structure by the standard action on each factor. (See [7] for representation theory of $\mathrm{GL}(n;\mathbb{Z})$.) For $\mu \vdash r+1$, let $\mathcal{L}_{r+1}(n)_\mu$ denote the isotypic component of $\mathcal{L}_{r+1}(n)$ corresponding to μ; that is,

$$\mathcal{L}_{r+1}(n)_\mu \cong H^{\otimes(r+1)} \otimes_{\mathbb{Z}\mathfrak{S}_{r+1}} \mathbb{Z}\mathfrak{S}_{r+1} c_\mu L_r.$$

We have $\mathcal{L}_{r+1}(n) = \bigoplus_{\mu\vdash r+1} \mathcal{L}_{r+1}(n)_\mu$.

For partitions λ and μ, we write $\lambda \not\supseteq \mu$ if the Young diagram of λ does not contain that of μ.

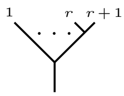
Proposition 5.3. *For $2d-k \geq r+1$, let $\lambda \vdash 2d-k$ and $\mu \vdash r+1$. We have*

$$c(B_{d,k}(n)_\lambda \otimes (H^* \otimes \mathcal{L}_{r+1}(n)_\mu)) \subset \bigoplus_{\rho : LR^\lambda_{\mu,\nu} LR^\rho_{\nu,(1)} \neq 0 \text{ for some } \nu} B_{d,k+r}(n)_\rho.$$

In particular, if $\lambda \not\supseteq \mu$, then we have

$$c(B_{d,k}(n)_\lambda \otimes (H^* \otimes \mathcal{L}_{r+1}(n)_\mu)) = 0.$$

Proof. Any element of $B_{d,k}(n)_\lambda$ is a linear sum of $(c_\lambda \cdot u)(w_1, \cdots, w_{2d-k})$, where $u(w_1, \cdots, w_{2d-k}) \in B_{d,k}(n)$. Any element of L_r is a linear sum of

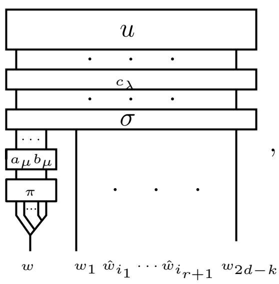
$L = \pi^{-1} \cdot$ 

for $\pi \in \mathfrak{S}_{r+1}$. Thus, any element of $H^* \otimes \mathcal{L}_{r+1}(n)_\mu$ is a linear sum of $w \otimes ((y_1 \otimes \cdots \otimes y_{r+1}) \otimes c_\mu \cdot L)$ for $w \in H^*, y_1, \cdots, y_{r+1} \in H$.

For any $I = (i_1, \cdots, i_{r+1}) \in [2d-k]^{r+1}$ such that $i_1, \cdots, i_{r+1}$ are distinct, we have

$$c^I((c_\lambda \cdot u)(w_1, \cdots, w_{2d-k}) \otimes (w \otimes ((y_1 \otimes \cdots \otimes y_{r+1}) \otimes c_\mu \cdot L))) = \prod_{j=1}^{r+1} \langle w_{i_j}, y_j \rangle D,$$

where

$D =$  ,

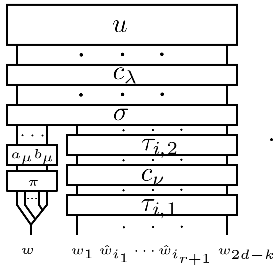
$$\sigma^{-1} = \begin{pmatrix} 1 & \cdots & r+1 & r+2 & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & 2d-k \\ i_1 & \cdots & i_{r+1} & 1 & \cdots & \hat{i}_1 & \cdots & \hat{i}_{r+1} & \cdots & 2d-k \end{pmatrix}.$$

Let $l = 2d-k-r-1$. By Lemma 5.1, we have

$$\mathrm{id}_l = \sum_{\nu \vdash l, 1 \leq i \leq \dim S^\nu} \tau_{i,1} c_\nu \tau_{i,2},$$

where $\tau_{i,1}, \tau_{i,2} \in \Bbbk\mathfrak{S}_l$. Thus, we have

$$D = \sum_{\nu \vdash l, 1 \leq i \leq \dim S^\nu}$$



If $LR^{\lambda}_{\mu,\nu} = 0$ for any $\nu \vdash l$, then we have $D = 0$ by Lemma 5.2. Otherwise, since we have

$$\mathrm{id}_1 \otimes c_\nu \in \bigoplus_{\rho \vdash l+1} (S^\rho)^{LR^{\rho}_{\nu,(1)}},$$

by the Littlewood–Richardson rule, it follows that

$$D \in \bigoplus_{\rho : LR^{\lambda}_{\mu,\nu} LR^{\rho}_{\nu,(1)} \neq 0 \text{ for some } \nu} (B_{d,k+r}(n))_\rho .$$

If $\lambda \not\supseteq \mu$, then $LR^{\lambda}_{\mu,\nu} = 0$ for any $\nu \vdash l$. Thus, we have

$$c(B_{d,k}(n)_\lambda \otimes (H^* \otimes \mathcal{L}_{r+1}(n)_\mu)) = 0.$$

□

Remark 5.4. Note that we have $\mathcal{L}_2(n) = \mathcal{L}_2(n)_{(1^2)}$. Thus, the restriction

$$c : B_{d,k}(n)_\lambda \otimes (H^* \otimes \mathcal{L}_2(n)_{(1^2)}) \to B_{d,k+1}(n)_\rho$$

of the contraction map vanishes unless ρ can be obtained from λ by taking away one box from each of two different rows of λ and then by adding one box.

6. Correspondence between the map $\tilde{\beta}^r_{d,k}$ and the map $\gamma^r_{d,k}$

In this section, we prove that the map $\tilde{\beta}^r_{d,k}$ defined in Section 4 can be identified with the map $\gamma^r_{d,k}$ defined in Section 5 via the Johnson homomorphism of $\mathrm{End}(F_n)$ defined in Section 3.

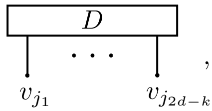
Theorem 6.1. *We have $\tilde{\beta}^r_{d,k} = (-1)^r \cdot \gamma^r_{d,k} \circ \tilde{\tau}_r$. That is, we have the following commutative diagram (up to sign):*

$$\begin{array}{ccc} \mathrm{gr}^r(\mathcal{E}_*(n)) & \xrightarrow{\tilde{\beta}^r_{d,k}} & \mathrm{Hom}(B_{d,k}(n), B_{d,k+r}(n)) \\ \cong \downarrow \tilde{\tau}_r & \nearrow_{\gamma^r_{d,k}} & \\ H^* \otimes \mathcal{L}_{r+1}(n). & & \end{array}$$

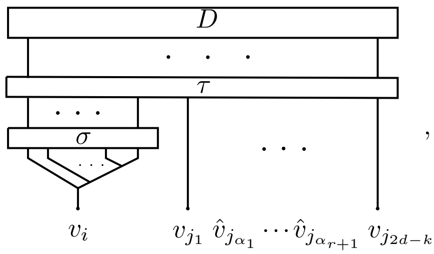
Proof. The $\mathbb{Z}$-module $H^* \otimes \mathcal{L}_{r+1}(n)$ is spanned by $v_i \otimes [\bar{x}_{i_1}, \cdots, [\bar{x}_{i_r}, \bar{x}_{i_{r+1}}] \cdots]$ for $i, i_1, \cdots, i_{r+1} \in [n]$. Define $\phi \in \mathrm{End}(F_n)$ by

$$\phi(x_i) = [x_{i_1}, \cdots, [x_{i_r}, x_{i_{r+1}}] \cdots] \cdot x_i, \quad \phi(x_j) = x_j \; (j \neq i).$$

It is easily checked that $\phi \in \mathcal{E}_r(n)$ and that $\tilde{\tau}_r([\phi]_r) = v_i \otimes [\bar{x}_{i_1}, \cdots, [\bar{x}_{i_r}, \bar{x}_{i_{r+1}}] \cdots]$, where $[\phi]_r \in \mathrm{gr}^r(\mathcal{E}_*(n))$ denotes the image of ϕ under the projection.

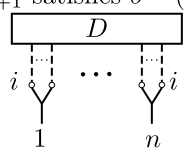
Any element of $B_{d,k}(n)$ can be written as a linear sum of $u =$ 

, where $1 \leq j_1 \leq \cdots \leq j_{2d-k} \leq n$, by arranging the univalent vertices according to the order of indices of the colorings from left to right. We have

$$\gamma^r_{d,k} \circ \tilde{\tau}_r([\phi]_r)(u)$$
$$= c(u \otimes (v_i \otimes [\bar{x}_{i_1}, \cdots, [\bar{x}_{i_r}, \bar{x}_{i_{r+1}}] \cdots]))$$
$$= \sum_{(\alpha_l) \in [2d-k]^{r+1}:\ \text{distinct}} \left(\prod_{l=1}^{r+1} \langle v_{j_{\alpha_l}}, \bar{x}_{i_l} \rangle \right)$$ 

,

where $\tau^{-1} \in \mathfrak{S}_{2d-k}$ is the $(r+1, 2d-k-r-1)$-shuffle that maps $[r+1] \subset [2d-k]$ to $\{\alpha_l\}$, and $\sigma \in \mathfrak{S}_{r+1}$ satisfies $\sigma^{-1}(l) = \tau(\alpha_l)$ for any $l \in [r+1]$.

Let $\tilde{u} =$ 

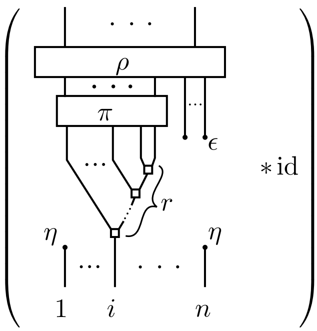
$\in A_{d,k}(n)$, which can be obtained from u by replacing univalent vertices with 

and combining solid lines whose corresponding colorings of u are the same. Then $\tilde{u}$ is a lift of u; that is, we have $\theta_{d,n,k}(\tilde{u}) = u$. By the definition of $\tilde{\beta}^r_{d,k}$, we have

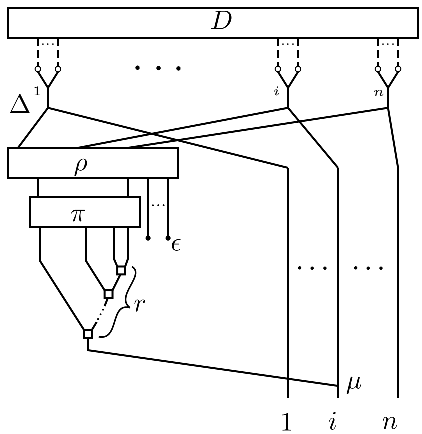
$$\tilde{\beta}^r_{d,k}([\phi]_r)(u) = [u, [\phi]_r] = \theta_{d,n,k+r}([\tilde{u}, \phi]).$$

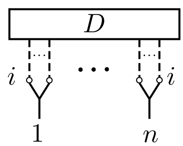
We have

$$[\tilde{u},\phi] = \tilde{u}\cdot\phi - \tilde{u}$$

$= \tilde{u}\cdot$ 

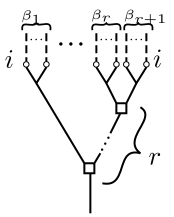
$-\tilde{u}$

$=$ 

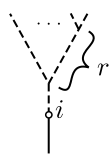
$-$ 

$\in A_{d,k+r}(n),$

where $\rho^{-1}\in\mathfrak{S}_n$ is the $(r+1,n-r-1)$-shuffle that maps $[r+1]\subset[n]$ to $\{i_1,\cdots,i_{r+1}\}$ and $\pi\in\mathfrak{S}_{r+1}$ satisfies $\pi^{-1}(j)=\rho(i_j)$ for any $j\in[r+1]$. By using Lemma 4.9, we have for $\beta_1,\cdots,\beta_{r+1}\geq 0$,



$=$

- $(\beta_j = 0 \text{ for all } j\in[r+1])$
- $(-1)^r$ 

$(\beta_j = 1 \text{ for all } j\in[r+1])$
- a linear sum of diagrams with at least $r+1$ trivalent vertices (otherwise).

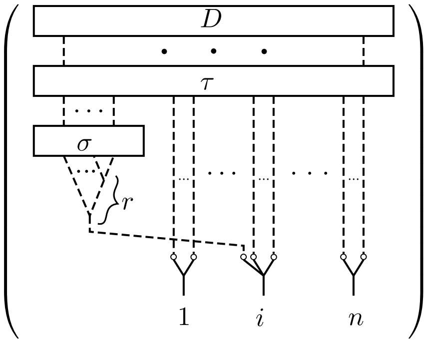
(6.1)

In the last case, the corresponding term of $[\tilde{u},\phi]$ is included in $A_{d,k+r+1}(n)$.

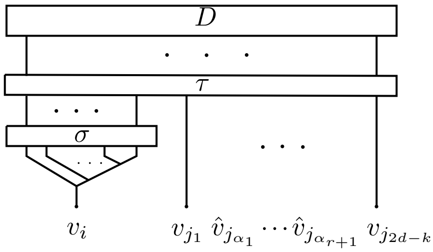
Thus, by equation (6.1) and Lemma 4.8 (2), we have

$$\tilde{\beta}^r_{d,k}([\phi]_r)(u)$$

$$= \sum_{(\alpha_l)\in[2d-k]^{r+1}} (-1)^r \left(\prod_{j=1}^{r+1} \langle v_{j_{\alpha_l}}, \bar{x}_{i_l}\rangle\right) \theta_{d,n,k+r}$$



$$= (-1)^r \sum_{(\alpha_l)\in[2d-k]^{r+1}} \left(\prod_{l=1}^{r+1} \langle v_{j_{\alpha_l}}, \bar{x}_{i_l}\rangle\right)$$



$$= (-1)^r \gamma^r_{d,k} \cdot \tilde{\tau}_r([\phi]_r)(u).$$

□

7. The $\mathrm{GL}(V_n)$-module structure of $B_d(n)$

In this section, we consider the $\mathrm{GL}(V_n)$-module structure of $B_d(n)$ and give a decomposition of $B_d(n)$ with respect to connected parts. Moreover, we compute the irreducible decomposition of $B_d(n)$ for $d = 3,4,5$ and that of $B_{d,0}(n), B_{d,1}(n)$ for any d. Lastly, we show the surjectivity of the bracket map which we defined in Section 4.

Let $B^c_{d,k}(n) \subset B_{d,k}(n)$ denote the *connected part* of $B_{d,k}(n)$, which is spanned by connected V_n-colored open Jacobi diagrams. Let $D^c_{d,k} \subset D_{d,k}$ denote the *connected part* of $D_{d,k}$, which is spanned by connected $[2d-k]$-colored open Jacobi diagrams. We have an isomorphism of $\mathrm{GL}(V_n)$-modules

$$B^c_{d,k}(n) \cong V_n^{\otimes 2d-k} \otimes_{\Bbbk\mathfrak{S}_{2d-k}} D^c_{d,k},$$

which is the connected version of equation (5.2).

The direct sum $\bigoplus_{d\geq 0} B_d(n)$ has the following coalgebra structure. This is an analogue of the coalgebra structure of the space of open Jacobi diagrams colored by one element [2]. Let $C = \bigcup_{i\in I} C_i$ be a presentation of a diagram $C \in \bigoplus_{d\geq 0} B_d(n)$ as the disjoint union of its connected components. The comultiplication Δ is defined by

$$\Delta(C) = \sum_{J\subset I} \left(\bigcup_{i\in J} C_i \right) \otimes \left(\bigcup_{i\in I\setminus J} C_i \right).$$

Note that the connected part $\bigoplus_{d,k\geq 0} B^c_{d,k}(n)$ coincides with the *primitive part* of the coalgebra $\bigoplus_{d\geq 0} B_d(n)$.

7.1. Decomposition of $B_d(n)$ with respect to connected parts

Note that $D^c_{d,k} \neq 0$ if and only if $d-1 \leq k \leq 2d-2$ because each element of $D^c_{d,k}$ has at least two univalent vertices and is connected. For $d \geq 1, k \geq 0$, the pair (d,k) is called a *good pair* if $d-1 \leq k \leq 2d-2$. We consider the following decomposition of a pair (d,k) to consider the decomposition of an element of $D_{d,k}$ into the connected parts.

Definition 7.1. Let d, $k \geq 0$. A *decomposition of* (d,k) *into good pairs* is a sequence of triples of integers

$$\pi = ((a_1,d_1,k_1), \cdots, (a_l,d_l,k_l))$$

such that (d_i,k_i) are good pairs, $a_i \geq 1$,

$$\sum_{i=1}^{l} a_i d_i = d, \quad \sum_{i=1}^{l} a_i k_i = k,$$

and

$$(d_1,k_1) > (d_2,k_2) > \cdots > (d_l,k_l)$$

in the lexicographical order.

Let $\Pi(d,k)$ be the set of all decompositions of (d,k) into good pairs.

For example, we have

$$\Pi(4,2) = \{((1,3,2),(1,1,0)),((1,2,2),(2,1,0)),((2,2,1))\}. \tag{7.1}$$

For any diagram $K \in D_{d,k}$, we can assign a decomposition of (d,k) into good pairs such that d_i and k_i correspond to the degree and the number of trivalent vertices of each connected component of K, respectively, and a_i corresponds to the multiplicity of (d_i,k_i). We call a coloring of $K = \bigsqcup_{1\leq i\leq l, 1\leq j\leq a_i} K_i^{(j)} \in D_{d,k}$ *standard* if the set of colorings of $K_i^{(j)} \in D^c_{d_i,k_i}$ is

$$\left\{ \sum_{p=1}^{i-1} (2d_p - k_p)a_p + (j-1)(2d_i - k_i) + 1, \cdots, \sum_{p=1}^{i-1} (2d_p - k_p)a_p + j(2d_i - k_i) \right\}$$

for each $i \in [l], j \in [a_i]$.

Theorem 7.2. *For $d,k,n \geq 0$, we have an isomorphism of* $\mathrm{GL}(V_n)$*-modules*

$$B_{d,k}(n) \cong \bigoplus_{\pi=((a_1,d_1,k_1),\cdots,(a_l,d_l,k_l))\in\Pi(d,k)} \left(\bigotimes_{i=1}^{l} \mathrm{Sym}^{a_i}(B^c_{d_i,k_i}(n))\right). \tag{7.2}$$

To prove this, we need the following proposition.

Proposition 7.3. *Let $d,k \geq 0$. We have an isomorphism of $\mathfrak{S}_{2d-k}$-modules*

$$D_{d,k} \cong \bigoplus_{\pi=((a_1,d_1,k_1),\cdots,(a_l,d_l,k_l))\in\Pi(d,k)} \mathrm{Ind}^{\mathfrak{S}_{2d-k}}_{\prod_{i=1}^{l}(\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i})}\Big(\bigotimes_{i=1}^{l}(D^c_{d_i,k_i})^{\otimes a_i}\Big), \tag{7.3}$$

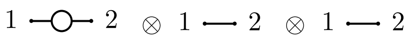
where $\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i} = {\mathfrak{S}_{2d_i-k_i}}^{a_i} \rtimes \mathfrak{S}_{a_i} \subset \mathfrak{S}_{(2d_i-k_i)a_i}$ is the wreath product.

For example, we have an isomorphism of $\mathfrak{S}_6$-modules for $(d,k)=(4,2)$, which corresponds to equation (7.1),

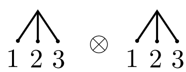
$$D_{4,2} \cong \mathrm{Ind}^{\mathfrak{S}_6}_{\mathfrak{S}_4\times\mathfrak{S}_2}(D^c_{3,2}\otimes D^c_{1,0}) \oplus \mathrm{Ind}^{\mathfrak{S}_6}_{\mathfrak{S}_2\times(\mathfrak{S}_2\wr\mathfrak{S}_2)}(D^c_{2,2}\otimes(D^c_{1,0})^{\otimes 2}) \oplus \mathrm{Ind}^{\mathfrak{S}_6}_{\mathfrak{S}_3\wr\mathfrak{S}_2}(D^c_{2,1})^{\otimes 2}.$$

For example,

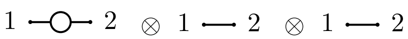
 $\in \mathrm{Ind}^{\mathfrak{S}_6}_{\mathfrak{S}_4\times\mathfrak{S}_2}(D^c_{3,2}\otimes D^c_{1,0})$,

 $\in \mathrm{Ind}^{\mathfrak{S}_6}_{\mathfrak{S}_2\times(\mathfrak{S}_2\wr\mathfrak{S}_2)}(D^c_{2,2}\otimes(D^c_{1,0})^{\otimes 2})$

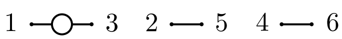
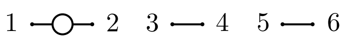
and

 $\in \mathrm{Ind}^{\mathfrak{S}_6}_{\mathfrak{S}_3\wr\mathfrak{S}_2}(D^c_{2,1})^{\otimes 2}$.

Via the above isomorphism, the element

$(2,3)(4,5)\cdot($  $) \in \mathrm{Ind}^{\mathfrak{S}_6}_{\mathfrak{S}_2\times(\mathfrak{S}_2\wr\mathfrak{S}_2)}(D^c_{2,2}\otimes(D^c_{1,0})^{\otimes 2})$

corresponds to the element

 $= (2,3)(4,5)\cdot($  $) \in D_{4,2}$.

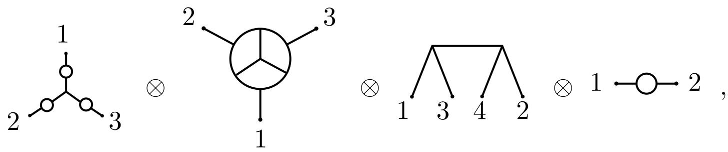
Proof of Proposition 7.3. Let $D'_{d,k}$ denote the right-hand side of equation (7.3).

For any coset $\sigma \in \mathfrak{S}_{2d-k}/\prod_{i=1}^{l}(\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i})$, we fix a representative $\tilde{\sigma}\in\mathfrak{S}_{2d-k}$ of σ. Any element of $D'_{d,k}$ can be written uniquely as a linear sum of

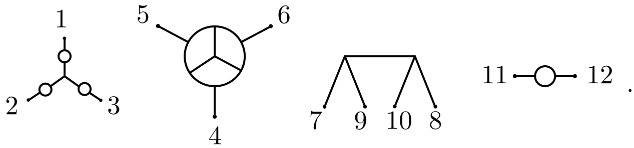
$$K = \tilde{\sigma}\cdot \bigotimes_{1\leq i\leq l, 1\leq j\leq a_i} K_i^{(j)},$$

where $K_i^{(j)} \in D^c_{d_i,k_i}$. We assign $\bigsqcup_{1\leq i\leq l,1\leq j\leq a_i} K_i^{(j)}$ a standard coloring in $[2d-k]$ according to the order of the colorings in $\bigsqcup_{i=1}^{l}[2d_i-k_i]^{a_i}$ of $\bigotimes_{1\leq i\leq l,1\leq j\leq a_i} K_i^{(j)}$. For example, if

$$\bigotimes_{1\leq i\leq l, 1\leq j\leq a_i} K_i^{(j)} =$$



then the corresponding coloring of $\bigsqcup_{1\leq i\leq l, 1\leq j\leq a_i} K_i^{(j)}$ is



Define a map $\Psi : D'_{d,k} \to D_{d,k}$ by

$$\Psi(K) = \tilde{\sigma} \cdot \bigsqcup_{1\leq i\leq l, 1\leq j\leq a_i} K_i^{(j)},$$

where $\tilde{\sigma} \in \mathfrak{S}_{2d-k}$ acts on the colorings in $[2d-k]$. We can check that the map Ψ is an $\mathfrak{S}_{2d-k}$-module map.

We need to check that Ψ is bijective. If we have $\Psi(K) = \Psi(L)$ for $K = \tilde{\sigma} \cdot \bigotimes_{1\leq i\leq l, 1\leq j\leq a_i} K_i^{(j)}$, $L = \tilde{\tau} \cdot \bigotimes_{1\leq i\leq l, 1\leq j\leq a_i} L_i^{(j)}$, then we have $\sigma = \tau$ by looking at the set of colorings of each connected component. Since we fix the representatives of cosets of $\mathfrak{S}_{2d-k}/\prod_{i=1}^{l}(\mathfrak{S}_{2d_i-k_i} \wr \mathfrak{S}_{a_i})$, we have $\tilde{\sigma} = \tilde{\tau}$. Thus, we have $K = L$ and Ψ is injective. For any element $K \in D_{d,k}$, we can take $\sigma \in \mathfrak{S}_{2d-k}/\prod_{i=1}^{l}(\mathfrak{S}_{2d_i-k_i} \wr \mathfrak{S}_{a_i})$ such that $K = \tilde{\sigma} \cdot \bigsqcup_{1\leq i\leq l, 1\leq j\leq a_i} K_i^{(j)}$, where $K_i^{(j)} \in D^c_{(d_i,k_i)}$ and $\bigsqcup_{1\leq i\leq l, 1\leq j\leq a_i} K_i^{(j)}$ has a standard coloring. Therefore, Ψ is surjective. □

Proof of Theorem 7.2. By Proposition 7.3, we have

$$\begin{aligned} B_{d,k}(n) &\cong V_n^{\otimes 2d-k} \otimes_{\Bbbk\mathfrak{S}_{2d-k}} D_{d,k} \\ &\cong \bigoplus_{\pi\in\Pi(d,k)} \left(V_n^{\otimes 2d-k} \otimes_{\Bbbk\mathfrak{S}_{2d-k}} \mathrm{Ind}^{\mathfrak{S}_{2d-k}}_{\prod_{i=1}^{l}(\mathfrak{S}_{2d_i-k_i} \wr \mathfrak{S}_{a_i})} \left(\bigotimes_{i=1}^{l} (D^c_{d_i,k_i})^{\otimes a_i} \right) \right). \end{aligned}$$

Moreover, we can check equation (7.2) as follows.

$$
\begin{aligned}
&V_n^{\otimes 2d-k} \otimes_{\Bbbk\mathfrak{S}_{2d-k}} \operatorname{Ind}^{\mathfrak{S}_{2d-k}}_{\prod_{i=1}^{l}(\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i})}\left(\bigotimes_{i=1}^{l}(D^c_{d_i,k_i})^{\otimes a_i}\right)\\
&\cong V_n^{\otimes 2d-k} \otimes_{\Bbbk\mathfrak{S}_{2d-k}} \operatorname{Ind}^{\mathfrak{S}_{2d-k}}_{\prod_{i=1}^{l}\mathfrak{S}_{a_i(2d_i-k_i)}}\left(\operatorname{Ind}^{\prod_{i=1}^{l}\mathfrak{S}_{a_i(2d_i-k_i)}}_{\prod_{i=1}^{l}(\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i})}\left(\bigotimes_{i=1}^{l}(D^c_{d_i,k_i})^{\otimes a_i}\right)\right)\\
&\cong V_n^{\otimes 2d-k} \otimes_{\Bbbk\mathfrak{S}_{2d-k}} \operatorname{Ind}^{\mathfrak{S}_{2d-k}}_{\prod_{i=1}^{l}\mathfrak{S}_{a_i(2d_i-k_i)}}\left(\bigotimes_{i=1}^{l}\operatorname{Ind}^{\mathfrak{S}_{a_i(2d_i-k_i)}}_{\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i}}\left((D^c_{d_i,k_i})^{\otimes a_i}\right)\right)\\
&\cong V_n^{\otimes 2d-k} \otimes_{\Bbbk(\prod_{i=1}^{l}\mathfrak{S}_{a_i(2d_i-k_i)})}\left(\bigotimes_{i=1}^{l}\operatorname{Ind}^{\mathfrak{S}_{a_i(2d_i-k_i)}}_{\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i}}\left((D^c_{d_i,k_i})^{\otimes a_i}\right)\right)\\
&\cong \bigotimes_{i=1}^{l}\left(V_n^{\otimes a_i(2d_i-k_i)} \otimes_{\Bbbk\mathfrak{S}_{a_i(2d_i-k_i)}}\left(\operatorname{Ind}^{\mathfrak{S}_{a_i(2d_i-k_i)}}_{\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i}}\left((D^c_{d_i,k_i})^{\otimes a_i}\right)\right)\right)\\
&\cong \bigotimes_{i=1}^{l}\left(V_n^{\otimes a_i(2d_i-k_i)} \otimes_{\Bbbk(\mathfrak{S}_{2d_i-k_i}\wr\mathfrak{S}_{a_i})}\left((D^c_{d_i,k_i})^{\otimes a_i}\right)\right)\\
&\cong \bigotimes_{i=1}^{l}\operatorname{Sym}^{a_i}\left(V_n^{\otimes(2d_i-k_i)} \otimes_{\Bbbk\mathfrak{S}_{2d_i-k_i}} D^c_{d_i,k_i}\right)\\
&\cong \bigotimes_{i=1}^{l}\operatorname{Sym}^{a_i}\left(B^c_{d_i,k_i}(n)\right).
\end{aligned}
$$

□

7.2. Irreducible decomposition of $B_d(n)$ as $\mathrm{GL}(V_n)$-modules

In this subsection, for simplicity, we write $V = V_n$, $B_{d,k} = B_{d,k}(n)$ and $B^c_{d,k} = B^c_{d,k}(n)$.

Let N be a nonnegative integer and $\lambda \vdash N$. Recall from Section 5.1 that S^λ denotes the Specht module, which is an irreducible representation of $\mathfrak{S}_N$ corresponding to λ. Let $V_\lambda = \mathbb{S}_\lambda V$ denote the image of V under the Schur functor $\mathbb{S}_\lambda$. Note that V_λ is a simple $\mathrm{GL}(V)$-module if $n \geq r(\lambda)$ and that $V_\lambda = 0$ if $n < r(\lambda)$, where $r(\lambda)$ is the number of rows of λ.

We use the Littlewood–Richardson rule, plethysms and results by Bar-Natan [4] to compute the irreducible decompositions of the $\mathrm{GL}(V)$-modules B_d.

Proposition 7.4 (Bar-Natan [4]). *As $\mathfrak{S}_{2d-k}$-modules, we have isomorphisms*

$$D^c_{1,0} \cong S^{(2)},$$

$$D^c_{2,1} \cong S^{(1^3)}, \quad D^c_{2,2} \cong S^{(2)},$$

$$D^c_{3,2} \cong S^{(2^2)}, \quad D^c_{3,3} \cong S^{(1^3)}, \quad D^c_{3,4} \cong S^{(2)},$$

$$D^c_{4,3} \cong S^{(3,1^2)}, \quad D^c_{4,4} \cong S^{(4)} \oplus S^{(2^2)}, \quad D^c_{4,5} \cong S^{(1^3)}, \quad D^c_{4,6} \cong S^{(2)},$$

$$D^c_{5,4} \cong S^{(4,2)} \oplus S^{(2^3)} \oplus S^{(3,1^3)}, \quad D^c_{5,5} \cong (S^{(3,1^2)})^{\oplus 2},$$

$$D^c_{5,6} \cong S^{(4)} \oplus (S^{(2^2)})^{\oplus 2}, \quad D^c_{5,7} \cong (S^{(1^3)})^{\oplus 2}, \quad D^c_{5,8} \cong (S^{(2)})^{\oplus 2}.$$

Lemma 7.5. *We have the following isomorphisms of the* $\mathrm{GL}(V)$*-modules:*

$$B^c_{1,0} \cong V_{(2)},$$

$$B^c_{2,1} \cong V_{(1^3)}, \quad B^c_{2,2} \cong V_{(2)},$$

$$B^c_{3,2} \cong V_{(2^2)}, \quad B^c_{3,3} \cong V_{(1^3)}, \quad B^c_{3,4} \cong V_{(2)},$$

$$B^c_{4,3} \cong V_{(3,1^2)}, \quad B^c_{4,4} \cong V_{(4)} \oplus V_{(2^2)}, \quad B^c_{4,5} \cong V_{(1^3)}, \quad B^c_{4,6} \cong V_{(2)},$$

$$B^c_{5,4} \cong V_{(4,2)} \oplus V_{(2^3)} \oplus V_{(3,1^3)}, \quad B^c_{5,5} \cong (V_{(3,1^2)})^{\oplus 2},$$

$$B^c_{5,6} \cong V_{(4)} \oplus (V_{(2^2)})^{\oplus 2}, \quad B^c_{5,7} \cong (V_{(1^3)})^{\oplus 2}, \quad B^c_{5,8} \cong (V_{(2)})^{\oplus 2}.$$

Proof. These follow from Proposition 7.4. □

Proposition 7.6. *For* $d = 3,4,5$*, we have the following irreducible decompositions of the* $\mathrm{GL}(V)$*-modules* B_d*.*

(1) *We have* $B_3 = B_{3,0} \oplus \cdots \oplus B_{3,4}$*, where*

$$\begin{aligned}
B_{3,0} &\cong V_{(6)} \oplus V_{(4,2)} \oplus V_{(2^3)},\\
B_{3,1} &\cong V_{(3,1^2)} \oplus V_{(2,1^3)},\\
B_{3,2} &\cong V_{(4)} \oplus V_{(3,1)} \oplus (V_{(2^2)})^{\oplus 2},\\
B_{3,3} &= B^c_{3,3} \cong V_{(1^3)},\\
B_{3,4} &= B^c_{3,4} \cong V_{(2)}.
\end{aligned}$$

(2) *We have* $B_4 = B_{4,0} \oplus \cdots \oplus B_{4,6}$*, where*

$$\begin{aligned}
B_{4,0} &\cong V_{(8)} \oplus V_{(6,2)} \oplus V_{(4^2)} \oplus V_{(4,2^2)} \oplus V_{(2^4)},\\
B_{4,1} &\cong V_{(5,1^2)} \oplus V_{(4,1^3)} \oplus V_{(3^2,1)} \oplus V_{(3,2,1^2)} \oplus V_{(2^2,1^3)},\\
B_{4,2} &\cong V_{(6)} \oplus V_{(5,1)} \oplus (V_{(4,2)})^{\oplus 3} \oplus (V_{(3,2,1)})^{\oplus 2} \oplus (V_{(2^3)})^{\oplus 3} \oplus V_{(2,1^4)},\\
B_{4,3} &\cong (V_{(3,1^2)})^{\oplus 3} \oplus (V_{(2,1^3)})^{\oplus 2},\\
B_{4,4} &\cong (V_{(4)})^{\oplus 3} \oplus V_{(3,1)} \oplus (V_{(2^2)})^{\oplus 3},\\
B_{4,5} &\cong V_{(1^3)},\\
B_{4,6} &\cong V_{(2)}.
\end{aligned}$$

(3) *We have* $B_5 = B_{5,0} \oplus \cdots \oplus B_{5,8}$, *where*

$$\begin{aligned}
B_{5,0} &\cong V_{(10)} \oplus V_{(8,2)} \oplus V_{(6,4)} \oplus V_{(6,2^2)} \oplus V_{(4^2,2)} \oplus V_{(4,2^3)} \oplus V_{(2^5)},\\
B_{5,1} &\cong V_{(7,1^2)} \oplus V_{(6,1^3)} \oplus V_{(5,3,1)} \oplus V_{(5,2,1^2)} \oplus V_{(4,3,1^2)} \oplus V_{(4,2,1^3)}\\
&\quad \oplus V_{(3^3)} \oplus V_{(3^2,2,1)} \oplus V_{(3,2^2,1^2)} \oplus V_{(2^3,1^3)},\\
B_{5,2} &\cong V_{(8)} \oplus V_{(7,1)} \oplus (V_{(6,2)})^{\oplus 3} \oplus V_{(5,3)} \oplus (V_{(5,2,1)})^{\oplus 2} \oplus (V_{(4^2)})^{\oplus 2}\\
&\quad \oplus (V_{(4,3,1)})^{\oplus 2} \oplus (V_{(4,2^2)})^{\oplus 5} \oplus V_{(4,1^4)} \oplus V_{(3^2,1^2)} \oplus (V_{(3,2^2,1)})^{\oplus 3}\\
&\quad \oplus V_{(3,2,1^3)} \oplus V_{(3,1^5)} \oplus (V_{(2^4)})^{\oplus 3} \oplus V_{(2^2,1^4)},\\
B_{5,3} &\cong (V_{(5,1^2)})^{\oplus 3} \oplus (V_{(4,2,1)})^{\oplus 2} \oplus (V_{(4,1^3)})^{\oplus 4} \oplus (V_{(3^2,1)})^{\oplus 4} \oplus (V_{(3,2,1^2)})^{\oplus 5}\\
&\quad \oplus V_{(3,1^4)} \oplus (V_{(2^2,1^3)})^{\oplus 3},\\
B_{5,4} &\cong (V_{(6)})^{\oplus 3} \oplus (V_{(5,1)})^{\oplus 3} \oplus (V_{(4,2)})^{\oplus 8} \oplus (V_{(3,2,1)})^{\oplus 4} \oplus V_{(3,1^3)} \oplus (V_{(2^3)})^{\oplus 6}\\
&\quad \oplus V_{(2^2,1^2)} \oplus V_{(2,1^4)} \oplus V_{(1^6)},\\
B_{5,5} &\cong (V_{(3,1^2)})^{\oplus 5} \oplus (V_{(2,1^3)})^{\oplus 3},\\
B_{5,6} &\cong (V_{(4)})^{\oplus 3} \oplus (V_{(3,1)})^{\oplus 2} \oplus (V_{(2^2)})^{\oplus 4},\\
B_{5,7} &\cong (V_{(1^3)})^{\oplus 2},\\
B_{5,8} &\cong (V_{(2)})^{\oplus 2}.
\end{aligned}$$

Proof. By using Theorem 7.2, Lemma 7.5 and plethysm, we have

$$B_{3,0} \cong \mathrm{Sym}^3(B^c_{1,0}) \cong \mathbb{S}_{(3)}(\mathbb{S}_{(2)}V) \cong V_{(6)} \oplus V_{(4,2)} \oplus V_{(2^3)}.$$

By using Theorem 7.2, Lemma 7.5 and the Littlewood–Richardson rule, we have

$$B_{3,1} \cong B^c_{2,1} \otimes B^c_{1,0} \cong V_{(1^3)} \otimes V_{(2)} \cong V_{(3,1^2)} \oplus V_{(2,1^3)},$$

and

$$B_{3,2} \cong B^c_{3,2} \oplus (B^c_{2,2} \otimes B^c_{1,0}) \cong V_{(2^2)} \oplus (V_{(4)} \oplus V_{(3,1)} \oplus V_{(2^2)}).$$

The other isomorphisms of (1) follow from Lemma 7.5.

The irreducible decompositions (2) and (3) follow in a similar way. □

We need the irreducible decompositions of $B_{d,0}$ and $B_{d,1}$ to study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$. For $\lambda = (\lambda_1, \cdots, \lambda_r) \vdash N$, let 2λ denote the partition $(2\lambda_1, \cdots, 2\lambda_r)$ of $2N$.

Proposition 7.7. *For any* $d \geq 0$, *we have*

$$B_{d,0} \cong \bigoplus_{\lambda \vdash d} V_{2\lambda}.$$

For any $d \geq 2$, *we have*

$$B_{d,1} \cong \bigoplus_{\lambda \vdash 2d-1 \text{ with exactly 3 odd parts}} V_\lambda.$$

Proof. By Theorem 5.4.23 in [14], we have

$$\mathbb{S}_{(d)}(\mathbb{S}_{(2)}V) \cong \bigoplus_{\lambda \vdash d} V_{2\lambda}.$$

Therefore, by Theorem 7.2 and Lemma 7.5, we have

$$B_{d,0} \cong \mathrm{Sym}^d(B^c_{1,0}) \cong \mathbb{S}_{(d)}(\mathbb{S}_{(2)}V) \cong \bigoplus_{\lambda \vdash d} V_{2\lambda}.$$

By Theorem 7.2, Lemma 7.5, plethysm and the Littlewood–Richardson rule, we have

$$B_{d,1} \cong B^c_{2,1} \otimes \mathrm{Sym}^{d-2}(B^c_{1,0}) \cong V_{(1^3)} \otimes \bigoplus_{\mu \vdash d-2} V_{2\mu} \cong \bigoplus_{\lambda \vdash 2d-1 \text{ with exactly 3 odd parts}} V_\lambda.$$

□

7.3. Surjectivity of the bracket map $[\cdot,\cdot] : B_{d,k}(n) \otimes \mathrm{gr}^1(\mathrm{IA}(n)) \to B_{d,k+1}(n)$

Here, we show that the bracket map $[\cdot,\cdot] : B_{d,k}(n) \otimes \mathrm{gr}^1(\mathrm{IA}(n)) \to B_{d,k+1}(n)$ is surjective for $n \geq 2d$. Since we have abelian group isomorphisms (3.7), the bracket map of $\mathrm{gr}^1(\mathrm{IA}(n))$ coincides with that of $\mathrm{gr}^1(\mathcal{E}_*(n))$. Thus, we can compute the bracket map by using the contraction map c defined in Section 5.

Define $K_{i,j}, K_{i,j,k} \in \mathrm{IA}(n)$ by

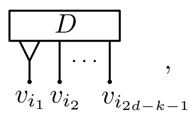
$$K_{i,j}(x_i) = x_j x_i x_j^{-1}, \quad K_{i,j}(x_l) = x_l \quad (l \neq i),$$

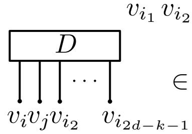
$$K_{i,j,k}(x_i) = x_i[x_j, x_k], \quad K_{i,j,k}(x_l) = x_l \quad (l \neq i). \tag{7.4}$$

Proposition 7.8. *For $n \geq 2d-k$, the bracket map*

$$[\cdot,\cdot] : B_{d,k}(n) \otimes \mathrm{gr}^1(\mathrm{IA}(n)) \to B_{d,k+1}(n)$$

is surjective.

Proof. Any element of $B_{d,k+1}(n)$ is a linear sum of $u =$ 

, where $i_1, \cdots, i_{2d-k-1} \in [n]$. Since $n \geq 2d-k$, we can take $\tilde{u} =$ 

$\in B_{d,k}(n)$, where $i, j \in [n] \setminus \{i_2, \cdots, i_{2d-k-1}\}$ are distinct. We have $[\tilde{u}, K_{i_1,j,i}] = u$, and therefore, the bracket map is surjective. □

As in Section 5.3, for $\lambda \vdash 2d-k$, let $B_{d,k}(n)_\lambda$ denote the isotypic component of $\mathrm{GL}(n;\mathbb{Z})$-module $B_{d,k}(n)$ corresponding to λ.

In Proposition 7.7, we computed a decomposition of $B_{d,0}(n)$. Since the Young diagram of $(2d)$ does not contain that of (1^2), by Remark 5.4, we have the following corollary.

Corollary 7.9. *The restriction of the bracket map*

$$[\cdot,\cdot] : \bigoplus_{\lambda\vdash d,\,\lambda\neq(d)} B_{d,0}(n)_{2\lambda}\otimes \mathrm{gr}^1(\mathrm{IA}(n)) \to B_{d,1}(n)$$

is surjective for $n \geq 2d$.

Lastly, we consider the condition for $\lambda\vdash 2d-k$ that the isotypic component $B_{d,k}(n)_\lambda$ of $B_{d,k}(n)$ does not vanish. Let $o(\lambda)$ be the number of odd parts of λ. We have

$$o(\lambda)\equiv 2d-k\equiv k \pmod 2.$$

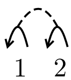
In Proposition 7.7, we observed that $o(\lambda)=0$ $(k=0)$ and $o(\lambda)=3$ $(k=1)$. Moreover, by Proposition 7.8 and Remark 5.4, we have $o(\lambda)\leq 3k$ if $B_{d,k}(n)_\lambda\neq 0$.

8. The $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$

In this section, we study the $\mathrm{Aut}(F_n)$-module structure of $A_d(n)$. We have $A_0(n)=\Bbbk$ for any $n\geq 0$, and we studied the cases where $d=1,2$ in [16]. Note that we have $A_d(0)=0$ for $d\geq 1$. Thus, we have only to consider $n\geq 1$. Here, we construct a direct decomposition of $A_d(n)$ as $\mathrm{Aut}(F_n)$-modules for any $d\geq 3, n\geq 1$, which is indecomposable for $n\geq 2d$. Moreover, we study the degree 3 case in detail.

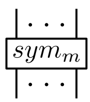
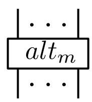
8.1. A direct decomposition of $A_d(n)$

Here, we give a direct decomposition of the $\mathrm{Aut}(F_n)$-module $A_d(n)$.

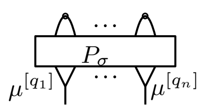
Let $c=$  $\in A_1(2)=\mathbf{A}_1(0,2)$, and depict it as . Here, we use the same graphical notation of morphisms $\mu,\eta,\Delta,\epsilon,S$ in the category $\mathbf{A}$ as in the category $\mathbf{A}^L$. As in Section 4.3, we can define the iterated multiplications $\mu^{[q]}\in\mathbf{A}(q,1)$ for $q\geq 0$. For $m\geq 0$, there is a group homomorphism

$$\mathfrak{S}_m\to\mathbf{A}(m,m),\quad \sigma\mapsto P_\sigma,$$

where P_σ is the symmetry in $\mathbf{A}$ corresponding to σ. Set


$$:=\sum_{\sigma\in\mathfrak{S}_m} P_\sigma,$$

$$:=\sum_{\sigma\in\mathfrak{S}_m}\mathrm{sgn}(\sigma)P_\sigma\in\mathbf{A}(m,m).$$

By Habiro–Massuyeau [11, Lemma 5.16], every element of $A_d(n)$ is a linear combination of morphisms of the form

$$(\mu^{[q_1]}\otimes\cdots\otimes\mu^{[q_n]})\circ P_\sigma\circ c^{\otimes d}=$$


for $\sigma\in\mathfrak{S}_{2d}$ and $q_1,\cdots,q_n\geq 0$ such that $q_1+\cdots+q_n=2d$. The following lemma easily follows.

Lemma 8.1. *For $n \geq 0$, we have*

$$A_d(n) = \mathrm{Span}_{\Bbbk}\{A_d(f)(c^{\otimes d}) \mid f \in \mathbf{F}^{\mathrm{op}}(2d,n)\}.$$

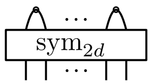
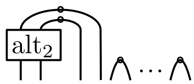
For $X \in A_d(m)$, let

$$A_d X : \mathbf{F}^{\mathrm{op}} \to \mathbf{fVect}$$

denote the subfunctor of A_d generated by X. That is, for any $n \in \mathbb{N}$, $A_d X(n)$ is the $\mathrm{Aut}(F_n)$-submodule of $A_d(n)$ defined by

$$A_d X(n) := \mathrm{Span}_{\Bbbk}\{A_d(f)(X) \mid f \in \mathbf{F}^{\mathrm{op}}(m,n)\}.$$

Set

$P =$  , $Q =$  $\in A_d(2d)$.

Note that we have $A_1 Q = 0$.

Theorem 8.2. *We have*

$$A_d(n) = A_d P(n) \oplus A_d Q(n). \tag{8.1}$$

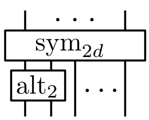
Proof. By Lemma 8.1, any element of $A_d(n)$ is a linear sum of $A_d(f)(c^{\otimes d})$ for $f \in \mathbf{F}^{\mathrm{op}}(2d,n)$. Define an $\mathrm{Aut}(F_n)$-module map

$$e_n : A_d(n) \to A_d(n)$$

by $e_n(A_d(f)(c^{\otimes d})) = \frac{1}{(2d)!} A_d(f)(P)$ for $f \in \mathbf{F}^{\mathrm{op}}(2d,n)$. This is well defined because the 4T relation is sent to 0. Since $A_d P$ is generated by P, we have $\mathrm{im}(e_n) = A_d P(n)$.

Since we have $e_n(A_d(f)(P)) = A_d(f)(P)$ for any $f \in \mathbf{F}^{\mathrm{op}}(2d,n)$, the $\mathrm{Aut}(F_n)$-endomorphism e_n is an idempotent in $\mathrm{End}(A_d(n))$, where we consider $A_d(n)$ as a right $\mathrm{Aut}(F_n)$-module. Therefore, we have

$$A_d(n) = \mathrm{im}(e_n) \oplus \ker(e_n), \quad \ker(e_n) = \mathrm{im}(1 - e_n).$$

Since  $= 0$, we have $A_d Q(n) \subset \ker(e_n)$. Finally, we need to check that

$\mathrm{im}(1 - e_n) \subset A_d Q(n)$. Since we have for $f \in \mathbf{F}^{\mathrm{op}}(2d,n)$,

$$\begin{aligned}(1 - e_n)(A_d(f)(c^{\otimes d})) &= A_d(f)(c^{\otimes d}) - \frac{1}{(2d)!} A_d(f)(P) \\ &= \frac{1}{(2d)!} \sum_{\sigma \in \mathfrak{S}_{2d}} A_d(f)(c^{\otimes d} - \sigma c^{\otimes d}),\end{aligned}$$

we need to show that, for any $\sigma \in \mathfrak{S}_{2d}$, there exists $\tau \in \Bbbk \mathfrak{S}_{2d}$ such that

$$c^{\otimes d} - \sigma c^{\otimes d} = \tau Q \in A_d Q(2d). \tag{8.2}$$

It suffices to show the existence of τ satisfying equation (8.2) when σ is an adjacent transposition because any permutation is generated by adjacent transpositions, and we have such τ by inductively using

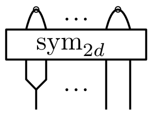
$$c^{\otimes d} - \sigma\rho c^{\otimes d} = c^{\otimes d} - \sigma c^{\otimes d} + \sigma(c^{\otimes d} - \rho c^{\otimes d}).$$

If σ is an adjacent transposition $(2i, 2i+1)$ for $i \in [n-1]$, then we set

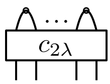
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & 2d \\ 2i-1 & 2i+2 & 2i+1 & 2i & 1 & \cdots \widehat{2i-1} \cdots \widehat{2i+2} \cdots & 2d \end{pmatrix}.$$

If σ is an adjacent transposition $(2i-1, 2i)$ for $i \in [n]$, then we set $\tau = 0$. The proof is complete. □

Lemma 8.3. *The* $\mathrm{Aut}(F_n)$*-module* $A_dP(n)$ *is irreducible and thus indecomposable.*

Proof. Since 

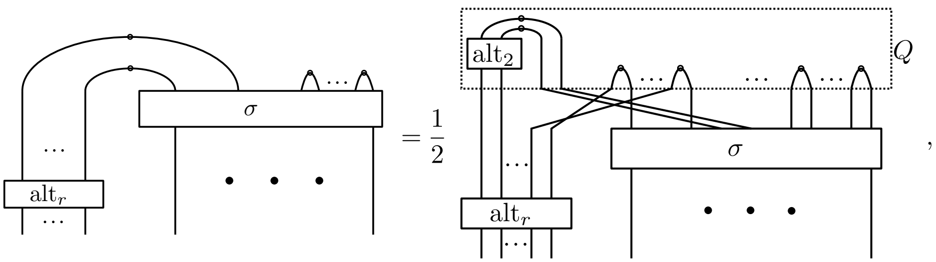
$= 0$, we have $\theta_{d,n}(A_dP(n)) = B_{d,0}(n)_{(2d)}$ by the PBW map. Therefore, $A_dP(n)$ is an irreducible $\mathrm{Aut}(F_n)$-module. □

For $\lambda \vdash d$, set $Q_\lambda =$ 

, where $c_{2\lambda} \in \Bbbk\mathfrak{S}_{2d}$ is the Young symmetrizer. Note that we have $Q_{(d)} = P$.

Lemma 8.4. *For* $\lambda \vdash d, \lambda \neq (d)$*, we have* $Q_\lambda \in A_dQ(2d)$*.*

Proof. For $\lambda = (\lambda_1, \cdots, \lambda_r) \neq (d)$, we have $r \geq 2$. By expanding a_λ and b_λ except for the first column, we can write Q_λ as a linear sum of



where $\sigma \in \mathfrak{S}_{2d-r}$. The latter diagram is obtained from Q by composing a morphism of $\Bbbk\mathbf{F}^{\mathrm{op}}(2d, 2d)$, so is included in $A_dQ(2d)$. □

By Lemma 8.4, we have $A_dQ(n) \supset \sum_{\lambda \vdash d, \lambda \neq (d)} A_dQ_\lambda(n)$. Moreover, we have the following corollary.

Corollary 8.5. *The* $\mathrm{Aut}(F_n)$*-module* $A_dQ(n)$ *is generated by* $\{Q_\lambda \mid \lambda \vdash d, \lambda \neq (d)\}$ *for* $n \geq 2d$*. That is, we have* $A_dQ(n) = \sum_{\lambda \vdash d, \lambda \neq (d)} A_dQ_\lambda(n)$*.*

Proof. For simplicity, let A denote $\sum_{\lambda\vdash d,\lambda\neq(d)} A_d Q_\lambda(n)$. By Lemma 8.3, we have $\theta_{d,n}(A_d P(n)) = B_{d,0}(n)_{(2d)}$. Thus, by Theorem 8.2, we have

$$\theta_{d,n}(A_d Q(n)) = \left(\bigoplus_{\lambda\vdash d,\lambda\neq(d)} B_{d,0}(n)_{2\lambda} \right) \oplus \left(\bigoplus_{k\geq 1} B_{d,k}(n) \right).$$

On the other hand, by the PBW theorem, we have

$$\theta_{d,n}(A) \supset \left(\bigoplus_{\lambda\vdash d,\lambda\neq(d)} B_{d,0}(n)_{2\lambda} \right).$$

By Corollary 7.9 and Proposition 7.8, we have

$$\theta_{d,n}(A) \supset \left(\bigoplus_{\lambda\vdash d,\lambda\neq(d)} B_{d,0}(n)_{2\lambda} \right) \oplus \left(\bigoplus_{k\geq 1} B_{d,k}(n) \right).$$

Therefore, we have $A_d Q(n) \subset A$. Hence, we have $A_d Q(n) = A$. □

8.2. Radical filtration of $A_d(n)$

For an $\mathrm{Aut}(F_n)$-module M, let $\mathrm{Rad}(M)$ denote the *radical* of M; that is,

$$\mathrm{Rad}(M) = \bigcap \{K \subset M \mid K \text{ is maximal in } M\}.$$

We have a radical filtration of $A_d(n)$

$$A_d(n) \supset \mathrm{Rad}(A_d(n)) \supset \mathrm{Rad}^2(A_d(n)) = \mathrm{Rad}(\mathrm{Rad}(A_d(n))) \supset \cdots.$$

Theorem 8.6. *Let $n \geq 2d$. Then, the filtration of $A_d(n)$ by the number of trivalent vertices coincides with the radical filtration. That is, we have* $\mathrm{Rad}(A_{d,k}(n)) = A_{d,k+1}(n)$ *for any $k \geq 0$.*

Proof. For $\lambda \vdash 2d-k$, we have $B_{d,k}(n)_\lambda \cong \bigoplus_{i=1}^{r_\lambda} (V_\lambda)_i$ as $\mathrm{GL}(n;\mathbb{Z})$-modules. Let $B_{d,k}(n)_{\lambda,i} \subset B_{d,k}(n)_\lambda$ be a $\mathrm{GL}(n;\mathbb{Z})$-submodule corresponding to $(V_\lambda)_i$. Let $A_{d,k}(n)_{\lambda,i} \subset A_{d,k}(n)$ be the $\mathrm{Aut}(F_n)$-submodule generated by $\theta_{d,n}^{-1}(B_{d,k}(n)_{\lambda,i})$. For each $\lambda \vdash 2d-k, i \in [r_\lambda]$, we have a maximal submodule

$$R_{\lambda,i} = \left(\sum_{(\mu,j)\neq(\lambda,i)} A_{d,k}(n)_{\mu,j} \right) + A_{d,k+1}(n).$$

Since we have $\bigcap_{(\lambda,i)} R_{\lambda,i} = A_{d,k+1}(n)$, it follows that $\mathrm{Rad}(A_{d,k}(n)) \subset A_{d,k+1}(n)$.

For any maximal submodule K of $A_{d,k}(n)$, the quotient $A_{d,k}(n)/K$ is an irreducible $\mathrm{Aut}(F_n)$-module, which factors through an irreducible polynomial $\mathrm{GL}(n;\mathbb{Z})$-module. It follows that $\theta_{d,n}(A_{d,k}(n))/\theta_{d,n}(K)$ is isomorphic to one of the irreducible components of the $\mathrm{GL}(n;\mathbb{Z})$-module $\bigoplus_{i\geq k} B_{d,i}(n)$. If $B_{d,k}(n) \subset \theta_{d,n}(K)$, then by Proposition 7.8, we have $K = A_{d,k}(n)$, which contradicts to the maximality of K. Therefore, $\theta_{d,n}(A_{d,k}(n))/\theta_{d,n}(K)$ is isomorphic to one of the irreducible components of $B_{d,k}(n)$, and we have $K \supset A_{d,k+1}(n)$. This implies that $\mathrm{Rad}(A_{d,k}(n)) \supset A_{d,k+1}(n)$, and the proof is complete. □

It is possible that Theorem 8.6 holds for some $n < 2d$. However, it does not hold for all n. (See Remark 8.13.)

8.3. Indecomposability of the decomposition of $A_d(n)$

Here, we consider the indecomposability of the decomposition (8.1) of $A_d(n)$.

In Proposition 7.7, we observed that

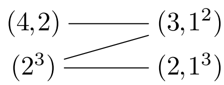
$$B_{d,0}(n) \cong \bigoplus_{\lambda \vdash d} B_{d,0}(n)_{2\lambda}, \quad B_{d,1}(n) \cong \bigoplus_{\mu \vdash 2d-1 \text{ with exactly 3 odd parts}} B_{d,1}(n)_{\mu}.$$

In order to study the indecomposability of equation (8.1), we observe certain connectivity at the level of partitions.

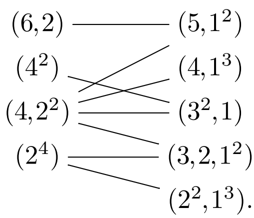
Let $X_d = \{2\lambda \mid \lambda \vdash d, \lambda \neq (d)\}$ and $Y_d = \{\mu \vdash 2d-1 \mid \mu \text{ has exactly 3 odd parts}\}$. We consider the bipartite graph G_d with vertex sets X_d and Y_d and with an edge between each pair of vertices 2λ and μ if μ is obtained from 2λ by taking away one box from each of two different rows of 2λ and then by adding one box to another row. For example, G_2 is

$$(2^2) \text{ ——— } (1^3),$$

G_3 is



and G_4 is



Proposition 8.7. *The graph G_d is path-connected.*

Proof. For $\lambda \vdash d, \lambda \neq (d)$, let $r(\lambda)$ be the number of rows of λ. We write $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \cdots, \lambda_l^{a_l})$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_l$, $\sum_{i=1}^{l} a_i = r(\lambda)$, $a_i \geq 1$.

We show that for $\lambda \vdash d$ such that $r(\lambda) < d$, there is a path between 2λ and some $2\lambda' \in X_d$ such that $r(\lambda') = r(\lambda)+1$. Then, since (2^d) is the only partition that has d rows, it follows by induction on $k = r(\lambda)$ that all vertices in X_d are path-connected.

If $a_1 = k$, then we have $2\lambda = ((2\lambda_1)^k)$ and $2\lambda_1 \geq 4$ because we assume that $k < d$. Thus, we have

$$2\lambda \text{———} \mu',$$

where $\mu' = ((2\lambda_1)^{k-2}, (2\lambda_1 - 1)^2, 1)$ is obtained from 2λ by taking away a box from each of the $(k-1)$-st and k-th row and adding one box to the $(k+1)$-st row, and

$$\mu' \text{———} 2\lambda',$$

where $2\lambda' = ((2\lambda_1)^{k-1}, 2\lambda_1 - 2, 2)$ is obtained from μ' by taking away a box from the k-th row and adding a box to each of the $(k-1)$-st and $(k+1)$-st row. Therefore, we have a path between 2λ and $2\lambda'$ such that $r(\lambda') = k+1$.

If $a_1 < k$, then we have

$$2\lambda \text{———} \mu'',$$

where μ'' is obtained from 2λ by taking away a box from each of the a_1-th and (a_1+a_2)-th row, and adding a box to the $(k+1)$-st row, and

$$\mu'' \text{———} 2\lambda'',$$

where $2\lambda''$ is obtained from μ'' by taking away a box from the a_1-th row and adding a box to each of the (a_1+a_2)-th and $(k+1)$-st row. Therefore, we have a path between 2λ and $2\lambda''$ such that $r(\lambda'') = k+1$.

Lastly, we will show that each vertex of Y_d is connected to a vertex of X_d. Any element $\mu \in Y_d$ is a partition of $2d-1$ and has three odd parts. Therefore, by taking away a box from the last odd row and then adding one box to each of the other two odd rows, we obtain a partition of $2d$ with only even parts, which is a vertex of X_d. The proof is complete. □

If $n \geq d$, then for any $2\lambda \in X_d$, $B_{d,0}(n)_{2\lambda}$ is a nonzero $\mathrm{GL}(n;\mathbb{Z})$-submodule of $B_d(n)$. If $n \geq d$, then for any $\mu \in Y_d$ (except $\mu = (2^{d-2}, 1^3)$ if $n = d$), $B_{d,1}(n)_\mu$ is a nonzero $\mathrm{GL}(n;\mathbb{Z})$-submodule of $B_d(n)$.

Let $\pi_\mu : B_{d,1}(n) \twoheadrightarrow B_{d,1}(n)_\mu$ be the projection.

Proposition 8.8. *Let $n \geq 2d$. Let $2\lambda \in X_d$, $\mu \in Y_d$ be two endpoints of an edge of the bipartite graph G_d. Then the composition of the bracket map and the projection π_μ*

$$B_{d,0}(n)_{2\lambda} \otimes \mathrm{gr}^1(\mathrm{IA}(n)) \xrightarrow{[\cdot,\cdot]} B_{d,1}(n) \xrightarrow{\pi_\mu} B_{d,1}(n)_\mu \tag{8.3}$$

does not vanish.

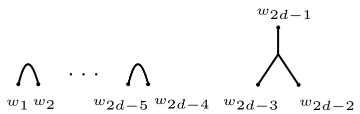
Note that this proposition holds for $d = 1, 2$ because we have $X_1 = Y_1 = \emptyset, X_2 = \{(2^2)\}, Y_2 = \{(1^3)\}$ and by Lemma 6.7 in [16].

Recall that we have

$$B_{d,0}(n) = \frac{\mathrm{Span}_{\Bbbk}\{ \underset{w_1\, w_2}{\bigcap} \cdots \underset{w_{2d-1}\, w_{2d}}{\bigcap} \mid w_1, \cdots, w_{2d} \in V_n\}}{\text{multilinearity}}$$

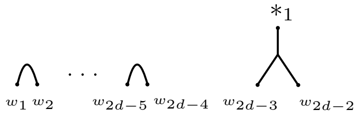
and

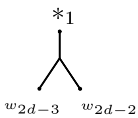
$$B_{d,1}(n) = \frac{\mathrm{Span}_{\Bbbk}\{\ \text{(diagram)}\ \mid w_1,\cdots,w_{2d-1} \in V_n\}}{\text{AS relation and multilinearity}}.$$



What the bracket map does is to contract two of the univalent vertices of a diagram of an element of $B_{d,0}(n)$ with two leaves of a trivalent tree in $\mathrm{gr}^1(\mathrm{IA}(n))$, which corresponds to the operation on partitions of taking away two boxes from different rows and then adding a box. Here, we introduce an intermediate vector space $B'_d(n)$ between $B_{d,0}(n)$ and $B_{d,1}(n)$, whose elements correspond to partitions which are obtained by the operation of taking away two boxes from different rows. Define $B'_d(n)$ by

$$\frac{\mathrm{Span}_{\Bbbk}\{\ \text{(diagram)}\ \mid w_1,\cdots,w_{2d-2} \in V_n\}}{\text{AS relation and multilinearity}},$$



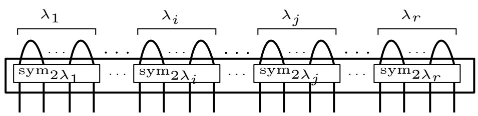
where  is a based trivalent tree of degree 1. Then, $B'_d(n)$ is a $\mathrm{GL}(n;\mathbb{Z})$-module,

and we have an irreducible decomposition

$$B'_d(n) \cong \mathbb{S}_{(d-2)}(\mathbb{S}_{(2)}V_n) \otimes \mathbb{S}_{(1^2)}V_n \cong \bigoplus_{\nu \vdash 2d-2 \text{ with exactly 2 odd parts}} V_\nu$$

in a way similar to Proposition 7.7. Let $B'_d(n)_\nu$ be the isotypic component of $B'_d(n)$ corresponding to ν.

Recall that a_λ, b_λ and $\diamond$ are defined in Section 5.1. In the proof of Proposition 8.8, we use the following notation



,

which represents the linear sum of permutations $a_{2\lambda}$.

Proof of Proposition 8.8. Let $2\lambda = (2\lambda_1,\cdots,2\lambda_r) \vdash 2d \in X_d$. Any vertex $\mu \in Y_d$ that is connected to 2λ by an edge of G_d is obtained from 2λ by taking away a box from each of the i-th and j-th row of 2λ and adding a box to the k-th row of 2λ for some $i,j \in [r], i<j, k \in [r+1], k \neq i,j$. We write $\mu = (\mu_1,\cdots,\mu_s)$. Then we have $\mu_i = 2\lambda_i - 1, \mu_j = 2\lambda_j - 1, \mu_k = 2\lambda_k + 1$ and $\mu_l = 2\lambda_l$ for $l \in [s], l \neq i,j,k$.

Since we have $\mathrm{gr}^1(\mathrm{IA}(n)) \cong H^* \otimes \mathcal{L}_2(n)$, we can write equation (8.3) by

$$h_{\lambda,\mu} : B_{d,0}(n)_{2\lambda} \otimes H^* \otimes \mathcal{L}_2(n) \to B_{d,1}(n) \xrightarrow{\pi_\mu} B_{d,1}(n)_\mu.$$

We will show that $h_{\lambda,\mu}$ does not vanish.

Let $\nu \vdash 2d-2$ be the partition that is obtained from 2λ by taking away a box from each of the i-th and j-th row of 2λ. We decompose $h_{\lambda,\mu}$ into the composition

$$h_{\lambda,\mu} = h_{\nu,\mu} h_{\lambda,\nu},$$

where $h_{\nu,\mu}$ and $h_{\lambda,\nu}$ are $\mathrm{GL}(n;\mathbb{Z})$-module maps defined as follows.

Let

$$h'_\lambda : B_{d,0}(n)_{2\lambda} \otimes \mathcal{L}_2(n) \to B'_d(n)$$

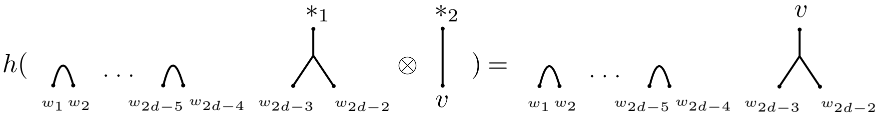
be a $\mathrm{GL}(n;\mathbb{Z})$-module map defined in a way similar to the contraction map in Section 5.2. Define

$$h_\lambda : B_{d,0}(n)_{2\lambda} \otimes H^* \otimes \mathcal{L}_2(n) \to B'_d(n) \otimes H^*$$

by $h_\lambda(x \otimes y \otimes z) = h'_\lambda(x \otimes z) \otimes y$ for $x \in B_{d,0}(n)_{2\lambda}, y \in H^*, z \in \mathcal{L}_2(n)$. We also define a $\mathrm{GL}(n;\mathbb{Z})$-module map

$$h : B'_d(n) \otimes H^* \to B_{d,1}(n)$$

by connecting two bases $*_1, *_2$, that is, for $w_1, \cdots, w_{2d-2} \in V_n, v \in H^*$,



Let $\pi_\nu : B'_d(n) \otimes H^* \to B'_d(n)_\nu \otimes H^*$ be the tensor product of the projection and id_{H^*}. Then we have two $\mathrm{GL}(n;\mathbb{Z})$-module maps

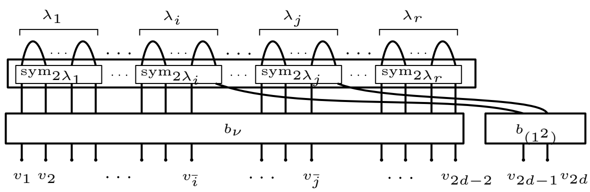
$$h_{\lambda,\nu} : B_{d,0}(n)_{2\lambda} \otimes H^* \otimes \mathcal{L}_2(n) \xrightarrow{h_\lambda} B'_d(n) \otimes H^* \xrightarrow{\pi_\nu} B'_d(n)_\nu \otimes H^*$$

and

$$h_{\nu,\mu} : B'_d(n)_\nu \otimes H^* \xrightarrow{h} B_{d,1}(n) \xrightarrow{\pi_\mu} B_{d,1}(n)_\mu.$$

Since $h_{\lambda,\nu}$ and $h_{\nu,\mu}$ are $\mathrm{GL}(n;\mathbb{Z})$-module maps and since $B_{d,0}(n)_{2\lambda}$ and $B'_d(n)_\nu$ are irreducible, it suffices to prove that $h_{\lambda,\nu} \neq 0$ and $h_{\nu,\mu} \neq 0$.

We will prove that $h_{\lambda,\nu}$ does not vanish. Let

$u =$  $\in B_{d,0}(n),$

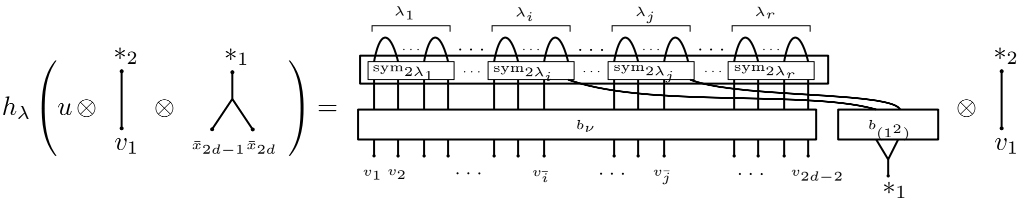
where $\bar{i} = \sum_{l=1}^{i} 2\lambda_l - 1, \bar{j} = \sum_{l=1}^{j} 2\lambda_l - 2$. Since we have

$$c_\nu \diamond c_{(1^2)} \in S^\nu \diamond S^{(1^2)} = \bigoplus_{\rho \vdash 2d} (S^\rho)^{LR^\rho_{\nu,(1^2)}}$$

and

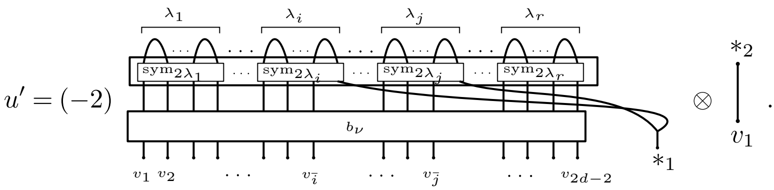
$$\{\rho \vdash 2d \mid LR^\rho_{\nu,(1^2)} \neq 0\} \cap X_d = \{2\lambda\},$$

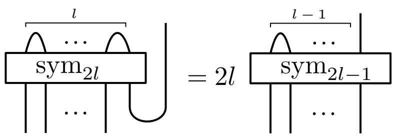
we have $u \in B_{d,0}(n)_{2\lambda}$. Moreover, we have

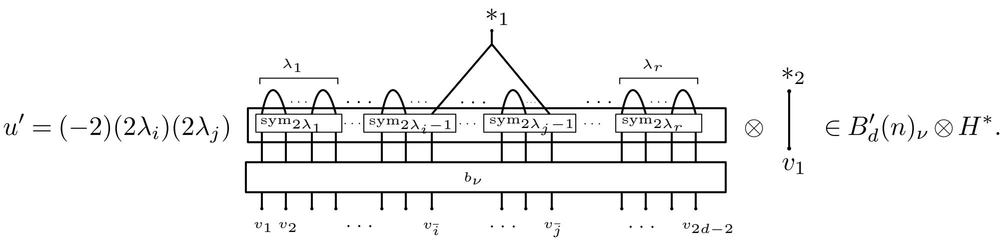


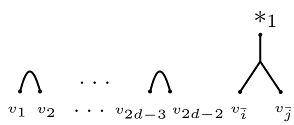
(8.4)

By the relation $b_{(1^2)} = \mathrm{id} - (1,2)$ and the AS relation, the right-hand side of equation (8.4) is



Since we have  locally, by pulling $*_1$ to the top, we have

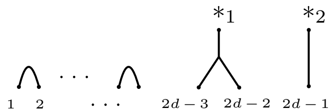


We will look at the coefficient in u' of $u_0 =$  to show that u' does not vanish. Note that the upper box corresponds to a_ν and that $b_\nu a_\nu = \sum_{\tau \in C_{t_0}, \rho \in R_{t_0}} \mathrm{sgn}(\tau)\tau\rho$, where t_0 is the canonical ν-tableau. If $\lambda_i \neq \lambda_j$, then there is no $\tau \in C_{t_0}$ such that $\tau(\bar{i}) = \bar{j}, \tau(\bar{j}) = \bar{i}$. Thus, the diagram u_0 appears only when τ is an even permutation which fixes $\bar{i}$ and $\bar{j}$. Then, the coefficient of u_0 in u' is negative. If $\lambda_i = \lambda_j$, then the diagram u_0 appears when τ preserves the subset $\{\bar{i},\bar{j}\}$ and the parity of τ coincides with that of the restriction of τ to $\{\bar{i},\bar{j}\}$. Hence, by the AS relation, the coefficient of u_0 in u' is negative. Therefore, $h_{\lambda,\nu}$ does not vanish.

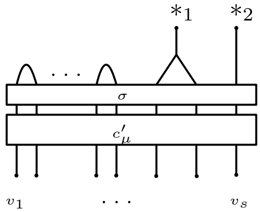
We will prove that $h_{\nu,\mu}$ does not vanish. Let $N \in \mathbb{N}$. Set $c'_\rho = a_\rho b_\rho \in \Bbbk\mathfrak{S}_N$ for $\rho \vdash N$. From basic facts of representation theory, we have an isomorphism of $\Bbbk\mathfrak{S}_N$-modules

$$\Bbbk\mathfrak{S}_N c_\rho \cong \Bbbk\mathfrak{S}_N c'_\rho.$$

In what follows, we use c'_ρ instead of c_ρ as the Young symmetrizer. Let

$$Z_\mu = \left(c'_\mu \sigma \cdot \right.$$


$$\left. \right) (v_1^{\otimes \mu_1} \otimes \cdots \otimes v_s^{\otimes \mu_s})$$

$$=$$


$$\in B'_d(n) \otimes H^*,$$

where $\sigma \in \mathfrak{S}_{2d-1}$ is defined by

$$\sigma = \begin{pmatrix} 1 & \cdots & 2d-3 & 2d-2 & 2d-1 \\ 1 & \cdots & i' & j' & k' \end{pmatrix} \text{ for } i' = \sum_{l=1}^{i} \mu_l,\, j' = \sum_{l=1}^{j} \mu_l,\, k' = \sum_{l=1}^{k} \mu_l.$$

We will show that $h(\pi_\nu(Z_\mu)) \in B_{d,1}(n)_\mu$ and that $h(\pi_\nu(Z_\mu)) \neq 0$.

If the diagram that is obtained from μ by taking away a box from the i-th (resp. j-th) row of μ is a partition of $2d-2$, then write it ν_i (resp. ν_j). Since any partition $\rho \vdash 2d-2$ with exactly two odd parts other than ν, ν_i, ν_j is not included in μ, it follows that

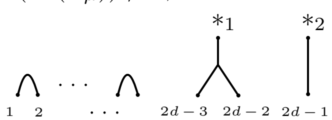
$$Z_\mu \in (B'_d(n)_\nu \otimes H^*) \oplus (B'_d(n)_{\nu_i} \otimes H^*) \oplus (B'_d(n)_{\nu_j} \otimes H^*).$$

By using an argument similar to Proposition 5.3, we have

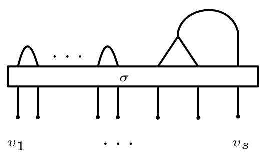
$$h(B'_d(n)_\nu \otimes H^*) \subset \bigoplus_{\alpha = \nu \sqcup \square} B_{d,1}(n)_\alpha, \quad h(B'_d(n)_{\nu_i} \otimes H^*) \subset \bigoplus_{\alpha = \nu_i \sqcup \square} B_{d,1}(n)_\alpha,$$
$$h(B'_d(n)_{\nu_j} \otimes H^*) \subset \bigoplus_{\alpha = \nu_j \sqcup \square} B_{d,1}(n)_\alpha.$$

Since $\{\nu \sqcup \square\} \cap \{\nu_i \sqcup \square\} \cap \{\nu_j \sqcup \square\} = \{\mu\}$ and since $h(Z_\mu) \in B_{d,1}(n)_\mu$, we have $h(\pi_\nu(Z_\mu)) \in B_{d,1}(n)_\mu$.

In order to prove that $h(\pi_\nu(Z_\mu)) \neq 0$, we will look at the coefficient in $h(\pi_\nu(Z_\mu))$ of

$$z = h\left(\left(\sigma \cdot \right.\right.$$


$$\left.\left. \right) (v_1^{\otimes \mu_1} \otimes \cdots \otimes v_s^{\otimes \mu_s}) \right)$$

$$=$$


$$.$$

Note that $c'_\mu = \sum_{\rho \in R_{s_0}, \tau \in C_{s_0}} \mathrm{sgn}(\tau) \rho\tau$, where s_0 is the canonical μ-tableau.

Firstly, we consider the case where μ_i, μ_j, μ_k are distinct. Then z appears only when τ is an even permutation which fixes i', j' and k'. Therefore, the coefficient of z in $h(Z_\mu)$ is positive. Moreover, the linear sum of terms in Z_μ such that $*_2$ is connected to v_k lies in $\pi_\nu(Z_\mu)$, so the coefficient of z in $h(\pi_\nu(Z_\mu))$ is equal to that of z in $h(Z_\mu)$, which is nonzero.

The other cases, where at least two of μ_i,μ_j and μ_k are equal, follow in a similar argument. The only thing that differs from the above case is that z appears when τ preserves the subset $\{i',j',k'\} \subset [2d-1]$, and the parity of τ coincides with that of the restriction of τ to $\{i',j',k'\}$. Since we have the AS relation, the sign due to the permutation of $\{i',j',k'\}$ is cancelled. Therefore, the coefficient of z in $h(Z_\mu)$ is positive in any case. The proof is complete. □

Theorem 8.9. *Let $d \geq 2$. The direct decomposition*

$$A_d(n) = A_dP(n) \oplus A_dQ(n)$$

of $\mathrm{Aut}(F_n)$*-modules is indecomposable for $n \geq 2d$.*

Proof. By Lemma 8.3, it suffices to show that $A_dQ(n)$ is indecomposable. Since the radical preserves the direct sum, we have only to show that $A_dQ(n)/\mathrm{Rad}^2(A_dQ(n))$ is indecomposable. Suppose that we have a nontrivial decomposition of $\mathrm{Aut}(F_n)$-modules

$$\begin{aligned} A_dQ(n)/\mathrm{Rad}^2(A_dQ(n)) &= A_dQ(n)/A_{d,2}(n) \\ &= (M_1 + A_{d,2}(n))/A_{d,2}(n) \oplus (M_2 + A_{d,2}(n))/A_{d,2}(n), \end{aligned}$$

where M_i is an $\mathrm{Aut}(F_n)$-submodule of $A_dQ(n)$ for $i = 1,2$. Let

$$N_i = \theta_{d,n}(M_i + A_{d,2}(n))/\theta_{d,n}(A_{d,2}(n))$$

for $i = 1,2$. We have

$$N_1 \oplus N_2 = \theta_{d,n}(A_dQ(n))/\theta_{d,n}(A_{d,2}(n)) = \left(\bigoplus_{\lambda \vdash d,\, \lambda \neq (d)} B_{d,0}(n)_{2\lambda} \right) \oplus B_{d,1}(n).$$

For any $2\lambda \in X_d$, there uniquely exists $i \in \{1,2\}$ such that N_i includes a $\mathrm{GL}(n;\mathbb{Z})$-submodule $(N_i)_{2\lambda} \cong V_{2\lambda}$. Let $x \in (N_i)_{2\lambda}$ be a generator of the irreducible $\mathrm{GL}(n;\mathbb{Z})$-module $(N_i)_{2\lambda}$. Then, the image x' of x under the composition of $\mathrm{GL}(n;\mathbb{Z})$-module maps

$$(N_i)_{2\lambda} \hookrightarrow N_i \hookrightarrow B_{d,0}(n) \oplus B_{d,1}(n) \twoheadrightarrow B_{d,0}(n)$$

is an element of $B_{d,0}(n)_{2\lambda}$. For any $\mu \in Y_d$ that is connected to 2λ by an edge of G_d, by Proposition 8.8, there exists $g \in \mathrm{gr}^1(\mathrm{IA}(n))$ such that $[x',g] \neq 0 \in B_{d,1}(n)_\mu$. Therefore, we have

$$[x,g] = [x',g] + [x - x',g] = [x',g] \neq 0 \in B_{d,1}(n)_\mu.$$

It follows that N_i includes a $\mathrm{GL}(n;\mathbb{Z})$-submodule $(N_i)_\mu$ that is isomorphic to V_μ for any $\mu \in Y_d$ that is connected to 2λ by an edge of G_d. Hence, by Proposition 8.7, we have $N_1 \cap N_2 \neq \{0\}$, a contradiction. Therefore, $A_dQ(n)$ is indecomposable. □

Note that the assumption $n \geq 2d$ is needed for the surjectivity of the bracket map and the nontriviality of the bracket map for each pair of nonzero irreducible $\mathrm{GL}(n;\mathbb{Z})$-submodules. Thus, if we have the surjectivity and the nontriviality of the bracket map for some $n < 2d$, we can loose the assumption.

8.4. The $\mathrm{Aut}(F_n)$-module structure of $A_3(n)$

Here, we consider the $\mathrm{Aut}(F_n)$-module structure of $A_3(n)$ in detail.

In degree 3, the restrictions of the bracket map to each isotypic component induce $\mathrm{GL}(n;\mathbb{Z})$-module homomorphisms

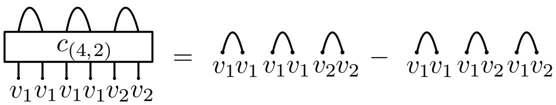
$$\rho_1 : B_{3,0}(n)_{(4,2)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{3,1}(n)_{(3,1^2)}),$$
$$\rho_2 : B_{3,0}(n)_{(2^3)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{3,1}(n)_{(3,1^2)}),$$
$$\rho_3 : B_{3,0}(n)_{(2^3)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{3,1}(n)_{(2,1^3)}).$$

Proposition 8.10. *The* $\mathrm{GL}(n;\mathbb{Z})$*-module homomorphisms* ρ_1 *and* ρ_2 *are injective for* $n \geq 3$ *and* ρ_3 *for* $n \geq 4$*.*

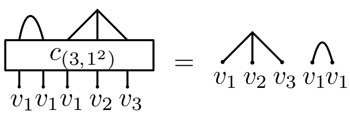
Proof. Recall that c_λ denotes the Young symmetrizer defined in equation (5.1) and that $K_{i,j,k} \in \mathrm{IA}(n)$ is defined by equation (7.4). For $n \geq 3$, we have

$$\rho_1(u)(K_{3,2,1}) = [u, K_{3,2,1}] = -10w \neq 0 \in B_{3,1}(n)_{(3,1^2)},$$

where

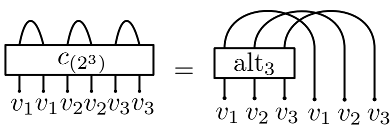
$u = \frac{1}{64}$  $\in B_{3,0}(n)_{(4,2)}$

and

$w = \frac{1}{20}$  $\neq 0 \in B_{3,1}(n)_{(3,1^2)}.$

Thus, we have $\rho_1 \neq 0$ for $n \geq 3$. Since $B_{3,0}(n)_{(4,2)}$ is irreducible, ρ_1 is injective.

Let

$x = \frac{1}{48}$  $\in B_{3,0}(n)_{(2^3)}.$

We have

$$\rho_2(x)(K_{1,3,2}) = [x, K_{1,3,2}] = -6w \neq 0 \in B_{3,1}(n)_{(3,1^2)}.$$

Thus, we have $\rho_2 \neq 0$ for $n \geq 3$. Since $B_{3,0}(n)_{(2^3)}$ is irreducible, ρ_2 is injective.

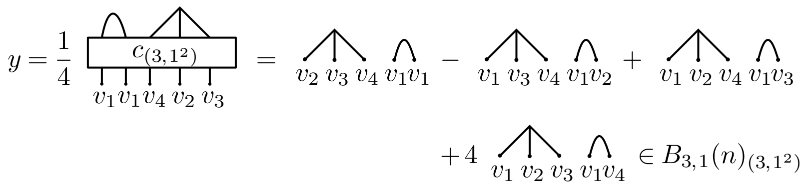
For $n \geq 4$, we have

$$[x, K_{4,3,2}] = -\frac{6}{5}y - \frac{24}{5}z,$$

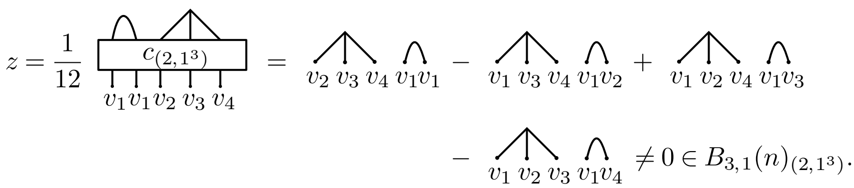
and thus,

$$\rho_3(x)(K_{4,3,2}) = -\frac{24}{5}z \neq 0 \in B_{3,1}(n)_{(2,1^3)},$$

where



and



Therefore, we have $\rho_3 \neq 0$ for $n \geq 4$. Since $B_{3,0}(n)_{(2^3)}$ is irreducible, ρ_3 is injective. □

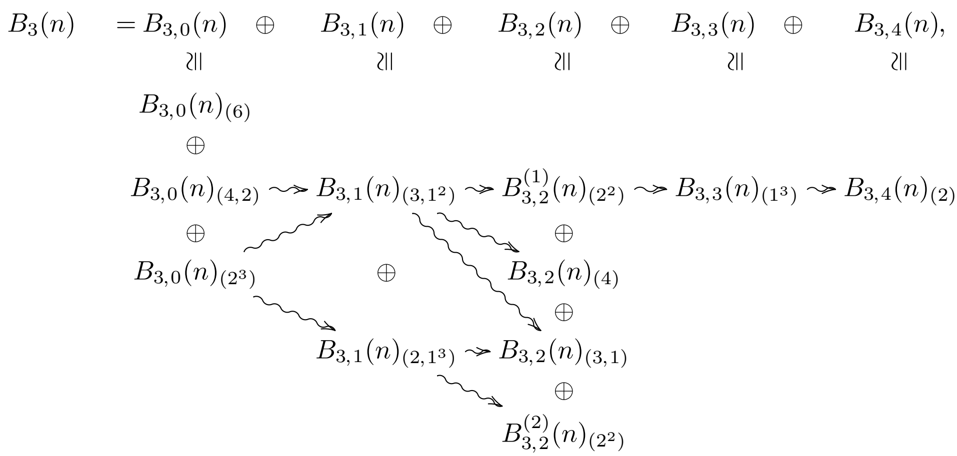
Remark 8.11. We consider a restriction of the bracket map

$$[\cdot,\cdot] : V_\lambda \otimes \mathrm{gr}^1(\mathrm{IA}(n)) \to V_\mu \tag{8.5}$$

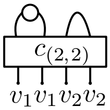
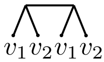
for each irreducible $\mathrm{GL}(n;\mathbb{Z})$-submodule V_λ (resp. V_μ) of $B_{d,k}(n)$ (resp. $B_{d,k+1}(n)$). We write a *wavy arrow*

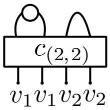
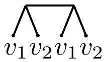
$$V_\lambda \rightsquigarrow V_\mu$$

if the restriction map (8.5) does not vanish. Then, we have the following diagram for $n \geq 4$:



where $B^{(i)}_{3,2}(n)_{(2^2)}$ is the irreducible component of $B_{3,2}(n)_{(2^2)}$ generated by

 -24  $(i=1)$,

 $+16$  $(i=2)$,

respectively. Note that, for $n=3$, $B_3(3)$ includes all of the above irreducible subrepresentations but $B_{3,1}(3)_{(2,1^3)}=0$, and there are all of the wavy arrows but the three wavy arrows that are directed to or coming from $B_{3,1}(3)_{(2,1^3)}$. For $n=2$, we have

$$B_3(2) = (B_{3,0}(2)_{(6)} \oplus B_{3,0}(2)_{(4,2)}) \oplus B_{3,2}(2) \oplus B_{3,4}(2).$$

For $n=1$, we have

$$B_3(1) = B_{3,0}(1)_{(6)} \oplus B_{3,2}(1)_{(4)} \oplus B_{3,4}(1)_{(2)}.$$

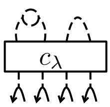
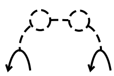
For $n=1,2$, there are no wavy arrows because $B_{3,1}(n)=B_{3,3}(n)=0$.

By Proposition 8.10 and Remark 8.11, we have the surjectivity and the nontriviality of the bracket map for $n \geq 3$. Thus, by Theorem 8.9, one can obtain the following theorem, which improves Theorem 8.9 for $d=3$.

Theorem 8.12. *We have an indecomposable decomposition*

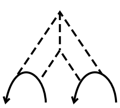
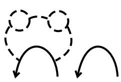
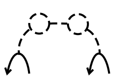
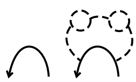
$$A_3(n) = A_3P(n) \oplus A_3Q(n)$$

of $\mathrm{Aut}(F_n)$*-modules for* $n \geq 3$.

For $\lambda \vdash 4$, let $R_\lambda =$  $\in A_3(4)$, where c_λ is the Young symmetrizer. Let $S = R_{(2,2)} + 16$  $\in A_3(4)$ and $T =$  $\in A_3(2)$.

For $n=2$, we can check that $A_{3,2}(2)$ is semisimple as $\mathrm{Aut}(F_2)$-modules, that is,

$$A_{3,2}(2) = A_3R_{(4)}(2) \oplus A_3R_{(3,1)}(2) \oplus A_3S(2) \oplus A_3U(2) \oplus A_3T(2),$$

where $U =$  $-\frac{1}{8}$  $-\frac{1}{8}$  $-\frac{1}{8}$  $\in A_3(2)$. We do not know whether or not the $\mathrm{Aut}(F_2)$-module $A_3(2)$ is semisimple.

Remark 8.13. Since $A_{3,2}(2)$ is semisimple, we have $\mathrm{Rad}(A_{3,2}(2))=0$. On the other hand, we have $A_{3,3}(2)=A_{3,4}(2)\cong B_{3,4}(2)\neq 0$. Therefore, we have $\mathrm{Rad}(A_{3,2}(2)) \neq A_{3,3}(2)$.

For $n = 1$, we have $\mathrm{Aut}(F_1) = \mathbb{Z}/2\mathbb{Z}$. We can easily check the following proposition.

Proposition 8.14. *The* $\mathrm{Aut}(F_1)$*-action on* $A_3(1)$ *is trivial. Therefore, we have* $A_3(1) = A_3P(1) \oplus A_3R_{(4)}(1) \oplus A_3T(1)$.

8.5. The socle of $A_d(n)$ for small d

For an $\mathrm{Aut}(F_n)$-module M, let $\mathrm{Soc}(M)$ denote the *socle* of M; that is,

$$\mathrm{Soc}(M) = \sum \{K \subset M \mid K \text{ is simple}\}.$$

Let us consider the cases for small d. Since $A_1(n) \cong \mathrm{Sym}^2(V_n)$ is simple, we have

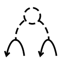
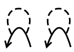
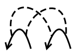
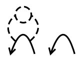
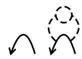
$$\mathrm{Soc}(A_1(n)) = A_1(n) \quad (n \geq 1).$$

By Theorem 6.9 of [16], we have

$$\mathrm{Soc}(A_2(n)) = A_2P(n) \oplus A_2\tilde{T}(n) \quad (n \geq 3, n = 1),$$

$$\mathrm{Soc}(A_2(n)) = A_2(n) = A_2P(n) \oplus A_2W(n) \oplus A_2\tilde{T}(n) \quad (n = 2),$$

where

$\tilde{T} =$  , $W = 2$  $-$  $-$  $-\frac{1}{2}$  $-\frac{1}{2}$  $\in A_2(2)$.

Note that $A_2\tilde{T}(n) = A_{2,2}(n)$

By Proposition 8.14, we have $\mathrm{Soc}(A_3(1)) = A_3(1)$.

Proposition 8.15. *For* $n \geq 3$*, we have*

$$\mathrm{Soc}(A_3(n)) = A_3P(n) \oplus A_3R_{(4)}(n) \oplus A_3R_{(3,1)}(n) \oplus A_3S(n) \oplus A_3T(n).$$

Proof. A simple $\mathrm{Aut}(F_n)$-submodule $K \subset A_3(n)$ corresponds to an irreducible component of $B_3(n)$ via the PBW map. Therefore, by Remark 8.11, we have

$$\mathrm{Soc}(A_3(n)) \subset A_3P(n) \oplus A_3R_{(4)}(n) \oplus A_3R_{(3,1)}(n) \oplus A_3S(n) \oplus A_3T(n).$$

Moreover, we can check that

$$A_3P(n) \cong V_{(6)}, \quad A_3R_{(4)}(n) \cong V_{(4)}, \quad A_3R_{(3,1)}(n) \cong V_{(3,1)},$$
$$A_3S(n) \cong V_{(2,2)}, \quad A_3T(n) \cong V_{(2)}.$$

Hence, we have

$$\mathrm{Soc}(A_3(n)) \supset A_3P(n) \oplus A_3R_{(4)}(n) \oplus A_3R_{(3,1)}(n) \oplus A_3S(n) \oplus A_3T(n),$$

and the proof is complete. $\square$

8.6. The indecomposable decomposition of $A_4(n)$

Here, we consider the indecomposable decomposition of $A_4(n)$.

Similarly, in degree 4, we have $\mathrm{GL}(n;\mathbb{Z})$-module homomorphisms

$$\begin{aligned}
\rho_1 &: B_{4,0}(n)_{(6,2)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{4,1}(n)_{(5,1^2)}),\\
\rho_2 &: B_{4,0}(n)_{(4^2)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{4,1}(n)_{(3^2,1)}),\\
\rho_3 &: B_{4,0}(n)_{(4,2^2)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{4,1}(n)_{(5,1^2)}),\\
\rho_4 &: B_{4,0}(n)_{(4,2^2)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{4,1}(n)_{(4,1^3)}),\\
\rho_5 &: B_{4,0}(n)_{(4,2^2)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{4,1}(n)_{(3^2,1)}),\\
\rho_6 &: B_{4,0}(n)_{(4,2^2)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{4,1}(n)_{(3,2,1^2)}),\\
\rho_7 &: B_{4,0}(n)_{(2^4)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{4,1}(n)_{(3,2,1^2)}),\\
\rho_8 &: B_{4,0}(n)_{(2^4)} \to \mathrm{Hom}(\mathrm{gr}^1(\mathrm{IA}(n)), B_{4,1}(n)_{(2^2,1^3)}).
\end{aligned}$$

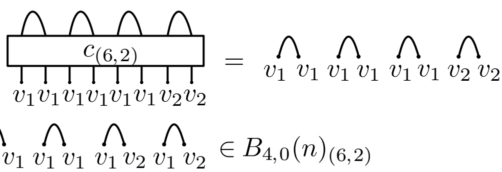
Proposition 8.16. *The* $\mathrm{GL}(n;\mathbb{Z})$*-module homomorphisms* ρ_1, ρ_2, ρ_3 *and* ρ_5 *are injective for* $n \geq 3$*,* ρ_4, ρ_6 *and* ρ_7 *for* $n \geq 4$ *and* ρ_8 *for* $n \geq 5$*.*

Proof. As in the proof of Proposition 8.10 in degree 3, we will check that ρ_1 is injective for $n \geq 3$, ρ_7 for $n \geq 4$ and ρ_8 for $n \geq 5$. The others can be obtained in a similar way.

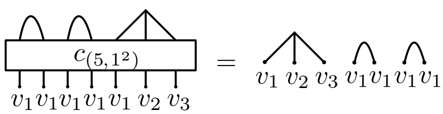
For $n \geq 3$, we have

$$[u, K_{3,1,2}] = 14w \neq 0 \in B_{4,1}(n)_{(5,1^2)},$$

where

$u = \frac{1}{1728}$  $\in B_{4,0}(n)_{(6,2)}$

and

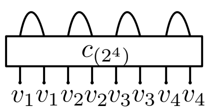
$w = \frac{1}{336}$  $\neq 0 \in B_{4,1}(n)_{(5,1^2)}$.

Thus, we have $\rho_1 \neq 0$ for $n \geq 3$. Since $B_{4,0}(n)_{(6,2)}$ is irreducible, ρ_1 is injective.

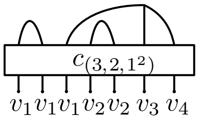
For $n \geq 4$, we have

$$[x, K_{1,4,3}] = -48y \neq 0 \in B_{4,1}(n)_{(3,2,1^2)},$$

where

$x =$  $\in B_{4,0}(n)_{(2^4)}$

and

$y =$ 

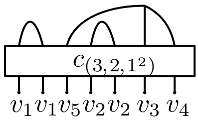
$\in B_{4,1}(n)_{(3,2,1^2)}.$

Thus, ρ_7 is injective for $n \geq 4$.

For $n \geq 5$, we have

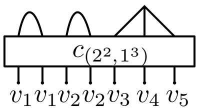
$$[x, K_{5,4,3}] = -48y' - 32z,$$

where

$y' =$ 

$\in B_{4,1}(n)_{(3,2,1^2)}$

and

$z =$ 

$\neq 0 \in B_{4,1}(n)_{(2^2,1^3)}.$

Therefore, we have

$$\rho_8(x)(K_{5,4,3}) = -32z \neq 0 \in B_{4,1}(n)_{(2^2,1^3)},$$

and thus, ρ_8 is injective for $n \geq 5$. □

By using Theorem 8.9 and Proposition 8.16 carefully, one can obtain the following theorem, which improves Theorem 8.9 for $d = 4$.

Theorem 8.17. *We have an indecomposable decomposition*

$$A_4(n) = A_4P(n) \oplus A_4Q(n)$$

of $\mathrm{Aut}(F_n)$*-modules for* $n \geq 7$.

We expect that Theorem 8.17 holds for $n \geq 3$.

9. The $\mathrm{Out}(F_n)$-module structure of $A_d(n)$

In [16], we observed that the $\mathrm{Aut}(F_n)$-action on $A_d(n)$ induces an action of $\mathrm{Out}(F_n)$ on $A_d(n)$. In this section, we obtain some results for $A_d(n)$ as $\mathrm{Out}(F_n)$-modules, which is induced by the results in Section 8.

Since the $\mathrm{Aut}(F_n)$-action on $A_d(n)$ factors through $\mathrm{Out}(F_n)$, any submodule of $A_d(n)$ as $\mathrm{Aut}(F_n)$-modules is a submodule of $A_d(n)$ as $\mathrm{Out}(F_n)$-modules, and vice versa. By Theorem 8.6, we obtain the radical filtration of $A_d(n)$ as $\mathrm{Out}(F_n)$-modules.

Theorem 9.1. *Let $n \geq 2d$. Then, the filtration of $A_d(n)$ by the number of trivalent vertices coincides with the radical filtration of $A_d(n)$ as* $\mathrm{Out}(F_n)$*-modules.*

By Theorem 8.9, we obtain an indecomposable decomposition of $A_d(n)$ as $\mathrm{Out}(F_n)$-modules.

Theorem 9.2. *Let $d \geq 2$. We have a direct decomposition*

$$A_d(n) = A_dP(n) \oplus A_dQ(n)$$

of $\mathrm{Out}(F_n)$*-modules, which is indecomposable for $n \geq 2d$.*

Theorems 8.12, 8.17 also hold as $\mathrm{Out}(F_n)$-modules. Other results for $A_d(n)$ as $\mathrm{Aut}(F_n)$-modules such as Proposition 8.15 also hold.

10. Indecomposable decomposition of the functor A_d

In this section, we obtain an indecomposable decomposition of the functor A_d by using results in Section 8.

By Theorem 8.2, we obtain the following direct decomposition of the functor A_d.

Theorem 10.1. *We have a direct decomposition*

$$A_d = A_dP \oplus A_dQ$$

in the functor category $\mathbf{fVect}^{\mathbf{F}^{\mathrm{op}}}$.

For $d=1$, we have $A_1Q=0$ and the functor $A_1 = A_1P$ is simple. For $d=2$, we obtained this direct decomposition in Theorem 6.5 of [16]. Moreover, we proved that this direct decomposition of the functor A_2 is indecomposable (see Theorem 6.14 of [16]).

By Theorem 8.9, we obtain the indecomposability of the direct decomposition of the functor A_d.

Proposition 10.2. *Let $d \geq 2$. The decomposition*

$$A_d = A_dP \oplus A_dQ$$

of the functor A_d is indecomposable in the functor category $\mathbf{fVect}^{\mathbf{F}^{\mathrm{op}}}$.

Proof. Suppose that we have a decomposition

$$A_dQ = G \oplus G' \in \mathbf{fVect}^{\mathbf{F}^{\mathrm{op}}}.$$

Then we have $A_dQ(2d) = G(2d) \oplus G'(2d)$ as $\mathrm{Aut}(F_{2d})$-modules. By Theorem 8.9, the $\mathrm{Aut}(F_{2d})$-module $A_dQ(2d)$ is indecomposable. Therefore, we can assume that $G'(2d)=0$ and $A_dQ(2d) = G(2d)$. Since the subfunctor A_dQ is generated by $Q \in A_dQ(2d)$, we have $A_dQ = G$. Hence, the subfunctor A_dQ is also indecomposable. By Lemma 8.3, $A_dP(2d)$ is also indecomposable. Therefore, by the similar argument, the subfunctor A_dP is also indecomposable. □

Appendix A. Presentation of the category $\mathbf{A}^L$

In this section, we construct a category $\widetilde{\mathbf{A}^L}$ and a full functor $F : \widetilde{\mathbf{A}^L} \to \mathbf{A}^L$ to study a presentation of the category $\mathbf{A}^L$, which we construct in Section 4.2.

A.1. The category $\widetilde{\mathbf{A}^L}$

In this section, we construct a category $\widetilde{\mathbf{A}^L}$, which has a generating set and some relations of the category $\mathbf{A}^L$.

In a linear symmetric strict monoidal category $\mathcal{C}$, let H be a Hopf algebra and L a Lie algebra. Define the *adjoint action* $ad_H : H \otimes H \to H$ by

$$ad_H = \mu^{[3]}(\mathrm{id}_{H^{\otimes 2}} \otimes S)(\mathrm{id}_H \otimes P_{H,H})(\Delta \otimes \mathrm{id}_H).$$

We call a morphism $c : I \to L^{\otimes 2}$ a *symmetric invariant* 2*-tensor* if c satisfies

$$P_{L,L}c = c$$

and

$$([\cdot,\cdot] \otimes \mathrm{id}_L)(\mathrm{id}_L \otimes c) = (\mathrm{id}_L \otimes [\cdot,\cdot])(c \otimes \mathrm{id}_L).$$

Define $\widetilde{\mathbf{A}^L}$ to be the category which is as a linear symmetric strict monoidal category, generated by

- a cocommutative Hopf algebra $(H,\mu,\eta,\Delta,\epsilon,S)$
- a Lie algebra with a symmetric invariant 2-tensor $(L,[\cdot,\cdot],c)$
- morphisms $i : L \to H$ and $ad_L : H \otimes L \to L$

with the following nine relations:

($\widetilde{\mathbf{A}^L}$.1) $i\,[\cdot,\cdot] = -\mu(i \otimes i) + \mu P_{H,H}(i \otimes i)$,

($\widetilde{\mathbf{A}^L}$.2) $\Delta i = i \otimes \eta + \eta \otimes i$,

($\widetilde{\mathbf{A}^L}$.3) $\epsilon i = 0$,

($\widetilde{\mathbf{A}^L}$.4) $ad_L(\mu \otimes \mathrm{id}_L) = ad_L(\mathrm{id}_H \otimes ad_L)$,

($\widetilde{\mathbf{A}^L}$.5) $ad_L(\eta \otimes \mathrm{id}_L) = \mathrm{id}_L$,

($\widetilde{\mathbf{A}^L}$.6) $(ad_L \otimes ad_L)(\mathrm{id}_H \otimes P_{H,L} \otimes \mathrm{id}_L)(\Delta \otimes c) = c\epsilon$,

($\widetilde{\mathbf{A}^L}$.7) $ad_L(\mathrm{id}_H \otimes [\cdot,\cdot]) = [\cdot,\cdot](ad_L \otimes ad_L)(\mathrm{id}_H \otimes P_{H,L} \otimes \mathrm{id}_L)(\Delta \otimes \mathrm{id}_{L^{\otimes 2}})$,

($\widetilde{\mathbf{A}^L}$.8) $i\,ad_L = ad_H\,i$,

($\widetilde{\mathbf{A}^L}$.9) $ad_L(i \otimes \mathrm{id}_L) = -[\cdot,\cdot]$.

Lemma A.1. *In the category* $\widetilde{\mathbf{A}^L}$*, the following relations hold.*

(1) $Si = -i$.

(2) $ad_H(i \otimes i) = -i\,[\cdot,\cdot]$.

Proof. By ($\widetilde{\mathbf{A}^L}$.2) and ($\widetilde{\mathbf{A}^L}$.3) of the category $\widetilde{\mathbf{A}^L}$ and relations of Hopf algebras, we have

$$i + Si = \mu(i \otimes S\eta) + \mu(\eta \otimes Si) = \mu(\mathrm{id}_H \otimes S)\Delta i = \eta\epsilon i = 0.$$

Thus, we have equation (1). By $(\widetilde{\mathbf{A}^L}.8), (\widetilde{\mathbf{A}^L}.9)$, we have equation (2) as follows:

$$ad_H(i \otimes i) = i\, ad_L(i\ \mathrm{id}_L) = -i\, [\cdot,\cdot]. \qquad \square$$

We review the definition of a Casimir Hopf algebra. Let $\mathcal{C}$ be a linear symmetric strict monoidal category and H be a cocommutative Hopf algebra in $\mathcal{C}$. A *Casimir 2-tensor* for H is a morphism $c : I \to H^{\otimes 2}$ which is primitive, symmetric and invariant:

$$(\Delta \otimes \mathrm{id}_H)c = c_{13} + c_{23}, \tag{A.1}$$

$$P_{H,H}c = c, \tag{A.2}$$

$$(ad_H \otimes ad_H)(\mathrm{id}_H \otimes P_{H,H} \otimes \mathrm{id}_H)(\Delta \otimes c) = c\epsilon, \tag{A.3}$$

where $c_{13} := (\mathrm{id} \otimes \eta \otimes \mathrm{id})c$ and $c_{23} := \eta \otimes c$. By a *Casimir Hopf algebra*, we mean a cocommutative Hopf algebra H equipped with a Casimir 2-tensor.

Lemma A.2. *$(H, \mu, \eta, \Delta, \epsilon, S, \tilde{c} := (i \otimes i)c)$ is a Casimir Hopf algebra in $\widetilde{\mathbf{A}^L}$.*

Proof. Since H is a cocommutative Hopf algebra in $\widetilde{\mathbf{A}^L}$, it suffices to check that $\tilde{c}$ is a Casimir 2-tensor. By $(\widetilde{\mathbf{A}^L}.\ 2)$, we have equation (A.1) because

$$(\Delta \otimes \mathrm{id}_H)\tilde{c} = ((i \otimes \eta + \eta \otimes i) \otimes i)c = \tilde{c}_{13} + \tilde{c}_{23}.$$

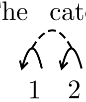
By the symmetricity of c, we have equation (A.2) because

$$P_{H,H}\tilde{c} = P_{H,H}(i \otimes i)c = (i \otimes i)P_{L,L}c = (i \otimes i)c = \tilde{c}.$$

By $(\widetilde{\mathbf{A}^L}.\ 6)$ and $(\widetilde{\mathbf{A}^L}.\ 8)$, we have equation (A.3) because

$$\begin{aligned}
&(ad_H \otimes ad_H)(\mathrm{id}_H \otimes P_{H,H} \otimes \mathrm{id}_H)(\Delta \otimes \tilde{c}) \\
&= (ad_H \otimes ad_H)(\mathrm{id}_H \otimes P_{H,H} \otimes \mathrm{id}_H)(\Delta \otimes (i \otimes i))(\mathrm{id}_H \otimes c) \\
&= (i \otimes i)(ad_L \otimes ad_L)(\mathrm{id}_H \otimes P_{H,L} \otimes \mathrm{id}_L)(\Delta \otimes c) \\
&= (i \otimes i)c\epsilon \\
&= \tilde{c}\epsilon.
\end{aligned}$$

$\square$

The category $\mathbf{A}$ has a Casimir Hopf algebra $(H, c) = (1, \mu, \eta, \Delta, \epsilon, S, c)$, where $c =$ 

. Moreover, Theorem 5.11 in [11] implies that as a linear symmetric strict monoidal category, the category $\mathbf{A}$ is free on the Casimir Hopf algebra (H, c). Therefore, we have a unique linear symmetric monoidal functor $F_{(H,\tilde{c})} : \mathbf{A} \to \widetilde{\mathbf{A}^L}$.

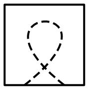
A.2. Structure of the category $\mathbf{A}^L$

In Section 4.3, we observed that the category $\mathbf{A}^L$ has a cocommutative Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$ and morphisms

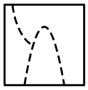
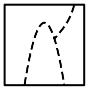
$$[\cdot,\cdot] : L \otimes L \to L, \quad c_L : I \to L \otimes L, \quad i : L \to H, \quad ad_L : H \otimes L \to L.$$

Lemma A.3. *In the category* $\mathbf{A}^L$, $(L,[\cdot,\cdot],c_L)$ *is a Lie algebra with a symmetric invariant 2-tensor.*

Proof. By the AS and IHX relations, it follows that $(L,[\cdot,\cdot])$ is a Lie algebra. Since we have

$P_{L,L}c_L =$  $=$  $= c_L$

and

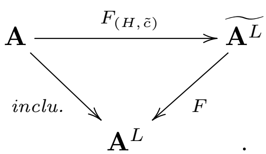
$([\cdot,\cdot]\otimes \mathrm{id}_L)(\mathrm{id}_L \otimes c_L) =$  $=$  $= (\mathrm{id}_L \otimes [\cdot,\cdot])(c_L \otimes \mathrm{id}_L),$

it follows that c_L is a symmetric invariant 2-tensor. □

Remark A.4. The full subcategory of $\mathbf{A}^L$ with the free monoid generated by L as the set of objects is isomorphic to the PROP LIE^c for Casimir Lie algebras (see [13] for details).

For each $m \geq 1, n \in \mathbb{N}$, the degree 0 part $\mathbf{A}^L_0(L^{\otimes m}, H^{\otimes n})$ of the hom-set $\mathbf{A}^L(L^{\otimes m}, H^{\otimes n})$ has an $\mathrm{Aut}(F_n)$-module structure which is defined in a way similar to that of $A_d(n)$. For general m,n, the $\mathrm{Aut}(F_n)$-action on $\mathbf{A}^L_0(L^{\otimes m}, H^{\otimes n})$ does not factors through the outer automorphism group $\mathrm{Out}(F_n)$.

Proposition A.5. *There exists a unique linear symmetric monoidal functor* $F : \widetilde{\mathbf{A}^L} \to \mathbf{A}^L$ *which maps* $(L,[\cdot,\cdot],c_L,i,ad_L)$ *in* $\widetilde{\mathbf{A}^L}$ *to* $(L,[\cdot,\cdot],c,i,ad_L)$ *in* $\mathbf{A}^L$ *and which makes the following diagram commutative*

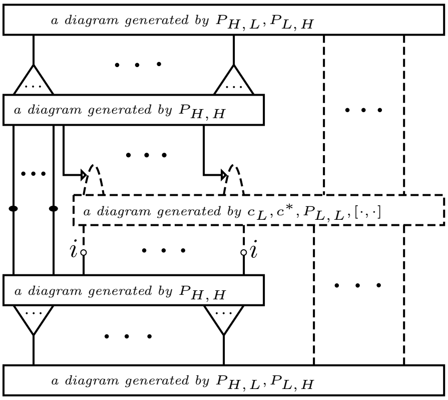


Proof. We can check that morphisms of $\mathbf{A}^L$ satisfy the relations $(\widetilde{\mathbf{A}^L}.1), \cdots, (\widetilde{\mathbf{A}^L}.9)$ by diagrammatic computation. Since $\widetilde{\mathbf{A}^L}$ is the linear symmetric strict monoidal category generated by H, L and morphisms i, ad_L with relations $(\widetilde{\mathbf{A}^L}.1), \cdots, (\widetilde{\mathbf{A}^L}.9)$, we can construct a unique linear symmetric monoidal functor $F : \widetilde{\mathbf{A}^L} \to \mathbf{A}^L$ which maps (H,L,c,i,ad_L) in $\widetilde{\mathbf{A}^L}$ to (H,L,c_L,i,ad_L) in $\mathbf{A}^L$. □

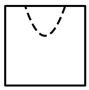
A.3. The full functor $F : \widetilde{\mathbf{A}^L} \to \mathbf{A}^L$

We prove that the functor F in Proposition A.5 is full.

Lemma A.6. *A morphism in $\mathbf{A}^L$ can be written as a linear sum of the following diagrams:*



, (A.4)

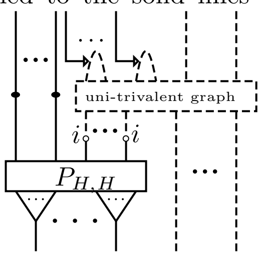
where  *denotes S or* id_H *and $c^* =$*  .

Note that c^* is not a morphism in $\mathbf{A}^L$ but just a diagram.

Proof. By using symmetries $P_{H,L}, P_{L,H}$, we can deform any diagram $f \in \mathbf{A}^L$ into a morphism in $\mathbf{A}^L(H^{\otimes m} \otimes L^{\otimes n}, H^{\otimes m'} \otimes L^{\otimes n'})$, so it suffices to consider a diagram f in $\mathbf{A}^L(H^{\otimes m} \otimes L^{\otimes n}, H^{\otimes m'} \otimes L^{\otimes n'})$.

We can decompose f as follows: $f = f' \circ ((P \circ \Delta^{[c_1,\cdots,c_m]}) \otimes \mathrm{id}_{L^{\otimes n}})$, where P is a tensor product of copies of $P_{H,H}$ and id_H, $c_1, \cdots, c_m \geq 0$, and f' is a diagram such that each handle has only one solid or dashed line. We can assume that handles of U_m which include a dashed line are arranged right-hand side of U_m.

By pulling univalent vertices that are attached to the solid lines toward the upper right-hand side of U_m, we can decompose f' as



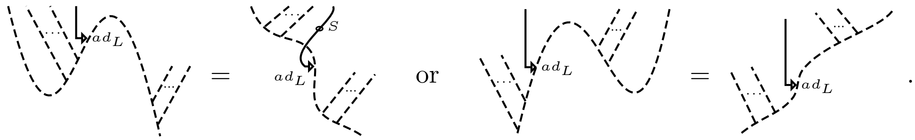
(see Lemma 5.16 [11]).

Furthermore, any uni-trivalent graph can be obtained from morphisms c_L, $P_{L,L}$, $[\cdot,\cdot]$, $\mathrm{id}_L \in \mathbf{A}^L$ and c^* by the tensor product and the composition, so the proof is complete. □

Proposition A.7. *The linear symmetric monoidal functor $F : \widetilde{\mathbf{A}^L} \to \mathbf{A}^L$ in Proposition A.5 is full.*

Proof. It suffices to show that morphisms of $\mathbf{A}^L$ are generated by $\mu,\eta,\Delta,\epsilon,S$, $[\cdot,\cdot],c_L$, i, ad_L and symmetries. By Lemma A.6, we need to prove that we can eliminate c^* from the diagram (A.4) by using the above morphisms in $\mathbf{A}^L$.

By the definition of the category $\mathbf{A}^L$, for any c^* in the diagram (A.4), if exists, either of the endpoints of c^* is finally attached to one of the lower dashed lines. Therefore, there is c_L between c^* and the lower dashed line. If there are more than one such c_L, then we choose one such that there are the least trivalent vertices between c^* and itself. By the AS relation, we have only to consider the case where the neighborhood of the c_L and the c^* is either



Hence, we can eliminate c^* from the diagram (A.4) and the proof is complete. □

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References

[1] S. Andreadakis, 'On the automorphisms of free groups and free nilpotent groups', *Proc. London Math. Soc. (3)* **15** (1965), 239–268. https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms/s3-15.1.239.

[2] D. Bar-Natan, 'On the Vassiliev knot invariants', *Topology* **34**(2) (1995), 423–472. https://www.sciencedirect.com/science/article/pii/0040938395932372?via.

[3] D. Bar-Natan, 'Vassiliev homotopy string link invariants', *J. Knot Theory Ramifications* **4**(1) (1995), 13–32. https://www.worldscientific.com/doi/abs/10.1142/S021821659500003X.

[4] D. Bar-Natan, 'Some computations related to Vassiliev invariants,' *Preprint*, 1996.

[5] L. Bartholdi, 'Automorphisms of free groups I—erratum', *New York J. Math.* **22** (2016), 1135–1137 [MR3084710]. https://nyjm.albany.edu/j/2016/22-52.html.

[6] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts, Vol. **35** (Cambridge University Press, Cambridge, 1997).

[7] W. Fulton and J. Harris, *Representation Theory* A First Course, Readings in Mathematics, Graduate Texts in Mathematics, Vol. **129** (Springer-Verlag, New York, 1991).

[8] N. Habegger and X.-S. Lin, 'The classification of links up to link-homotopy', *J. Amer. Math. Soc.* **3**(2) (1990), 389–419. https://www.ams.org/journals/jams/1990-03-02/S0894-0347-1990-1026062-0/home.html.

[9] K. Habiro, 'Bottom tangles and universal invariants', *Algebr. Geom. Topol.* **6** (2006), 1113–1214. https://msp.org/agt/2006/6-3/p05.xhtml.

[10] K. Habiro and G. Massuyeau, 'Generalized Johnson homomorphisms for extended N-series', *J. Algebra* **510** (2018), 205–258. https://www.sciencedirect.com/science/article/pii/S0021869318303569?via.

[11] K. Habiro and G. Massuyeau, 'The Kontsevich integral for bottom tangles in handlebodies', *Quantum Topol.* **12**(4) (2021), 593–703. https://mathscinet.ams.org/mathscinet-getitem?mr=4321214.

[12] M. Hartl, T. Pirashvili and C. Vespa, 'Polynomial functors from algebras over a set-operad and nonlinear Mackey functors', *Int. Math. Res. Not. IMRN* **6** (2015), 1461–1554. https://academic.oup.com/imrn/article/2015/6/1461/668187.

[13] V. Hinich and A. Vaintrob, 'Cyclic operads and algebra of chord diagrams', *Selecta Math. (N.S.)* **8**(2) (2002), 237–282. https://doi.org/10.1007/s00029-002-8106-2.

[14] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, Vol. **16** (Addison-Wesley Publishing Co., Reading, Mass., 1981), With a foreword by P. M. Cohn and an introduction by Gilbert de B. Robinson.

[15] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, Vol. **155** (Springer-Verlag, New York, 1995).

[16] M. Katada, 'Actions of automorphism groups of free groups on spaces of Jacobi diagrams, **I**', *Ann. Inst. Fourier* (2021) (to appear), Preprint, arXiv: 2102.06382. https://arxiv.org/abs/2102.06382.

[17] N. Kawazumi, 'Cohomological aspects of Magnus expansions', Preprint, 2005, arXiv: math/0505497. https://arxiv.org/abs/math/0505497.

[18] M. Kontsevich, 'Vassiliev's knot invariants', In *I. M. Gel'fand Seminar*, Adv. Soviet Math., Vol. **16** (Amer. Math. Soc., Providence, RI, 1993), 137–150.

[19] T. Ohtsuki, *Quantum Invariants*, Series on Knots and Everything, Vol. **29** (World Scientific Publishing Co., Inc., River Edge, NJ, 2002).

[20] G. Powell and C. Vespa, 'Higher Hochschild homology and exponential functors', Preprint, 2018, arXiv: 1802.07574. https://arxiv.org/abs/1802.07574.

[21] B. E. Sagan, *The Symmetric Group*, 2^{nd} ed., Graduate Texts in Mathematics, Vol. **203** (Springer-Verlag, New York, 2001).

[22] T. Satoh, 'A survey of the Johnson homomorphisms of the automorphism groups of free groups and related topics', In *Handbook of Teichmüller theory. Vol. V*, IRMA Lect. Math. Theor. Phys., Vol. **26** (Eur. Math. Soc., Zürich, 2016), 167–209. https://arxiv.org/abs/1204.0876.

[23] T. Satoh, 'The third subgroup of the Andreadakis-Johnson filtration of the automorphism group of a free group', *J. Group Theory* **22**(1) (2019), 41–61. https://www.degruyter.com/document/doi/10.1515/jgth-2018-0037/html.