Locally Defined Independence Systems on Graphs

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The maximization for independence system is one of the most fundamental combinatorial optimization problems [4, 16, 17]. An independence system is a pair $(E, \mathcal{I})$ of a finite set E and a family $\mathcal{I} \subseteq 2^E$ that satisfies

$$\mathcal{I} \text{ contains empty set, i.e., } \emptyset \in \mathcal{I}, \text{ and} \tag{1}$$

$$J \in \mathcal{I} \text{ implies } I \in \mathcal{I} \text{ for any } I \subseteq J \subseteq E. \tag{2}$$

Here a member I in $\mathcal{I}$ is called an *independent set*. Property (2) means that $\mathcal{I}$ is downward closed. The maximization problem for an independence system is to find an independent set with the maximum cardinality. This problem includes, as a special case, the maximum independent set of a graph, the maximum matching, the maximum set packing and the matroid (intersection) problems [14, 16, 17].

In the paper, we consider the following independence systems defined on graphs. Let $G = (V, E)$ be a graph with a vertex set V and an edge set E. For a vertex v in V, let E_v denote the set of edges incident to v. In our problem setting, each vertex v has a local independence system $(E_v, \mathcal{I}_v)$, i.e., $\mathcal{I}_v \subseteq 2^{E_v}$, and we consider the independence system $(E, \mathcal{I})$ defined by

$$\mathcal{I} = \{I \subseteq E \mid I \cap E_v \in \mathcal{I}_v \text{ for all } v \in V\}. \tag{3}$$

Namely, $(E, \mathcal{I})$ is obtained by concatenating local independence systems $(E_v, \mathcal{I}_v)$, and is called *an independence system defined on a graph* G. In the paper, we consider the maximization problem for it, i.e., for a given graph $G = (V, E)$ with local independence systems $(E_v, \mathcal{I}_v)$, our problem is described as

$$\begin{array}{rl} \text{maximize} & |I| \\ \text{subject to} & I \cap E_v \in \mathcal{I}_v \text{ for all } v \in V \\ & I \subseteq E. \end{array} \tag{4}$$

Note that any independence system $(E, \mathcal{I})$ is viewed as an independence system defined on a star.

In the paper, we consider problem (4) by making use of local oracles $\mathcal{A}_v$ for each v in V. For an independence system $(E, \mathcal{I})$ and a subset $F \subseteq E$, $\mathcal{I}[F]$ denotes the family of independent sets of $\mathcal{I}$ restricted to F, i.e., $\mathcal{I}[F] = \{I \cap F \mid I \in \mathcal{I}\}$. For a vertex $v \in V$ and a subset $F \subseteq E_v$, $\mathcal{A}_v(F)$ is an α-approximate independent set of the maximization for $(F, \mathcal{I}_v[F])$. That is, the oracle $\mathcal{A}_v : 2^{E_v} \to 2^{E_v}$ satisfies

$$\mathcal{A}_v(F) \in \mathcal{I}_v[F] \tag{5}$$

$$\alpha\, |\mathcal{A}_v(F)| \geq \max_{J \in \mathcal{I}_v[F]} |J|. \tag{6}$$

We call $\mathcal{A}_v$ an *α-approximation local* oracle. It is also called an *exact local oracle* if $\alpha = 1$. In the paper, we assume the monotonicity of $\mathcal{A}_v$, i.e., $|\mathcal{A}_v(S)| \leq |\mathcal{A}_v(T)|$ holds for the subsets $S \subseteq T \subseteq E_v$, which is a natural assumption on the oracle since it deals with independence system. We study this oracle model to investigate the global approximability of problem (4) by using the local approximability.

In the paper, we first propose two natural algorithms for problem (4), where the first one applies local oracles $\mathcal{A}_v$ in the order of the vertices v that is fixed in advance, while the second one applies local oracles in the greedy order of vertices $v_1, \ldots, v_n$, where $n = |V|$ and

$$v_i \in \arg\max\{|\mathcal{A}_v(E_v \cap F^{(i)})| \mid v \in V \setminus \{v_1, \ldots, v_{i-1}\}\} \qquad \text{for } i = 1, \ldots, n.$$

Here the subset $F^{(i)} \subseteq E$ is a set of available edges during the i-th iteration.

We show that the first algorithm guarantees an approximation ratio $(\alpha+n-2)$, and the second algorithm guarantees an approximation ratio $\rho(\alpha, n)$, where ρ is the function of α and n defined as

$$\rho(\alpha, n) = \begin{cases} \alpha + \dfrac{2\alpha - 1}{2\alpha}(n-1) - \dfrac{1}{2} & \text{if } (\alpha-1)(n-1) \geq \alpha(\alpha+1) \\ \alpha + \dfrac{\alpha}{\alpha+1}(n-1) & \text{if } \alpha \leq (\alpha-1)(n-1) < \alpha(\alpha+1) \\ \dfrac{n}{2} & \text{if } (\alpha-1)(n-1) < \alpha. \end{cases}$$

We also show that both of approximation ratios are *almost tight* for these algorithms.

We then consider two subclasses of problem (4). We provide two approximation algorithms for the k-degenerate graphs, whose approximation ratios are $\alpha + 2k - 2$ and αk. Here, a graph is *k-degenerate* if any subgraph has a vertex of degree at most k. This implies for example that the algorithms find an α-approximate independent set for the problem if a given graph is a tree. This is best possible, because the local maximization is not approximable with c ($< \alpha$). We also show that the second algorithm can be generalized to the hypergraph setting.

We next provide an $(\alpha + k)$-approximation algorithm for the problem when a given graph is bipartite and local independence systems for one side are all k-systems with independence oracles. Here an independence system $(E, \mathcal{I})$ is called a *k-system* if for any subset $F \subseteq E$, any two maximal independent sets I and J in $\mathcal{I}[F]$ satisfy $k|I| \geq |J|$, and its independence oracle is to decide if a given subset $J \subseteq E$ belongs to $\mathcal{I}$ or not.

All of statements and proofs can be found in the full version [1]. The full paper is organized as follows. In Section 2, we describe two natural algorithms for problem (4) and analyze their approximation ratios. Section 3 provides approximation algorithms for the problem in which a given graph G has bounded degeneracy. Section 4 also provides an approximation algorithm for the problem in which a given graph G is bipartite, and all the local independence systems of the one side of vertices are k-systems. Section 5 defines independence systems defined on hypergraphs and generalizes algorithms to the hypergraph case.

Acknowledgments

I give special gratitude to Kazuhisa Makino for his valuable discussions and carefully proofreading of the manuscript. I am also grateful to Yusuke Kobayashi, Akitoshi Kawamura, and Satoru Fujishige for constructive suggestions. This work was supported by the joint project of Kyoto University and Toyota Motor Corporation, titled "Advanced Mathematical Science for Mobility Society".

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