A summary of “The pro-$\mathcal{C}$ anabelian geometry of number fields”

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This is a summary of [Shimizu3]. Let K be a number field and $\mathcal{C}$ a full class of finite groups. We write $K^{\mathcal{C}}/K$ for the maximal pro-$\mathcal{C}$ Galois extension of K, and $G_K^{\mathcal{C}}$ for its Galois group, which is the maximal pro-$\mathcal{C}$ quotient of the absolute Galois group of K.

The Neukirch-Uchida theorem, which is one of the most important results in anabelian geometry, states that the isomorphisms of the absolute Galois groups of number fields arise functorially from unique isomorphisms of fields (cf. [Uchida]). Moreover, various generalizations of the Neukirch-Uchida theorem, where one replaces the absolute Galois groups by their various quotients, have been studied by many mathematicians (e.g. [Ivanov], [Ivanov2], [Saïdi-Tamagawa], [Shimizu] and [Shimizu2]). These results prompt the following natural question:

For $i = 1, 2$, let K_i be a number field, $\mathcal{C}_i$ a nontrivial full class of finite groups, and $\sigma : G_{K_1}^{\mathcal{C}_1} \xrightarrow{\sim} G_{K_2}^{\mathcal{C}_2}$ an isomorphism. Is σ induced by a unique isomorphism between $K_2^{\mathcal{C}_2}/K_2$ and $K_1^{\mathcal{C}_1}/K_1$?

In this paper, we answer this question under some assumptions. Write $\Sigma(\mathcal{C})$ for the set of prime numbers p with $\mathbb{Z}/p\mathbb{Z} \in \mathcal{C}$. For a set S of nonarchimedean primes of K, set

$$\delta_{\sup}(S) \overset{\text{def}}{=} \limsup_{s \to 1+0} \frac{\sum_{\mathfrak{p} \in S} \mathfrak{N}(\mathfrak{p})^{-s}}{\log \frac{1}{s-1}}, \quad \delta_{\inf}(S) \overset{\text{def}}{=} \liminf_{s \to 1+0} \frac{\sum_{\mathfrak{p} \in S} \mathfrak{N}(\mathfrak{p})^{-s}}{\log \frac{1}{s-1}}$$

and if $\delta_{\sup}(S) = \delta_{\inf}(S)$, then write $\delta(S)$ (the Dirichlet density of S) for them. The most important result is the following:

Theorem A ([Shimizu3, Theorem 7.1])**.** Assume that $\delta_{\sup}(\Sigma(\mathcal{C}_i)) > 0$ for at least one $i \in \{1, 2\}$. Then σ is induced by a unique isomorphism between $K_2^{\mathcal{C}_2}/K_2$ and $K_1^{\mathcal{C}_1}/K_1$.

The goal of this paper is to prove this theorem. As in proofs of the Neukirch-Uchida theorem (cf. [NSW]) and other previous results, we first characterize group-theoretically some data in $G_K^{\mathcal{C}}$ (e.g. the decomposition groups), and then obtain an isomorphism of fields using them. In the following, we describe the structure of this paper in more detail.

In §1, we begin by collecting results on the decomposition groups of $G_K^{\mathcal{C}}$. For example, we prove that the decomposition groups of $G_K^{\mathcal{C}}$ are canonically isomorphic to the maximal pro-$\mathcal{C}$ quotients of the absolute Galois groups of p-adic fields, and that for two distinct primes of

$K^{\mathcal{C}}$, the intersection of their decomposition groups is trivial. The uniqueness in the question follows immediately from this.

In §2, we recover group-theoretically a certain subset of the set of decomposition groups of $G_K^{\mathcal{C}}$. More precisely:

Theorem B ([Shimizu3, Theorem 2.11])**.** Let $l \in \Sigma(\mathcal{C})$. Set $P_{K,f}^{\mathcal{C},l} \stackrel{\text{def}}{=} \{\mathfrak{p} \in P_{K,f} \setminus P_{K,l} \mid \mu_l \subset K_{\mathfrak{p}}^{\mathcal{C}}\}$, where $P_{K,f}$ (resp. $P_{K,l}$) and $K_{\mathfrak{p}}^{\mathcal{C}}$ stand for the set of nonarchimedean primes (resp. nonarchimedean primes above l) of K and the maximal pro-$\mathcal{C}$ Galois extension of $K_{\mathfrak{p}}$, respectively. Then the set of decomposition groups in $G_K^{\mathcal{C}}$ at primes in $P_{K,f}^{\mathcal{C},l}$ can be recovered group-theoretically from $G_K^{\mathcal{C}}$.

Further, by using this result, the pro-$\mathcal{C}$ factor of the l-adic cyclotomic character $G_K^{\mathcal{C}} \to \mathrm{Aut}(\mu_{l^\infty})^{\mathcal{C}} = {\mathbb{Z}_l}^{*,\mathcal{C}}$ can be recovered group-theoretically from $G_K^{\mathcal{C}}$.

In §3, we study fundamental properties of the "local correspondence":

Definition C ([Shimizu3, Definition 3.1])**.** For $i = 1, 2$, let $S_i \subset P_{K_i,f}$. We say that a bijection $\phi\colon S_1(K_1^{\mathcal{C}_1}) \xrightarrow{\sim} S_2(K_2^{\mathcal{C}_2})$ is a local correspondence between S_1 and S_2 for σ if $\sigma(D_{\overline{\mathfrak{p}}_1}(K_1^{\mathcal{C}_1}/K_1)) = D_{\phi(\overline{\mathfrak{p}}_1)}(K_2^{\mathcal{C}_2}/K_2)$ for any $\overline{\mathfrak{p}}_1 \in S_1(K_1^{\mathcal{C}_1})$. Let $\phi\colon S_1(K_1^{\mathcal{C}_1}) \xrightarrow{\sim} S_2(K_2^{\mathcal{C}_2})$ be a local correspondence between S_1 and S_2 for σ. We say that ϕ satisfies condition (Char) (resp. condition (Deg)) if for any $\overline{\mathfrak{p}}_1 \in S_1(K_1^{\mathcal{C}_1})$, the residue characteristics (resp. the local degrees) of $\overline{\mathfrak{p}}_1|_{K_1}$ and $\phi(\overline{\mathfrak{p}}_1)|_{K_2}$ coincide.

By Theorem B, we obtain a local correspondence between $P_{K_1,f}^{\mathcal{C}_1}$ and $P_{K_2,f}^{\mathcal{C}_2}$ for σ, where $P_{K,f}^{\mathcal{C}} \stackrel{\text{def}}{=} \cup_{l\in\Sigma(\mathcal{C})} P_{K,f}^{\mathcal{C},l}$.

In §4, we study separatedness of the decomposition groups in $G(\mathbb{Q}(\cup_{l\in\Sigma(\mathcal{C})}\mu_l)/\mathbb{Q})^{\mathcal{C}} = G(\mathbb{Q}^{\mathcal{C}} \cap \mathbb{Q}(\cup_{l\in\Sigma(\mathcal{C})}\mu_l)/\mathbb{Q})$. Together with a certain part of the cyclotomic character recovered in §2, we prove that if $\delta_{\sup}(\Sigma(\mathcal{C}_i)) > 0$ for at least one $i \in \{1, 2\}$, then the local correspondence between $P_{K_1,f}^{\mathcal{C}_1}$ and $P_{K_2,f}^{\mathcal{C}_2}$ (resp. $(\underline{P_{K_1,f}^{\mathcal{C}_1}} \cap \mathrm{cs}(K_1/\mathbb{Q}))(K_1)$ and $(\underline{P_{K_2,f}^{\mathcal{C}_2}} \cap \mathrm{cs}(K_2/\mathbb{Q}))(K_2)$) for σ satisfies condition (Char) (resp. conditions (Char) and (Deg)), where $\underline{P_{K,f}^{\mathcal{C}}} \stackrel{\text{def}}{=} \{p \in P_{\mathbb{Q},f} \mid P_{K,p} \subset P_{K,f}^{\mathcal{C}}\}$.

In §5, we develop two ways to show that isomorphisms of Galois groups of number fields are induced by field isomorphisms under some assumptions. The two results ([Shimizu3, Proposition 5.3 and Proposition 5.8] are based on [Uchida2] and [Shimizu2], respectively. However, these previous works cannot be applied to the proof of our question as they are, so that the goal of §5 is to modify them by inventing some new methods. In the proof of Theorem A, we use the former:

Proposition D ([Shimizu3, Proposition 5.3])**.** Let $S_0 \subset P_{\mathbb{Q},f}$. Assume that the following conditions hold:

(a) $\#\Sigma(\mathcal{C}_1) = \infty$.

(b) $\delta_{\inf}(\mathrm{cs}(K_1/\mathbb{Q}) \setminus S_0) = 0$.

(c) There exists a local correspondence between $S_0(K_1)$ and $S_0(K_2)$ for σ satisfying conditions (Char) and (Deg).

Then σ is induced by a unique isomorphism between $K_2^{\mathcal{C}_2}/K_2$ and $K_1^{\mathcal{C}_1}/K_1$.

In the latter result, which will be used in the proof of Theorem E, we need to assume the existence of a complex prime for the number field, however, the assumption on the Dirichlet densities of the sets between which the local correspondence exists is weaker than the former.

In §6, as a preparation for Theorem A, we prove that if $\delta_{\sup}(\Sigma(\mathcal{C})) > 0$, then $\delta(\underline{P_{K,f}^{\mathcal{C}}}) = 1$.

In §7, using the results obtained so far, we prove Theorem A and its corollaries, for example about the group of outer isomorphisms of $G_K^{\mathcal{C}}$ (cf. [Shimizu3, Corollary 7.4]). Further, even when $\delta(\Sigma(\mathcal{C}_i)) = 0$ for each $i \in \{1, 2\}$, most of the results in this paper are still valid, so that we can answer the question under some technical conditions:

Theorem E ([Shimizu3, Theorem 7.5])**.** For $i = 1, 2$, let $S_i \subset P_{K_i,f}$, and write $\underline{S_i} \overset{\text{def}}{=} \{p \in P_{\mathbb{Q},f} \mid P_{K_i,p} \subset S_i\}$. Assume that the following conditions hold:

(a) For one i, $2 \in \Sigma(\mathcal{C}_i)$ or K_i has a complex prime.

(b) There exists a local correspondence between S_1 and S_2 for σ, satisfying condition (Char).

(c) For any finite Galois subextension L_1 of $K_1^{\mathcal{C}_1}/K_1$, $\delta_{\sup}(\underline{S_1} \cap \mathrm{cs}(L_1/\mathbb{Q})) > 0$.

(d) $\underline{S_2} \cap \mathrm{cs}(K_2/\mathbb{Q}) \neq \emptyset$.

Then σ is induced by a unique isomorphism between $K_2^{\mathcal{C}_2}/K_2$ and $K_1^{\mathcal{C}_1}/K_1$.

By using this result, we can prove the following analog of "the relative Grothendieck Conjecture":

Corollary F ([Shimizu3, Corollary 7.9])**.** Assume that the following conditions hold:

(a) For one i, $2 \in \Sigma(\mathcal{C}_i)$ or K_i has a complex prime.

(b) The following diagram commutes:

$$\begin{tikzcd} G_{K_1}^{\mathcal{C}_1} \arrow[rr, "\simeq", "\sigma"'] \arrow[dr] & & G_{K_2}^{\mathcal{C}_2} \arrow[dl] \\ & G_{\mathbb{Q}}^{\mathrm{ab},\mathcal{C}_1}, & \end{tikzcd}$$

where the diagonal arrows are the restrictions.

Then σ is induced by a unique isomorphism between $K_2^{\mathcal{C}_2}/K_2$ and $K_1^{\mathcal{C}_1}/K_1$.

Moreover, by Theorem E, Conjecture G is reduced to Conjecture H (cf. [Shimizu3, Proposition 7.12]).

Conjecture G ([Shimizu3, Conjecture 7.10])**.** Assume that the following condition holds:

(a) For one i, $2 \in \Sigma(\mathcal{C}_i)$ or K_i has a complex prime.

Then σ is induced by a unique isomorphism between $K_2^{\mathcal{C}_2}/K_2$ and $K_1^{\mathcal{C}_1}/K_1$.

Conjecture H ([Shimizu3, Conjecture 7.11])**.** As we have seen after Definition C, there exists a local correspondence ϕ between $P_{K_1,f}^{\mathcal{C}_1}$ and $P_{K_2,f}^{\mathcal{C}_2}$ for σ. Then ϕ satisfies condition (Char).

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