A summary of "The m-step solvable Grothendieck conjecture for affine hyperbolic curves over finitely generated fields"

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n anabelian geometry, the following conjecture, called the Grothendieck conjecture for hyperbolic curves, is a central problem: If the (tame) arithmetic fundamental groups of two hyperbolic curves over a field k are isomorphic as profinite groups (over the absolute Galois group G_k), are these hyperbolic curves are isomorphic (over k)? For this conjecture, the case where k is finitely generated over the rational number field $\mathbb{Q}$ and the hyperbolic curves have genus 0 was settled by Hiroaki Nakamura, the case where k is either a finite field or finitely generated over $\mathbb{Q}$ and the hyperbolic curves are affine was settled by Akio Tamagawa, and finally, this conjecture was completely proved by Shinichi Mochizuki when k is a sub-ℓ-adic field (i.e., a subfield of a field finitely generated over the ℓ-adic local field $\mathbb{Q}_\ell$ for a fixed prime ℓ). Thus, the Grothendieck conjecture has been proved. However, we can consider the following extension of this conjecture:

(Q) *If certain quotients of the (tame) arithmetic fundamental groups of two hyperbolic curves over k are isomorphic as profinite groups (over G_k), are these hyperbolic curves isomorphic (over k)?*

In particular, when the quotients are the maximal geometrically m-step solvable quotients of the (tame) arithmetic fundamental group, we call (Q) the m-step solvable Grothendieck conjecture, where m is a positive integer. This conjecture was proved in the case where "k is an algebraic number field satisfying certain conditions, $m \geq 2$, and the hyperbolic curves are 4-punctured projective lines over k" by Hiroaki Nakamura, where "k is a sub-ℓ-adic field and $m \geq 5$" by Shinichi Mochizuki, and where "k is finitely generated over the prime field, $m \geq 3$, and the hyperbolic curves have genus 0 and satisfy a certain condition" by the author.

In this paper, we prove the m-step solvable Grothendieck conjecture for affine hyperbolic curves in most cases, as follows.

(Notation) Let m be a positive integer. For $i = 1, 2$, let k_i be a field of characteristic $p_i \geq 0$ and G_{k_i} the absolute Galois group of k_i. Let X_i be a proper, smooth curve over k_i and E_i a closed subscheme of X_i which is finite, étale over k_i. Let g_i be the genus of X_i and r_i the degree of E_i over k_i. Set $U_i := X_i - E_i$. If $p_i > 0$, then, for $n \in \mathbb{Z}_{\geq 0}$, we write $U_i(n)$ for the n-th Frobenius twist of U_i over k_i. For any extension l over k_i, we write $U_{i,l} := U_i \times_{k_i} l$. We write Π_{U_i} for the tame arithmetic fundamental group $\pi_1^{\mathrm{tame}}(U_i, *)$ of U_i and $\overline{\Pi}_{U_i}$ for the tame geometric fundamental group $\pi_1^{\mathrm{tame}}(U_{i,k^{\mathrm{sep}}}, *)$ of U_i. We define $\overline{\Pi}_{U_i}^{[m]}$ as the m-step derived subgroup of $\overline{\Pi}_{U_i}$ and set $\overline{\Pi}_{U_i}^{m} := \overline{\Pi}_{U_i}/\overline{\Pi}_{U_i}^{[m]}$ and $\Pi_{U_i}^{(m)} := \Pi_{U_i}/\overline{\Pi}_{U_i}^{[m]}$.

Finite field case

When k_i is finite, the m-step solvable Grothendieck conjecture has not been proved in any single case. Thus, the following theorem is a completely new result and even the first result on the conjecture over finite fields.

Theorem A (Theorem 2.16, Corollary 2.22). Assume that k_1, k_2 are finite and that U_1 is affine hyperbolic.

(1) Assume that m satisfies

$$\begin{cases} m \geq 2 & (\text{if } r_1 \geq 3 \text{ and } (g_1, r_1) \neq (0,3), (0,4)) \\ m \geq 3 & (\text{if } r_1 < 3 \text{ or } (g_1, r_1) = (0,3), (0,4)) \end{cases}$$

Then $\Pi_{U_1}^{(m)}$ and $\Pi_{U_2}^{(m)}$ are isomorphic as profinite groups if and only if U_1 and U_2 are isomorphic as schemes.

(2) Assume that $m \geq 3$. Let n be an integer satisfying $m > n \geq 2$. Let Φ be an isomorphism $\Pi_{U_1}^{(m-n)} \xrightarrow{\sim} \Pi_{U_2}^{(m-n)}$ of profinite groups which is induced by an isomorphism $\Pi_{U_1}^{(m)} \xrightarrow{\sim} \Pi_{U_2}^{(m)}$ of profinite groups. Then Φ is induced (up to inner automorphisms of $\Pi_{U_2}^{(m-n)}$) by a unique isomorphism $U_1 \xrightarrow{\sim} U_2$ of schemes.

(*Sketch of Proof*) First, we reconstruct the decomposition groups of $\Pi_{U_i}^{(1)}$ from $\Pi_{U_i}^{(m)}$ by using the (quasi-)sections of the natural projection $\Pi_{U_i}^{(m)} \twoheadrightarrow G_{k_i}$. In this step, we always face the difficulty that comes from the fact that we can only use data from $\Pi_{U_i}^{(m)}$, not the whole Π_{U_i}. (Just for example, we face this difficulty when proving the separatedness of decomposition groups of $\Pi_{U_i}^{(m)}$.) Next, we reconstruct the curve U_i from $\Pi_{U_i}^{(1)}$ and its decomposition groups. Then we reconstruct the multiplicative group and the addition of the function field of U_i by using class field theory and a lemma of Tamagawa. This settles (1), and, by applying (1) to coverings of U_i, we prove (2). □

Finitely generated field case

Next, we consider the case that k_i is a field finitely generated over the prime field. In this case, we assume that $k_1 = k_2$ and write k and p instead of k_i and p_i, respectively. The following theorem is the main result in this case.

Theorem B (Theorem 4.12, Corollary 4.18). Assume that k is a field finitely generated over the prime field and that U_1 is affine hyperbolic. Assume that $U_{1,\overline{k}}$ does not descend to a curve over $\overline{\mathbb{F}}_p$ when $p > 0$.

(1) Assume that m satisfies

$$\begin{cases} m \geq 4 & (\text{if } r_1 \geq 3 \text{ and } (g_1, r_1) \neq (0,3), (0,4)) \\ m \geq 5 & (\text{if } r_1 < 3 \text{ or } (g_1, r_1) = (0,3), (0,4)). \end{cases}$$

Then, when $p = 0$ (resp. $p > 0$), $\Pi_{U_1}^{(m)}$ and $\Pi_{U_2}^{(m)}$ are isomorphic over G_k as profinite groups if and only if U_1 and U_2 are isomorphic as k-schemes (resp. $U_1(n_1)$ and $U_2(n_2)$ are isomorphic as k-schemes for some pair n_1, n_2 of non-negative integers).

(2) Assume that $m \geq 5$. Let n be an integer satisfying $m > n \geq 4$. Let Φ be an isomorphism $\Pi_{U_1}^{(m-n)} \xrightarrow{\sim} \Pi_{U_2}^{(m-n)}$ of profinite groups over G_k which is induced by an isomorphism $\Pi_{U_1}^{(m)} \xrightarrow{\sim} \Pi_{U_2}^{(m)}$ of profinite groups over G_k. Then, when $p = 0$ (resp. $p > 0$), Φ is induced (up to inner automorphisms of $\overline{\Pi}_{U_2}^{m-n}$) by a unique isomorphism $U_1 \xrightarrow{\sim} U_2$ of k-schemes (resp. a unique isomorphism $U_1(n_1) \xrightarrow{\sim} U_2(n_2)$ of k-schemes for a unique pair n_1, n_2 of non-negative integers satisfying $n_1 n_2 = 0$).

When $p = 0$, Theorem B(1) for $m = 4$, $g_1 \geq 1$ is a new result which is not covered by the three previous results (by Nakamura, Mochizuki, and the author). When $p > 0$, Theorem B for $g_1 \geq 1$ is a completely new result.

(*Sketch of Proof*) To show Theorem B, we need Theorem A(2) and the following theorem on the m-step solvable version of the Oda-Tamagawa good reduction criterion for affine hyperbolic curves.

Theorem C (Theorem 3.8) Assume that $m \geq 2$. Let R be a discrete valuation ring, s the closed point of $\mathrm{Spec}(R)$, p_s the characteristic of the residue field of s, and (X, E) a hyperbolic curve over the field of fractions $K(R)$. Set $U := X - E$. Then (X, E) has good reduction at s if and only if the image of the inertia group of G_K in the outer automorphism group of the maximal prime-to-p_s' quotient of $\overline{\Pi}_U^m$ is trivial.

By using Galois descent theory, we can reduce the proof of Theorem B to the case that the Jacobian variety of X_1 has a level N structure for an integer $N \geq 3$ which is not divisible by p and E_i consists of k-rational points. We take an integral regular scheme S of finite type over $\mathrm{Spec}(\mathbb{Z})$ with function field k. By replacing S with a suitable open subscheme if necessary, we may assume that N is invertible on S and that there exists a smooth curve $(\mathcal{X}_i, \mathcal{E}_i)$ over S whose generic fiber is isomorphic to (and identified with) (X_i, E_i). The main step of the proof is to show that the morphism $\zeta_i : S \to \mathcal{M}_{g,r}[N]$ classifying $(\mathcal{X}_i, \mathcal{E}_i)$ (with a suitable ordering of the sections in $\mathcal{E}_i$ and a suitable level N structure) for $i = 1, 2$ coincide set-theoretically. By using this, Theorem A, and Theorem C, (1) follows. By applying (1) to coverings of U_i, we prove (2). □