Resolution of sigma-fields for multiparticle finite-state action evolutions with infinite past [1]

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Abstract

For multiparticle finite-state action evolutions, we prove that the observation σ-field admits a resolution involving a third noise which is generated by a random variable with uniform law. The Rees decomposition from the semigroup theory and the theory of infinite convolutions are utilized in our proofs.

1 Introduction

Let us consider the stochastic recursive equation

$$X_k = N_k X_{k-1} \quad \text{for } k \in \mathbb{Z}, \tag{1.1}$$

which we call the *action evolution*, where the *observation* $X = (X_k)_{k\in\mathbb{Z}}$ taking values in a measurable space V evolves from X_{k-1} to X_k at each time k being acted by a random mapping N_k of V. Here we mean by $N_k X_{k-1}$ the evaluation $N_k(X_{k-1})$ of a random mapping N_k at X_{k-1}; we always abbreviate the parentheses to write fv simply for the evaluation $f(v)$. As our processes are indexed by $\mathbb{Z}$, the state X_k we observe at time k is a result after a long time has passed.

We would like to clarify the structure of the full noise $\mathcal{F}_k^{X,N} = \sigma(X_j, N_j : j \le k)$ and the observation noise $\mathcal{F}_k^X = \sigma(X_j : j \le k)$. For families of events, we write $\mathcal{A} \vee \mathcal{B} := \sigma(\mathcal{A} \bigcup \mathcal{B})$. For σ-fields, we say $\mathcal{F} \subset \mathcal{G}$ a.s. (resp. $\mathcal{F} = \mathcal{G}$ a.s.) if $\mathcal{F} \subset \mathcal{G} \vee \mathcal{N}$ (resp. $\mathcal{F} \vee \mathcal{N} = \mathcal{G} \vee \mathcal{N}$) with $\mathcal{N}$ being the family of null events. By iterating the equation (1.1), we have $X_k = N_k N_{k-1} \cdots N_{j+1} X_j$ a.s. for $j < k$. One may then expect that, for any $k \in \mathbb{Z}$,

$$\mathcal{F}_k^{X,N} = \bigcap_{j<k} \left(\mathcal{F}_k^N \vee \mathcal{F}_j^X\right) \overset{?}{=} \mathcal{F}_k^N \vee \left(\bigcap_{j<k} \mathcal{F}_j^X\right) = \mathcal{F}_k^N \vee \mathcal{F}_{-\infty}^X \quad \text{a.s.}, \tag{1.2}$$

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and may conclude that the full noise $\mathcal{F}_k^{X,N}$ can be known by the *driving noise* $\mathcal{F}_k^N := \sigma(N_j : j \le k)$ together with the *remote past noise* $\mathcal{F}_{-\infty}^X := \bigcap_k \mathcal{F}_k^X$, which plays the role of the initial noise at time $-\infty$. But the a.s. identity $\overset{?}{=}$ in (1.2) is false in general; see [11, (1) of Remark 1.4] for erroneous discussions by Kolmogorov and Wiener. We must refer to [2, Section 2.5] for careful treatment of exchanging the order of supremum and intersection between σ-fields.

1.1 Action evolutions and resolution of the observation

We would like to reveal the hidden extra noise. To this end let us introduce some terminology. The action evolution proposed in (1.1) is formulated as follows.

Definition 1.1. Let μ be a probability on a measurable space V^V of mappings of V into itself and call it a *mapping law* on V^V. A (mono-particle) μ-*evolution* is a pair (X, N) of a V-valued process $X = (X_k)_{k\in\mathbb{Z}}$ and an iid V^V-valued process $N = (N_k)_{k\in\mathbb{Z}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the following hold for each $k \in \mathbb{Z}$:
(i) $X_k = N_k X_{k-1}$ holds a.s.;
(ii) N_k is independent of $\mathcal{F}_{k-1}^{X,N} := \sigma(X_j, N_j : j \le k-1)$;
(iii) N_k has law μ.

It is easy to see that (X, N) is a μ-evolution if and only if the Markov property

$$\mathbb{P}\Big((X_k, N_k) \in B \mid \mathcal{F}_{k-1}^{X,N}\Big) = Q_\mu\Big(X_{k-1}; B\Big), \quad k \in \mathbb{Z},\ B \subset V \times V^V \tag{1.3}$$

holds with the joint transition probability:

$$Q_\mu\Big(x; B\Big) = \mu\Big\{f : (fx, f) \in B\Big\}, \quad x \in V,\ B \subset V \times V^V, \tag{1.4}$$

i.e., $Q_\mu(x; \cdot)$ is the image measure of μ by the map $f \mapsto (fx, f)$. If (X, N) is a μ-evolution, then the marginal process X satisfies the Markov property

$$\mathbb{P}\big(X_k \in A \mid \mathcal{F}_{k-1}^X\big) = P_\mu(X_{k-1}; A), \quad k \in \mathbb{Z},\ A \subset V \tag{1.5}$$

with the marginal transition probability:

$$P_\mu(x; A) = \mu\{f : fx \in A\}, \quad A \subset V, \tag{1.6}$$

i.e., $P_\mu(x; \cdot)$ is the image measure of μ by the map $f \mapsto fx$.

It is easy to see by definition that, if two μ-evolutions (X, N) and (X', N') satisfy $X_k \overset{\mathrm{d}}{=} X'_k$ for $k \in \mathbb{Z}$, then $(X, N) \overset{\mathrm{d}}{=} (X', N')$. This shows that the joint law of the μ-evolution (X, N) is determined by the family of marginal laws $\{\mathbb{P}(X_k \in \cdot) : k \in \mathbb{Z}\}$.

Unfortunately, it seems difficult to develop a complete investigation of the structure of the observation noise for a mono-particle μ-evolution. So we introduce a multi-particle counterpart. For a mapping $f : V \to V$ and a vector $\boldsymbol{x} = (x^1, \ldots, x^m) \in V^m$, we understand that f operates $\boldsymbol{x}$ componentwise, i.e., $f\boldsymbol{x} = (fx^1, \ldots, fx^m)$.

Definition 1.2. Let μ be a mapping law on V^V. An *m-particle μ-evolution* is a μ-evolution $(\mathbb{X}, N)$ with $\mathbb{X} = (\mathbb{X}_k)_{k\in\mathbb{Z}}$ taking values in V^m; precisely, the following hold for each $k \in \mathbb{Z}$:
(i) $\mathbb{X}_k = N_k\mathbb{X}_{k-1}$ holds a.s., i.e., $X^i_k = N_k X^i_{k-1}$ holds a.s. for $i = 1, \ldots, m$;
(ii) N_k is independent of $\mathcal{F}^{\mathbb{X},N}_{k-1} := \sigma(\mathbb{X}_j, N_j : j \le k-1)$;
(iii) N_k has law μ.

We will see in Proposition 5.1 that the number of distinct states among $\{X^1_k, \ldots, X^m_k\}$ does not depend upon $k \in \mathbb{Z}$ a.s.

We now propose our resolution problems.

Definition 1.3. For an m-particle μ-evolution $(\mathbb{X}, N)$, a *third noise* is a sequence of random variables $(U_k)_{k\in\mathbb{Z}}$ such that the following hold for each $k \in \mathbb{Z}$:
(i) the identity $\mathcal{F}^{\mathbb{X},N}_k = \mathcal{F}^N_k \vee \mathcal{F}^{\mathbb{X}}_{-\infty} \vee \sigma(U_k)$ holds a.s.;
(ii) the three σ-fields $\mathcal{F}^N_k$, $\mathcal{F}^{\mathbb{X}}_{-\infty}$ and $\sigma(U_k)$ are independent.
The identity in Condition (i) will be called the *resolution of the full noise.*

We remark that the third noise is not an innovation. Note that, if $(U_k)_{k\in\mathbb{Z}}$ is a third noise, then, for any $k_0 \in \mathbb{Z}$, the stopped sequence $(U_{k\wedge k_0})_{k\in\mathbb{Z}}$ is also a third noise, since $X_k = N_k N_{k-1} \cdots N_{k_0+1} X_{k_0}$ for $k > k_0$. We may suggest that the third noise emerges in the remote past and does not increase as time passes.

Provided that we can find a random variable ξ such that $\mathcal{F}^{\mathbb{X}}_{-\infty} = \sigma(\xi)$, then, for every $k \in \mathbb{Z}$, the identity $\mathcal{F}^{\mathbb{X},N}_k = \mathcal{F}^N_k \vee \mathcal{F}^{\mathbb{X}}_{-\infty} \vee \sigma(U_k)$ holds if and only if there exist measurable mappings F and G such that $(\mathbb{X}_k, \mathbb{X}_{k-1}, \ldots) = F(N_k, N_{k-1}, \ldots, \xi, U_k)$ and $U_k = G(\mathbb{X}_k, \mathbb{X}_{k-1}, \ldots, N_k, N_{k-1}, \ldots)$.

We also propose a finer resolution.

Definition 1.4. For an m-particle μ-evolution, a *reduced driving noise* is a sequence of σ-fields $(\mathcal{G}^N_k)_{k\in\mathbb{Z}}$ accompanying with a sequence of random variables $(U_k)_{k\in\mathbb{Z}}$ such that the following hold for each $k \in \mathbb{Z}$:
(i) the identity $\mathcal{F}^{\mathbb{X}}_k = \mathcal{G}^N_k \vee \mathcal{F}^{\mathbb{X}}_{-\infty} \vee \sigma(U_k)$ holds a.s.;
(ii) $\mathcal{G}^N_k \subset \mathcal{F}^N_k$ holds a.s.;
(iii) the three σ-fields $\mathcal{F}^N_k$, $\mathcal{F}^{\mathbb{X}}_{-\infty}$ and $\sigma(U_k)$ are independent.
The identity in Condition (i) will be called the *resolution of the observation.*

For every $k \in \mathbb{Z}$, provided that we can find random variables ξ and $\zeta_k, \zeta_{k-1}, \ldots$ such that $\mathcal{F}^{\mathbb{X}}_{-\infty} = \sigma(\xi)$ and $\mathcal{G}^N_k = \sigma(\zeta_k, \zeta_{k-1}, \ldots)$, then the identity $\mathcal{F}^{\mathbb{X}}_k = \mathcal{G}^N_k \vee \mathcal{F}^{\mathbb{X}}_{-\infty} \vee \sigma(U_k)$ holds if and only if there exist measurable mappings F and G such that $(\mathbb{X}_k, \mathbb{X}_{k-1}, \ldots) = F(\zeta_k, \zeta_{k-1}, \ldots, \xi, U_k)$ and $U_k = G(\mathbb{X}_k, \mathbb{X}_{k-1}, \ldots)$.

We remark that, if we have the resolution of the observation in the sense of Definition 1.4, then we also have the resolution of the full noise in the sense of Definition 1.3, and the sequence $(U_k)_{k\in\mathbb{Z}}$ in Definition 1.4 becomes a third noise in the sense of Definition 1.3.

In this paper, we shall give a general result of resolution of the observation for multi-particle action evolutions when the state space V is a finite set.

1.2 Infinite convolutions on finite semigroups

For our purpose we need several known facts from the theory of semigroups, which we recall without proofs. We may consult a celebrated textbook [6] and a concise expository note [13] for the details.

In what follows we assume S be a finite semigroup and we denote the set of all idempotents in S by $E(S) = \{f \in S : f^2 = f\}$. For $A, B \subset S$ and $f \in S$, we write $AB = \{ab : a \in A,\ b \in B\}$ and $Af = \{af : a \in A\}$, etc. We say that S is *completely simple* if S has no proper ideal, i.e., $\emptyset \neq IS \cup SI \subset I \subset S$ implies $I = S$, and if there exists $e \in E(S)$ which is *primitive*, i.e., $ef = fe = f \in E(S)$ implies $f = e$.

Proposition 1.5 (Rees decomposition). *Suppose S be a completely simple finite semigroup with a primitive idempotent e. Set $L = E(Se)$, $G = eSe$ and $R = E(eS)$. Then the following hold:*

(i) G is a group whose identity is e.

(ii) $eL = Re = \{e\}$.

(iii) $S = LGR$.

(iv) The product mapping $\psi : L \times G \times R \ni (f, g, h) \mapsto fgh \in S$ is bijective and its inverse is given as

$$\psi^{-1}(z)\big(=: (z^L, z^G, z^R)\big) = (ze(eze)^{-1}, eze, (eze)^{-1}ez). \tag{1.7}$$

The proof of Proposition 1.5 can be found, e.g., in [6, Theorem 1.1]. The product decomposition $S = LGR$ will be called the *Rees decomposition* of S at e, and G will be called the *group factor*. Note that the Rees decomposition depends upon the choice of a primitive idempotent; this is why we clarify the choice of the primitive idempotent by saying "at e" whenever we call the Rees decomposition.

Note by definition that $RL \subset eSSe \subset eSe = G$ and by the product bijectivity that $\psi^{-1}((fgh)(f'g'h')) = (f, ghf'g', h')$. It is obvious that the product $z = fgh \in S$ is idempotent if and only if $hf = g^{-1}$. We notice that *all idempotents of S are primitive*; in fact, if $e' = f'g'h' \in E(S)$ and $z = fgh \in S$ satisfies $e'z = ze' = z \in E(S)$, then we have $f' = f$ and $h' = h$ by the product bijectivity and thus we have $g' = (h'f')^{-1} = (hf)^{-1} = g$, which shows $e' = z$ so that e' is also primitive.

A subset K of S is called a *kernel* of S if it is a minimal ideal of S, i.e., K is an ideal of S and contains no proper ideal of S. Note that a kernel K of S is then automatically the least ideal of S; indeed, for any ideal I of S, we see that $K \cap I$ is an ideal of S containing KI and is contained in K, which implies $K \subset K \cap I \subset I$.

Proposition 1.6. *A finite semigroup S always contains a unique kernel K. In addition, the kernel K is completely simple. Fix $e \in E(K)$ and let $K = LGR$ denote the Rees decomposition of K at e. Then*

$$Se = Ke = LG, \quad eSe = eKe = G, \quad eS = eK = GR. \tag{1.8}$$

Proof. The proof of unique existence and complete simplicity of the kernel can be found, e.g., in [6, Proposition 1.7]. Let us prove (1.8). Since

$$Se = See \subset SKe \subset Ke = LGRe = LGe \subset Se, \tag{1.9}$$

we obtain $Se = Ke = LGe = LG$. By a similar argument we have $eS = GR$. Thus we obtain $eSe = eLG = G$. $\square$

Proposition 1.5 is fundamental in the theory of infinite convolutions. Let $\mathcal{P}(S)$ denote the set of probability measures on a finite semigroup S and write $\mu * \nu$ for the convolution of μ and ν in $\mathcal{P}(S)$:

$$(\mu * \nu)(A) = \sum_{f,g \in S} 1_A(fg)\mu\{f\}\nu\{g\}, \quad A \subset S. \tag{1.10}$$

We write μ^n for the n-fold convolution of μ: $\mu^1 = \mu$ and $\mu^n = \mu * \mu^{n-1}$ for $n = 2, 3, \dots$ We write $\mathcal{S}(\mu) = \{f \in S : \mu\{f\} > 0\}$ for the support of μ. It is easy to see that $\mathcal{S}(\mu * \nu) = \mathcal{S}(\mu)\mathcal{S}(\nu)$ for $\mu, \nu \in \mathcal{P}(S)$. We write ω_A for the uniform distribution on a finite set A; if G is a finite group, then ω_G is the normalized Haar measure of G.

Proposition 1.7 (Convolution idempotents). *Suppose that $\nu^2 = \nu \in \mathcal{P}(S)$. Then $\mathcal{S}(\nu)$ is a completely simple subsemigroup of S. Fix $e \in E(\mathcal{S}(\nu))$ and take $L = E(\mathcal{S}(\nu)e)$, $G = e\mathcal{S}(\nu)e$ and $R = E(e\mathcal{S}(\nu))$ so that $\mathcal{S}(\nu) = LGR$ gives the Rees decomposition of $\mathcal{S}(\nu)$ at e. Write $\nu^L(\cdot) = \nu\{z \in \mathcal{S}(\nu) : z^L \in \cdot\}$ and $\nu^R(\cdot) = \nu\{z : z^R \in \cdot\}$, i.e., ν^L and ν^R are the image measures of ν by the maps $z \mapsto z^L$ and $z \mapsto z^R$, respectively. Then ν has a factorization*

$$\nu = \nu^L * \omega_G * \nu^R. \tag{1.11}$$

Consequently, if Z is a random variable whose law is ν, then the projections Z^L, Z^G and Z^R are independent and Z^G is uniform on G.

The proof of Proposition 1.7 can be found, e.g., in [6, Theorem 2.2].

The following proposition plays a key role in our analysis.

Proposition 1.8 (Infinite convolutions). *Let $\mu \in \mathcal{P}(S)$ and suppose that S coincide with $\bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^n$, the semigroup generated by $\mathcal{S}(\mu)$. Then the following hold:*

(i) The set of subsequential limits of $\{\mu^n\}$ is a finite cyclic group of the form

$$\mathcal{K} := \{\eta, \mu * \eta, \dots, \mu^{p-1} * \eta\} \tag{1.12}$$

for some $p \in \mathbb{N}$, where η is the identity of $\mathcal{K}$ (so that $\eta^2 = \eta$), $\mu^p * \eta = \eta$ and $\eta, \mu * \eta, \dots, \mu^{p-1} * \eta$ are all different (and consequently $\mathcal{K}$ is a cyclic group with order p). The support $\mathcal{S}(\eta)$ is a completely simple subsemigroup of S (but not in general an ideal of S.)

(ii) It holds that

$$\frac{1}{n}\sum_{k=1}^{n}\mu^k \underset{n\to\infty}{\longrightarrow} \nu := \frac{1}{p}\sum_{k=0}^{p-1}\mu^k * \eta, \tag{1.13}$$

so that $\nu^2 = \nu$. The support $\mathcal{S}(\nu)$ is the kernel of S.

(iii) Let $e \in E(\mathcal{S}(\eta))$ be fixed. Then the Rees decompositions at e of $\mathcal{S}(\nu)$ and of $\mathcal{S}(\eta)$ are given as

$$\mathcal{S}(\nu) = LGR \qquad \textit{and} \quad \mathcal{S}(\eta) = LHR, \tag{1.14}$$

respectively, where $L = E(\mathcal{S}(\eta)e)$, $R = E(e\mathcal{S}(\eta))$, $G = e\mathcal{S}(\nu)e$ and $H = e\mathcal{S}(\eta)e$. Moreover, the group factor H of $\mathcal{S}(\eta)$ is a normal subgroup of the group factor G of $\mathcal{S}(\nu)$, and the convolution factorizations of ν and η are given as

$$\nu = \eta^L * \omega_G * \eta^R \qquad \textit{and} \quad \eta = \eta^L * \omega_H * \eta^R, \tag{1.15}$$

respectively, where $\eta^L(\cdot) = \eta\{z : z^L \in \cdot\}$ and $\eta^R(\cdot) = \eta\{z : z^R \in \cdot\}$, i.e., η^L and η^R are the image measures of η by the maps $z \mapsto z^L$ and $z \mapsto z^R$, respectively.

(iv) There exists $\gamma \in G$ such that $G/H = \{H, \gamma H, \ldots, \gamma^{p-1}H\}$ with $\gamma^p \in H$ and with $H, \gamma H, \ldots, \gamma^{p-1}H$ are all different (and consequently G/H is a cyclic group with order p and generated by γH). Moreover,

$$\mu^r * \eta = \eta^L * \delta_{\gamma^r} * \omega_H * \eta^R, \quad r = 0, 1, \ldots, p-1, \tag{1.16}$$

where δ_a stands for the Dirac mass at a.

The proof of Proposition 1.8 can be found in [13]; see also [6, Theorem 2.7].

Remark 1.9. By (i) and (iv) of Proposition 1.8, we see that $\mu^{pn} \to \eta$, $\mu^{pn+1} \to \mu * \eta, \ldots$, $\mu^{pn+p-1} \to \mu^{p-1} * \eta$. To see this fact, it suffices to prove $\mu^{pn} \to \eta$. Since $\mathcal{P}(S)$ is compact, it suffices to prove that η is the unique cluster point of $\{\mu^{pn}\}$. Suppose that a subsequence $\mu^{pn(k)}$ converge to an element of $\mathcal{K}$, say $\mu^r * \eta$ for $r = 0, 1, \ldots, p-1$. Since $\mu^{pn(k)} * \eta = \eta$, we have $\mu^r * \eta * \eta = \eta$, which implies $r = 0$. We thus obtain $\mu^{pn(k)} \to \eta$.

Remark 1.10. It may be useful to notice some connection between Proposition 1.8 and random walks generated by μ. By [6, Proposition 3.6] and by [6, Corollary 3.1, Theorem 3.2 and Proposition 3.4], we have the following facts:

- The set $\mathcal{S}(\nu)$, which is the kernel of S, is also the set of all recurrent elements for the unilateral, the bilateral or the mixed random walk generated by μ.
- For any $z \in \mathcal{S}(\nu)$, the greatest common divisor of $\{n - m : \mu^n\{z\} > 0,\ \mu^m\{z\} > 0\}$ coincides with p.

1.3 The semigroup consisting of mappings

Let V be a non-empty finite set and let V^V denote the set of mappings of V into itself. Note that V^V is also a finite semigroup with respect to composition as its product structure. In this concrete settings we recall several facts about the description of the kernel and the Rees decomposition (see e.g. [6, Example 1.1 and Proposition 1.8]). For $f \in V^V$, we write $\pi(f) := \{f^{-1}\{v\} : v \in V\}$ for the partition of S generated by the preimages of f.

Proposition 1.11. *Let S be a subsemigroup of V^V and denote*

$$m_S = \min\{\#(fV) : f \in S\}, \tag{1.17}$$

where $\#(A)$ denotes the number of elements of a set A.

(i) The kernel K of S is the set of all mappings in S with minimal rank, namely:

$$K = \{f \in S : \#(fV) = m_S\}. \tag{1.18}$$

(ii) e is an idempotent if and only if it is identity on eV.

(iii) e is a primitive idempotent of S if and only if $e \in E(K)$.

For a fixed $e \in E(K)$, the Rees decomposition $K = LGR$ at e may be characterized as follows:

(iv) Se is the set of all f in S such that $\pi(f) = \pi(e)$. Consequently, $L = E(Se)$ is the set of all idempotents f in S such that $\pi(f) = \pi(e)$.

(v) eS is the set of all f in S such that $fV = eV$. Consequently, $R = E(eS)$ is the set of all idempotents f in S such that $fV = eV$.

(vi) G is the set of all $f \in S$ such that $\pi(f) = \pi(e)$ and $fV = eV$.

For convenience of the readers, the proof of Proposition 1.11 will be given in the Appendix.

1.4 Main result

Let V be a non-empty finite set and let V^V denote the set of mappings of V into itself. For $f \in V^V$ and $\boldsymbol{x} = (x^1, \dots, x^m) \in V^m$, we understand $f\boldsymbol{x} = (fx^1, \dots, fx^m)$. For $\mu \in \mathcal{P}(V^V)$ and $\Lambda \in \mathcal{P}(V^m)$, we define $\mu * \Lambda \in \mathcal{P}(V^m)$ as

$$(\mu * \Lambda)(A) = \sum_{f \in V^V} \sum_{\boldsymbol{x} \in V^m} 1_A(f\boldsymbol{x})\mu\{f\}\Lambda\{\boldsymbol{x}\}, \quad A \subset V^m. \tag{1.19}$$

Note that, for independent random variables F and $\mathbb{X}$ whose laws are μ and Λ, respectively, the law of $F\mathbb{X}$ is $\mu * \Lambda$. Denote

$$V^m_\times = \{\boldsymbol{x} = (x^1, \dots, x^m) \in V^m : x^1, \dots, x^m \text{ are distinct}\}. \tag{1.20}$$

Proposition 1.12. *Let $\mu \in \mathcal{P}(V^V)$ and set $S = \bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^n$, the semigroup generated by $\mathcal{S}(\mu)$. We apply Proposition 1.8 and adopt its notations. Denote*

$$m_\mu = m_S = \min\{\#(fV) : f \in S\} \tag{1.21}$$

and define

$$W_\mu = \{\boldsymbol{x} \in V_\times^{m_\mu} : f\boldsymbol{x} \in V_\times^{m_\mu} \text{ for all } f \in S\}. \tag{1.22}$$

(Note that W_μ is not empty; in fact, if we write $eV = \{x_1, \ldots, x_{m_\mu}\}$, then $(x_1, \ldots, x_{m_\mu}) \in W_\mu$.) Let W be an arbitrary minimal subset of eW_μ such that $eW_\mu = GW$. Then the following hold:

(i) $W_\mu = LGW$.

(ii) *The product mapping $L \times G \times W \ni (f, g, \boldsymbol{w}) \mapsto fg\boldsymbol{w} \in W_\mu$ is bijective. Its inverse will be denoted by $\boldsymbol{x} \mapsto (\boldsymbol{x}^L, \boldsymbol{x}^G, \boldsymbol{x}^W)$.*

(iii) *Let $\Lambda \in \mathcal{P}(V_\times^{m_\mu})$. Then Λ is μ-*invariant*, i.e., $\Lambda = \mu * \Lambda$, if and only if $\Lambda = \eta^L * \omega_G * \Lambda_W$ for some $\Lambda_W \in \mathcal{P}(W)$.*

The proof of Proposition 1.12 will be given in Section 3.

If an m-particle μ-evolution $(\mathbb{X}, N)$ is *stationary*, i.e., $(\mathbb{X}_{\cdot+1}, N_{\cdot+1}) \overset{\mathrm{d}}{=} (\mathbb{X}, N)$, then the sequence $\mathbb{X}$ has a common law which is μ-invariant. Conversely, if $\Lambda \in \mathcal{P}(V^m)$ is μ-invariant, then there exists a stationary m-particle μ-evolution $(\mathbb{X}, N)$ such that the sequence $\mathbb{X}$ has Λ as its common law.

In Proposition 1.8, we write $C = \{e, \gamma, \ldots, \gamma^{p-1}\}$ so that $CH = \bigcup_{c \in C} cH = G$. By definition of C and H, we see that the product mapping $C \times H \ni (c, h) \mapsto ch \in G$ is bijective. Its inverse will be denoted by $g \mapsto (g^C, g^H)$.

We now state our main theorem, which will be proved in Section 4.

Theorem 1.13. *Suppose the same assumptions of Proposition 1.12 be satisfied. Suppose that $\Lambda \in \mathcal{P}(V_\times^{m_\mu})$ be μ-invariant and let $(\mathbb{X}, N)$ be a stationary m_μ-particle μ-evolution such that the sequence $\mathbb{X}$ has Λ as its common law. Then the following hold:*

(i) *For any fixed $k \in \mathbb{Z}$, it holds that $\mathbb{X}_k \in W_\mu = LGW$ a.s., $\mathbb{X}_k^L \overset{\mathrm{d}}{=} \eta^L$, $\mathbb{X}_k^G \overset{\mathrm{d}}{=} \omega_G$, $\mathbb{X}_k^W \overset{\mathrm{d}}{=} \Lambda_W$ and the three random variables $\mathbb{X}_k^L$, $\mathbb{X}_k^G$ and $\mathbb{X}_k^W$ are independent. (Note that the two processes $\mathbb{X}^L$ and $\mathbb{X}^G$ are* not *independent in general; see* (2.22)*.)*

(ii) *$\mathbb{X}_k^G = (\gamma^k Y_C)^C U_k$ a.s. for $k \in \mathbb{Z}$ for some C-valued random variable Y_C and some H-valued random variables U_k such that U_k is uniform on H.*

(iii) *$\mathbb{X}_k^W = \mathbb{Z}_W$ a.s. for $k \in \mathbb{Z}$ for some W-valued random variable $\mathbb{Z}_W$.*

(iv) *If we write* $M_j^G := \mathbb{X}_j^G(\mathbb{X}_{j-1}^G)^{-1}$ *for* $j \in \mathbb{Z}$ *and* $M_{k,j}^G := \mathbb{X}_k^G(\mathbb{X}_j^G)^{-1} = M_k^G M_{k-1}^G \cdots M_{j+1}^G$ *for* $j \le k$, *we have the following factorization:*

$$\mathbb{X}_j = \mathbb{X}_j^L (M_{k,j}^G)^{-1} (\gamma^k Y_C)^C U_k \mathbb{Z}_W \quad a.s.\ for\ j \le k. \tag{1.23}$$

(v) *A resolution of the observation holds in the sense that*

$$\mathcal{F}_k^{\mathbb{X}} = \mathcal{G}_k^N \vee \mathcal{F}_{-\infty}^{\mathbb{X}} \vee \sigma(U_k) \quad a.s., \tag{1.24}$$

where

$$\mathcal{G}_k^N = \sigma\big(\mathbb{X}_j^L,\ M_j^G : j \le k\big) \subset \mathcal{F}_k^N (\subset \sigma(N)) \quad a.s., \tag{1.25}$$

$$\text{the three } \sigma\text{-fields } \sigma(N),\ \mathcal{F}_{-\infty}^{\mathbb{X}} \text{ and } \sigma(U_k) \text{ are independent} \tag{1.26}$$

and

$$\mathcal{F}_{-\infty}^{\mathbb{X}} = \sigma(Y_C,\ \mathbb{Z}_W) \quad a.s. \tag{1.27}$$

(vi) $Y_C \overset{\mathrm{d}}{=} \omega_C$ *and* $\mathbb{Z}_W \overset{\mathrm{d}}{=} \Lambda_W$, *where* ω_C *denotes the uniform distribution on the set* C. *It holds that* Y_C *and* $\mathbb{Z}_W$ *are independent.*

We shall show in Section 5 that the non-stationary case can be reduced to the stationary case and satisfies Properties (i)-(v) of Theorem 1.13.

Note that, if we represent $Y_C = \gamma^{R_C}$ with $R_C \in \{0, 1, \ldots, p-1\}$, then

$$(\gamma^k Y_C)^C = (\gamma^{k+R_C})^C = \gamma^{(k+R_C) \bmod p} \quad \text{for } k \in \mathbb{Z}, \tag{1.28}$$

since $\gamma^p \in H$.

The following corollary to Proposition 1.12 ensures that a stationary mono-particle μ-evolution can always be extended to a stationary m_μ-particle μ-evolution.

Corollary 1.14. *Let* $\lambda \in \mathcal{P}(V)$ *and suppose that* λ *be* μ*-invariant, i.e.,* $\lambda = \mu * \lambda$. *Then there exists* $\Lambda \in \mathcal{P}(V_\times^{m_\mu})$ *such that* Λ *is* μ*-invariant and its marginal in the first coordinate equals to* λ, *i.e.,*

$$\Lambda\{(x^1, \ldots, x^{m_\mu}) \in V_\times^{m_\mu} : x^1 = v\} = \lambda\{v\}, \quad v \in V. \tag{1.29}$$

The proof of Corollary 1.14 will be given in Section 3. Unfortunately, the resolution of the observation obtained in Theorem 1.13 for an m_μ-paricle μ-evolution does not imply that for a mono-paricle μ-evolution, which will be illustrated in Section 2.

1.5 Historical remarks

The theories of Rees decomposition, convolution idempotents and infinite convolutions for finite semigroups are very old results and have nowadays been generalized to topological semigroups; see the textbook [6, Chapters 1 and 2] for the details. In particular, Proposition 1.8, which plays a fundamental tool for our results, dates back to Rosenblatt [8], Collins [4] and Schwarz [9].

Inspired by Tsirelson [3] of a stochastic differential equation, Yor [15] has made a thorough study of the action evolution $X_k = N_k X_{k-1}$ when both X and N take values in the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and N is not necessarily iid, where we understand $N_k X_{k-1}$ as the usual product between two complex values. He obtained a general result of the resolution of the observation. Hirayama and Yano [5] generalized Yor's results for the state space being a compact group. In these results the third noise is generated by a random variable with uniform law on a subgroup of the state space group. See also [14] for a survey of this topic.

If $m_\mu = 1$, then it is obvious that a stationary mono-particle μ-evolution (X, N) satisfies $\mathcal{F}^X_k \subset \mathcal{F}^N_k$ a.s. for all k. Yano [12] proved its converse: if a stationary mono-particle μ-evolution (X, N) satisfies $\mathcal{F}^X_k \subset \mathcal{F}^N_k$ a.s. for all k, then $m_\mu = 1$. For the proof, he proved existence of a non-trivial third noise when $m_\mu \geq 2$. He utilized several notions from the *road coloring theory*; for the details see Trahtman [10] and the references therein.

Brossard–Leuridan [1] have studied Markov chains indexed by $\mathbb{Z}$, which can be regarded as μ-evolutions on general state spaces. Let us pick up [1, Theorem 3], whose conclusion applied to our μ-evolutions is as follows: *For a stationary m_μ-particle μ-evolution* $(\mathbb{X}, N)$ *such that* $\mathbb{X}$ *has* Λ *as its common law, the conditional law of* $\mathbb{X}$ *given* $\sigma(N) \vee \mathcal{F}^{\mathbb{X}}_{-\infty}$ *is uniform on some random finite set.* Let us characterize the random finite set. By Theorem 1.13, we see that the conditional law $\mathbb{P}(\mathbb{X} \in \cdot \mid \sigma(N) \vee \mathcal{F}^{\mathbb{X}}_{-\infty})$ is uniform on the random finite set $\{\mathbb{X}^h : h \in H\} \subset (V^{m_\mu})^{\mathbb{Z}}$, where the process $\mathbb{X}^h = (\mathbb{X}^h_j)_{j \in \mathbb{Z}}$ is defined as

$$\mathbb{X}^h_j = \begin{cases} \mathbb{X}^L_j (M^G_{0,j})^{-1} Y_C h \mathbb{Z}_W & (j \leq 0) \\ N_j N_{j-1} \cdots N_1 \mathbb{X}^L_0 Y_C h \mathbb{Z}_W & (j \geq 1). \end{cases} \tag{1.30}$$

In particular, the conditional law $\mathbb{P}(\mathbb{X}_0 \in \cdot \mid \sigma(N) \vee \mathcal{F}^{\mathbb{X}}_{-\infty})$ is uniform on the random finite set $\{\mathbb{X}^L_0 Y_C h \mathbb{Z}_W : h \in H\} \subset V^{m_\mu}$. In Remark 4.3, we will discuss this conditional law as a special case of [1, Proposition 10 and Theorem 11].

1.6 Organization

The organization of this paper is as follows. In Section 2 we discuss an example. In Section 3 we prove Proposition 1.12 and discuss characterization of stationary probabilities. Section 4 is devoted to the proof of our main theorem, Theorem 1.13. In Section 5 we discuss the non-stationary case. In Section 6, as an appendix, we give the proofs for basic facts about the semigroup consisting of mappings. Finally, in Section 7, we discuss another example where the infinite convolution has at least two cluster points.

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2 Example

Let us investigate an example which was discussed in [12, Subsection 3.3] for mono-particle μ-evolution. We look at it from the viewpoint of multiparticle μ-evolution. See [7] for other examples.

Let $V = \{1, 2, 3, 4, 5\}$. We write $f = [y^1, y^2, y^3, y^4, y^5]$ if $f : V \to V$ is such that $f1 = y^1, \ldots, f5 = y^5$. Consider the two mappings

$$f = [2, 3, 4, 1, 5] \quad \text{and} \quad g = [2, 5, 5, 2, 4]. \tag{2.1}$$

Let $\mu = (\delta_f + \delta_g)/2$ be the uniform law on $\{f, g\}$, where δ_f stands for the Dirac mass at f. The marginal transition probability P_μ of (1.6) is given as

$$\begin{pmatrix} P_\mu(1,\{1\}) & P_\mu(1,\{2\}) & \cdots & P_\mu(1,\{5\}) \\ P_\mu(2,\{1\}) & P_\mu(2,\{2\}) & \cdots & P_\mu(2,\{5\}) \\ \vdots & \vdots & \ddots & \vdots \\ P_\mu(5,\{1\}) & P_\mu(5,\{2\}) & \cdots & P_\mu(5,\{5\}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{2.2}$$

Since the fifth power P_μ^5 has all positive entries, we see that P_μ is irreducible aperiodic. It is obvious that $\mu * \lambda = \lambda$ if and only if $\lambda P_\mu = \lambda$, and it is easy to see that there exists a unique μ-invariant probability measure given as

$$\lambda = \frac{1}{9}\delta_1 + \frac{2}{9}\delta_2 + \frac{1}{9}\delta_3 + \frac{2}{9}\delta_4 + \frac{3}{9}\delta_5. \tag{2.3}$$

In [12, Theorem 1], for a stationary mono-particle μ-evolution (X, N) with X having λ as its common law, it was proved that there exists a third noise $(U_k)_{k\in\mathbb{Z}}$ such that $\sigma(U_k) \subset \mathcal{F}_k^{X,N}$ a.s. for $k \in \mathbb{Z}$ and

$$\mathcal{F}_k^X \subset \mathcal{F}_k^N \vee \sigma(U_k) \quad \text{a.s. for } k \in \mathbb{Z} \tag{2.4}$$

with $\mathcal{F}_{-\infty}^X$ being trivial a.s. and $\sigma(U_k)$ being independent of $\mathcal{F}_k^N$.

Set $S = \bigcup_{n=1}^\infty \{f, g\}^n$ and we would like to apply Propositions 1.8, 1.11 and 1.12. Set

$$e := g^3 = [4, 2, 2, 4, 5]. \tag{2.5}$$

Note that e is an idempotent and that

$$efe = e, \quad eg = ge = g, \quad g^2 = [5, 4, 4, 5, 2], \quad ef^2 = f^2e = [2, 4, 4, 2, 5]. \tag{2.6}$$

Since $f^4 = \mathrm{id}_V$, the identity of V, we see that f is bijective. Note also that

$$gf^2 = f^2g^2 = [4, 5, 5, 4, 2], \quad g^2f^2 = f^2g = [5, 2, 2, 5, 4]. \tag{2.7}$$

If we write $A = \{1, 3, 5\}$ and $B = \{2, 4, 5\}$, then $fB = A$, $fA = B$ and $gV = gA = gB = A$, and hence we see that the minimum rank over all mappings of S is given as $m_\mu = 3$. We now see that the kernel of S is given as $K = \{h \in S : \#(hV) = 3\}$. Since K is an ideal containing g, we see that

$$S = \{\mathrm{id}_V, f, f^2, f^3\} \cup K. \tag{2.8}$$

Since $e \in E(K)$, we see that e is a primitive idempotent. Let $K = LGR$ denote the Rees decomposition of K at e.

Let us prove that

$$L = \{e, fe\}, \quad G = \{e, g, g^2, ef^2, gf^2, g^2f^2\}, \quad R = \{e, ef\}, \tag{2.9}$$

where

$$fe = [1, 3, 3, 1, 5], \quad ef = [2, 2, 4, 4, 5]. \tag{2.10}$$

We have already seen that $hV = A$ or B for all $h \in K$. Let $\pi_1 = \pi(e) = \{\{1, 4\}, \{2, 3\}, \{5\}\}$ and $\pi_2 = \pi(ef) = \{\{1, 2\}, \{3, 4\}, \{5\}\}$. Note that, for every $s \in S$ and $k = 0, 1, 2, 3$, the partition $\pi(sgf^k)$ is finer than the partition $\pi(gf^k)$ and hence equal since $\#(sgf^kV) = 3 = \#(gf^kV)$. We then see that $\pi(h) = \pi_1$ or π_2 for all $h \in K$; in fact, K is contained in $Sg \cup Sgf \cup Sgf^2 \cup Sgf^3$, $\pi(g) = \pi(gf^2) = \pi_1$ and $\pi(gf) = \pi(gf^3) = \pi_2$.

Since G is the set of all mappings h of S such that $\pi(h) = \pi(e) = A$ and $hV = eV = \pi_1$, it contains the six elements $e, g, g^2, ef^2, gf^2, g^2f^2$. On the other hand, each element of G induces a permutation on A and is determined by it, so that G has at most six elements, which shows $G = \{e, g, g^2, ef^2, gf^2, g^2f^2\}$. Note that $E(K)$ contains the four elements e, fe, ef, fef^3, where $fef^3 = (fe)(ef^2)(ef) = [1, 1, 3, 3, 5]$. On the other hand, each element h of $E(K)$ is determined by $hV = A$ or B and $\pi(h) = \pi_1$ or π_2, so that $E(K)$ has at most four elements, which shows $E(K) = \{e, fe, ef, fef^3\}$. We now see that $L = \{h \in E(K) : \pi(h) = \pi(e)\} = \{e, fe\}$ and $R = \{h \in E(K) : hV = eV\} = \{e, ef\}$. Therefore we obtain (2.9).

Let $H = e\mathcal{S}(\eta)e$ be the subgroup of G in Proposition 1.8. Then we have $\mu * \eta^L * \omega_H * \eta^R = \eta^L * \delta_\gamma * \omega_H * \eta^R$ so that $\mu * \eta^L * \omega_H = \eta^L * \delta_\gamma * \omega_H$, since $\eta^R * \delta_e = \delta_e$. Let $\eta^L = \alpha\delta_e + \beta\delta_{fe}$ for some $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Since $gfe = gefe = ge = g$ and $f^2e = ef^2$, we have

$$\mu * \eta^L = \left(\frac{1}{2}\delta_f + \frac{1}{2}\delta_g\right) * (\alpha\delta_e + \beta\delta_{fe}) = \frac{\alpha}{2}\delta_{fe} + \frac{1}{2}\delta_g + \frac{\beta}{2}\delta_{ef^2}. \tag{2.11}$$

Since $fe = (fe)e \in LH$ and $fe = fee \in \mathcal{S}(\mu * \eta^L * \omega_H) = \mathcal{S}(\eta^L * \delta_\gamma * \omega_H) = L\gamma H$, we have $H \cap \gamma H \neq \emptyset$, which shows $H = G$ and we may take $\gamma = e$. We now have

$$(\alpha\delta_e + \beta\delta_{fe}) * \omega_G = \eta^L * \omega_G = \eta^L * \delta_\gamma * \omega_G \tag{2.12}$$

$$= \mu * \eta^L * \omega_G = \left(\frac{\alpha}{2}\delta_{fe} + \frac{1+\beta}{2}\delta_e\right) * \omega_G, \tag{2.13}$$

since $\delta_e * \omega_G = \delta_g * \omega_G = \delta_{ef^2} * \omega_G = \omega_G$. Thus $\alpha = 2/3$ and $\beta = 1/3$, that is,

$$\eta^L = \frac{2}{3}\delta_e + \frac{1}{3}\delta_{fe}. \tag{2.14}$$

In the same way we have $\eta^R = \frac{2}{3}\delta_e + \frac{1}{3}\delta_{ef}$, and thus we have obtained that

$$\mu^n \to \eta = \nu = \eta^L * \omega_G * \eta^R. \tag{2.15}$$

Note that $fe = [1, 3, 3, 1, 5]$ and $ef = [2, 2, 4, 4, 5]$. For $(a, b, c) \in L \times G \times R$, we have

$$\begin{cases} a = e & \iff & aV = \{2, 4, 5\} \\ a = fe & \iff & aV = \{1, 3, 5\} \end{cases}, \qquad \begin{cases} c = e & \iff & c1 = c4, \ c2 = c3 \\ c = ef & \iff & c1 = c2, \ c3 = c4 \end{cases}. \tag{2.16}$$

We note that elements of G act as permutations over $\{2, 4, 5\}$:

$$e(2, 4, 5) = (2, 4, 5), \quad g(2, 4, 5) = (5, 2, 4), \quad h(2, 4, 5) = (4, 2, 5). \tag{2.17}$$

It is easy to see that

$$W_\mu = \{(x, y, z) : \text{a permutation of } (2, 4, 5) \text{ or } (1, 3, 5)\}. \tag{2.18}$$

We may take a set W of Proposition 1.12 as

$$W = \{(2, 4, 5)\}. \tag{2.19}$$

For example, for $\boldsymbol{x} = (3, 5, 1) \in W_\mu$, we see that $\boldsymbol{x}^L = fe$, $\boldsymbol{x}^G = gh$ and $\boldsymbol{x}^W = (2, 4, 5)$.

By (iii) of Proposition 1.12, we see that $\Lambda = \eta^L * \omega_G * \delta_{(2,4,5)}$ is the unique μ-invariant probability measure on $V_\times^3$. Let $(\mathbb{X}, N)$ be a stationary tri-particle μ-evolution such that $\mathbb{X}$ has Λ as its common law. Then we have the factorization

$$\mathbb{X}_j = \mathbb{X}_j^L \mathbb{X}_j^G (2, 4, 5) = \mathbb{X}_j^L (M_{k,j}^G)^{-1} U_k (2, 4, 5) \quad \text{a.s. for } j \leq k \tag{2.20}$$

with $U_k = \mathbb{X}_k^G$, $M_j^G = \mathbb{X}_j^G (\mathbb{X}_{j-1}^G)^{-1}$ and $M_{k,j}^G = \mathbb{X}_k^G (\mathbb{X}_j^G)^{-1} = M_k^G M_{k-1}^G \cdots M_{j+1}^G$, and consequently, we obtain the resolution

$$\mathcal{F}_k^{\mathbb{X}} = \mathcal{G}_k^N \vee \sigma(U_k) \quad \text{a.s.} \quad \text{with } \mathcal{G}_k^N = \sigma(\mathbb{X}_j^L, M_j^G : j \leq k) \tag{2.21}$$

where the two σ-fields $\mathcal{F}_k^N (\supset \mathcal{G}_k^N)$ and $\sigma(U_k)$ are independent.

We remark that

$$\text{the two processes } \mathbb{X}^L \text{ and } \mathbb{X}^G \text{ are } \textit{not} \text{ independent.} \tag{2.22}$$

In fact, we have

$$\mathbb{P}(\mathbb{X}^L_0 = e, \mathbb{X}^L_1 = fe, \mathbb{X}^G_0 = e, \mathbb{X}^G_1 = g) = \mathbb{P}(\mathbb{X}_0 = (2,4,5), \mathbb{X}_1 = feg(2,4,5)) = 0, \tag{2.23}$$

$$\mathbb{P}(\mathbb{X}^L_0 = e, \mathbb{X}^L_1 = fe) = \mathbb{P}(\mathbb{X}^L_0 = e, N_1 = f) = \mathbb{P}(\mathbb{X}^L_0 = e)\mathbb{P}(N_1 = f) = \frac{1}{3}, \tag{2.24}$$

$$\mathbb{P}(\mathbb{X}^G_0 = e, \mathbb{X}^G_1 = g) = \mathbb{P}(\mathbb{X}^G_0 = e, N_1 = g) = \mathbb{P}(\mathbb{X}^G_0 = e)\mathbb{P}(N_1 = g) = \frac{1}{12}, \tag{2.25}$$

which shows that the two events $\{\mathbb{X}^L_0 = e, \mathbb{X}^L_1 = fe\}$ and $\{\mathbb{X}^G_0 = e, \mathbb{X}^G_1 = g\}$ are not independent.

Note that the first component (X^1, N) is a mono-particle μ-evolution such that X^1 has a common law

$$\eta^L * \omega_G * \delta_2 = \left(\frac{2}{3}\delta_e + \frac{1}{3}\delta_{fe}\right) * \omega_{\{2,4,5\}} = \frac{2}{3}\omega_{\{2,4,5\}} + \frac{1}{3}\omega_{\{3,1,5\}} = \lambda, \tag{2.26}$$

where ω_A stands for the uniform law on a finite set A. Since $X^1_k = \mathbb{X}^L_k U_k 2$ and $eX^1_k = U_k 2$, we have

$$\mathcal{F}^{X^1,N}_k = \mathcal{F}^N_k \vee \sigma(U_k 2) \quad \text{a.s. for } k \in \mathbb{Z}, \tag{2.27}$$

where $U_k 2$ is independent of $\mathcal{F}^N_k$. We thus conclude that $(U_k 2)_{k\in\mathbb{Z}}$ is a third noise for (X^1, N).

3 F-cliques and stationary probabilities

Throughout this section we suppose all the assumptions of Proposition 1.12 be satisfied.

We borrow several notation from the *road coloring theory*. A pair $\{x, y\}$ from V will be called a *deadlock* if $gx \neq gy$ for all $g \in S := \bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^n$, or in other words, $f_n f_{n-1} \cdots f_1 x \neq f_n f_{n-1} \cdots f_1 y$ for all $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in \mathcal{S}(\mu)$. A subset F of V will be called an *F-clique* if every pair from F is a deadlock and $F = gV$ for some $g \in S$. Note that a subset F of V is an F-clique if and only if $F = gV$ for some $g \in S$ and if, for every $h \in S$, the restriction $h|_F$ is injective.

The F-cliques can be characterized as follows (see also [12, Lemma 1]).

Lemma 3.1. *For $g \in S$, the set gV is an F-clique if and only if $\#(gV) = m_\mu$. Consequently, the image of an F-clique by any mapping of S is still an F-clique. In addition, it holds that*

$$\mathcal{S}(\nu) = \{g \in S : gV \textit{ is an F-clique}\} = \{g \in S : \#(gV) = m_\mu\}. \tag{3.1}$$

Proof. If $\#(gV) = m_\mu$, for any $f \in S$ we have $m_\mu \le \#(fgV) \le \#(gV) = m_\mu$ so that $\#(fgV) = m_\mu$, which implies that gV is an F-clique. Conversely, if gV is an F-clique, then $\#(fV) \ge \#(fgV) = \#(gV) \ge m_\mu$ for any $f \in S$ so that $\#(gV) = m_\mu$.

Recall that $\mathcal{S}(\nu)$ is the kernel, i.e., the unique minimal ideal of S (see (ii) of Proposition 1.8). Therefore, to prove (3.1), it suffices to show that $K := \{g \in S : \#(gV) = m_\mu\}$ is a minimal ideal of S. It is obvious by definition that K is an ideal. Suppose $\emptyset \ne IS \cup SI \subset I \subset K$. Let $f \in I$ and $g \in K$. Since $gf|_{gV} : gV \to gV$ is bijective, the mapping $(gf|_{gV})^r$ is identity for some $r \in \mathbb{N}$ so that $(gf)^r g = g$. Hence $g = (gf)^{r-1} gfg \in SIS \subset I$, which shows $I = K$. □

By Lemma 3.1, the set W_μ defined in (1.22) can be represented as

$$W_\mu = \{\boldsymbol{x} = (x^1, \dots, x^{m_\mu}) : \{x^1, \dots, x^{m_\mu}\} \text{ is an F-clique}\}. \tag{3.2}$$

Lemma 3.2. *For any $\boldsymbol{x}, \boldsymbol{x}' \in eW_\mu$, the two sets $G\boldsymbol{x}$ and $G\boldsymbol{x}'$ are either equal or disjoint.*

Proof. Suppose $G\boldsymbol{x}$ and $G\boldsymbol{x}'$ have a common element $g\boldsymbol{x} = g'\boldsymbol{x}'$ for some $g, g' \in G$. We then obtain that

$$G\boldsymbol{x} = Gg\boldsymbol{x} = Gg'\boldsymbol{x}' = G\boldsymbol{x}'. \tag{3.3}$$

The proof is complete. □

We now prove Proposition 1.12.

Proof of Proposition 1.12. (i) By (3.1) and (3.2), we have $W_\mu = \mathcal{S}(\nu)W_\mu = LGRW_\mu$. Let us prove that

$$RW_\mu = GW_\mu = eW_\mu \quad \text{and} \quad LW_\mu = W_\mu. \tag{3.4}$$

Since $eW_\mu \subset RW_\mu \subset eLGRW_\mu = eW_\mu$ and $eW_\mu \subset GW_\mu \subset eLGRW_\mu = eW_\mu$, we have $RW_\mu = GW_\mu = eW_\mu$. Hence $LW_\mu = LLGRW_\mu = LGRW_\mu = W_\mu$. We now obtain (3.4).

We thus obtain

$$W_\mu = LGRW_\mu = LGGW = LGW. \tag{3.5}$$

(ii) Note that

$$eW_\mu = GW = \bigcup_{\boldsymbol{w} \in W} G\boldsymbol{w}. \tag{3.6}$$

By the minimality of W, we see that the sets $G\boldsymbol{w}$ for $\boldsymbol{w} \in W$ are disjoint; in fact, if $G\boldsymbol{w}$ and $G\boldsymbol{w}'$ are not disjoint with some distinct elements $\boldsymbol{w}$ and $\boldsymbol{w}'$ of W, then $G\boldsymbol{w} = G\boldsymbol{w}'$ by Lemma 3.2 and so $eW_\mu = G(W \setminus \{\boldsymbol{w}'\})$, which contradicts the minimality of W.

We have only to prove injectivity of the product $L \times G \times W \ni (f, g, \boldsymbol{w}) \mapsto fg\boldsymbol{w} \in W_\mu$. Suppose $fg\boldsymbol{w} = f'g'\boldsymbol{w}'$. Since $eL = \{e\}$, we have $g\boldsymbol{w} = g'\boldsymbol{w}'$, which implies $G\boldsymbol{w} = G\boldsymbol{w}'$. By the above argument, we have $\boldsymbol{w} = \boldsymbol{w}'$.

If we write $\boldsymbol{w} = (w^1, \dots, w^{m_\mu})$, then $\{w^1, \dots, w^{m_\mu}\} = eV$, because $(w^1, \dots, w^{m_\mu}) \in W \subset eW_\mu$ so that the points $w^1, \dots, w^{m_\mu}$ are distinct elements of eV, and $\#(eV) = m_\mu$ by (3.1). Hence the identity $fg\boldsymbol{w} = f'g'\boldsymbol{w}$ implies that $fg = f'g'$ on eV. Since $g = ge$ and $g' = g'e$, we see that $fg = f'g'$ on V, which implies $f = f'$ and $g = g'$.

(iii) Let $\Lambda_W \in \mathcal{P}(W)$ and set $\Lambda = \eta^L * \omega_G * \Lambda_W$. Since $Re = \{e\}$, we have $\nu * \delta_e = \eta^L * \omega_G * \eta^R * \delta_e = \eta^L * \omega_G * \delta_e = \eta^L * \omega_G$. We thus see that

$$\mu * \Lambda = \mu * \nu * \delta_e * \Lambda_W = \nu * \delta_e * \Lambda_W = \eta^L * \omega_G * \Lambda_W = \Lambda. \tag{3.7}$$

Conversely, suppose $\Lambda \in \mathcal{P}(V_\times^{m_\mu})$ be μ-invariant. Since $\Lambda = \mu * \Lambda$, we have $\nu * \Lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \mu^k * \Lambda = \Lambda$, and hence $\Lambda = \nu * \Lambda = \eta^L * \omega_G * \eta^R * \Lambda$. Since $\mathcal{S}(\nu)\mathcal{S}(\Lambda) = \mathcal{S}(\nu * \Lambda) = \mathcal{S}(\Lambda) \subset V_\times^{m_\mu}$ and by (3.2), we have $\mathcal{S}(\Lambda) \subset W_\mu$. Let us explain it in detail. If $(x_1, \dots, x_{m_\mu}) \in \mathcal{S}(\Lambda)$, then $x_1, \dots, x_{m_\mu}$ are distinct. Moreover, since $\mathcal{S}(\Lambda) = \mathcal{S}(\nu)\mathcal{S}(\Lambda)$, we have $(x_1, \dots, x_{m_\mu}) = k(y_1, \dots, y_{m_\mu})$ for some $k \in \mathcal{S}(\nu)$ and $(y_1, \dots, y_{m_\mu}) \in \mathcal{S}(\Lambda) \subset V_\times^{m_\mu}$. Thus $\{x_1, \dots, x_{m_\mu}\} \subset kV$, so $\{x_1, \dots, x_{m_\mu}\} = kV$ since $\#(kV) = m_\mu$. By Lemma 3.1, $\{x_1, \dots, x_{m_\mu}\}$ is an F-clique, i.e. $(x_1, \dots, x_{m_\mu}) \in W_\mu$ by (3.2).

Since $\mathcal{S}(\eta^R * \eta^L) \subset RL \subset G$ and thus $\eta^R * \Lambda = (\eta^R * \eta^L) * \omega_G * \eta^R * \Lambda = \omega_G * \eta^R * \Lambda$, we have $\mathcal{S}(\eta^R * \Lambda) = \mathcal{S}(\omega_G * \eta^R * \Lambda) \subset GRW_\mu = eW_\mu = GW$. Hence

$$\Lambda = (\eta^L * \omega_G) * (\eta^R * \Lambda) = \sum_{\boldsymbol{x} \in GW} (\eta^R * \Lambda)\{\boldsymbol{x}\} \, (\eta^L * \omega_G * \delta_{\boldsymbol{x}}) \tag{3.8}$$

$$= \sum_{\boldsymbol{x} \in GW} (\eta^R * \Lambda)\{\boldsymbol{x}\} \, (\eta^L * \omega_G * \delta_{\boldsymbol{x}^W}) = \eta^L * \omega_G * \Lambda_W, \tag{3.9}$$

where we take

$$\Lambda_W = \sum_{\boldsymbol{x} \in GW} (\eta^R * \Lambda)\{\boldsymbol{x}\} \delta_{\boldsymbol{x}^W}. \tag{3.10}$$

The proof is now complete. □

Let us prove Corollary 1.14.

Proof of Corollary 1.14. Let $\lambda \in \mathcal{P}(V)$ be μ-invariant. We then have

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \mu^k * \lambda = \nu * \lambda = \eta^L * \omega_G * \eta^R * \lambda. \tag{3.11}$$

For $w \in V$, we write

$$n(w) := \#(\{(x^1, \dots, x^{m_\mu}) \in W : x^1 = w\}), \tag{3.12}$$

which turns out to be positive by definition of W. We then define

$$\Lambda_W\{(v^1,\dots,v^{m_\mu})\} = \frac{1}{n(v^1)}(\eta^R * \lambda)\{v^1\}, \quad (v^1,\dots,v^{m_\mu}) \in W \tag{3.13}$$

so that $\Lambda_W \in \mathcal{P}(W)$, and define $\Lambda = \eta^L * \omega_G * \Lambda_W$. By (iii) of Proposition 1.12, we see that Λ is μ-invariant. Let us compute the marginal of Λ in the first coordinate. For any $v \in V$, we have

$$\Lambda\{(x^1,\dots,x^{m_\mu}) \in V_\times^{m_\mu} : x^1 = v\} \tag{3.14}$$

$$= \sum_{\substack{f\in L\\ g\in G}} \sum_{(v^1,\dots,v^{m_\mu})\in W} \eta^L\{f\}\omega_G\{g\}\Lambda_W\{(v^1,\dots,v^{m_\mu})\}1_{\{fgv^1=v\}} \tag{3.15}$$

$$= \sum_{\substack{f\in L\\ g\in G}} \sum_{(v^1,\dots,v^{m_\mu})\in W} \eta^L\{f\}\omega_G\{g\} \cdot \frac{1}{n(v^1)}(\eta^R * \lambda)\{v^1\}1_{\{fgv^1=v\}} \tag{3.16}$$

$$= \sum_{\substack{f\in L\\ g\in G}} \sum_{v^0\in V} \eta^L\{f\}\omega_G\{g\}(\eta^R * \lambda)\{v^0\}1_{\{fgv^0=v\}} \cdot \frac{1}{n(v^0)} \sum_{(v^1,\dots,v^{m_\mu})\in W} 1_{\{v^1=v^0\}} \tag{3.17}$$

$$= (\eta^L * \omega_G * \eta^R * \lambda)\{v\} = \lambda\{v\}. \tag{3.18}$$

The proof is complete. □

4 Proof of our main theorem

Throughout this section we suppose all the assumptions of Theorem 1.13 be satisfied. We divide the proof of Theorem 1.13 into several steps.

4.1 Factorizing $\mathbb{X}_k$ into LG- and W-factors

For any fixed $k \in \mathbb{Z}$, we see, by Proposition 1.12, that $\mathbb{X}_k \in LGW$ a.s. and the law of $\mathbb{X}_k$ is equal to $\eta^L * \omega_G * \Lambda_W$ for some $\Lambda_W \in \mathcal{P}(W)$, which shows that the three random variables $\mathbb{X}_k^L$, $\mathbb{X}_k^G$ and $\mathbb{X}_k^W$ are independent and for the marginal laws we have $\mathbb{X}_k^L \overset{\mathrm{d}}{=} \eta^L$, $\mathbb{X}_k^G \overset{\mathrm{d}}{=} \omega_G$ and $\mathbb{X}_k^W \overset{\mathrm{d}}{=} \Lambda_W$. Hence we have shown Claim (i) of Theorem 1.13.

Let us focus on the factor $\mathbb{X}_k^L\mathbb{X}_k^G$ in the factorization $\mathbb{X}_k = \mathbb{X}_k^L\mathbb{X}_k^G\mathbb{X}_k^W$ for $k \in \mathbb{Z}$.

Proposition 4.1. *Set $Y_k = \mathbb{X}_k^L\mathbb{X}_k^G$ for $k \in \mathbb{Z}$ and $Y = (Y_k)_{k\in\mathbb{Z}}$. Then the following hold:*

- *(i) (Y, N) is a μ-evolution such that the sequence Y has a common law $\eta^L * \omega_G$.*
- *(ii) There exists a W-valued random variable $\mathbb{Z}_W$ such that $\mathbb{X}_k^W = \mathbb{Z}_W$ a.s. for $k \in \mathbb{Z}$.*
- *(iii) (Y, N) and $\mathbb{Z}_W$ are independent.*

Proof. By the argument in the beginning of this subsection, we see that, for every $k \in \mathbb{Z}$, the law of $Y_k = \mathbb{X}_k^L \mathbb{X}_k^G$ is $\eta^L * \omega_G$.

Note that

$$Y_k \mathbb{X}_k^W = \mathbb{X}_k = N_k \mathbb{X}_{k-1} = (N_k Y_{k-1}) \mathbb{X}_{k-1}^W \quad \text{a.s.} \tag{4.1}$$

Since $Y_{k-1} \in LG$ and $N_k Y_{k-1} \in SLG = SSe \subset Se = LG$ by Proposition 1.6, we see, by Proposition 1.12, that

$$Y_k = N_k Y_{k-1} \quad \text{and} \quad \mathbb{X}_k^W = \mathbb{X}_{k-1}^W \quad \text{a.s.} \tag{4.2}$$

Since N_k is independent of $\mathcal{F}_{k-1}^Y (\subset \mathcal{F}_{k-1}^{\mathbb{X}})$, we see that (Y, N) is a μ-evolution. We now obtain Claims (i) and (ii) (and consequently we have shown Claim (iii) of Theorem 1.13).

Let $k \in \mathbb{Z}$ be fixed. By the above argument, we see that $Y_k = \mathbb{X}_k^L \mathbb{X}_k^G$ is independent of $\mathbb{Z}_W$. Since $\{N_j : j > k\}$ is independent of $\{Y_k, \mathbb{Z}_W\}$ and since $Y_j = N_j N_{j-1} \cdots N_{k+1} Y_k$ for $j > k$, we see that $\{(Y_j, N_j) : j > k\}$ is independent of $\mathbb{Z}_W$. Since $k \in \mathbb{Z}$ is arbitrary, we obtain Claim (iii). The proof is complete. □

4.2 Factorizing $\mathbb{X}_k^G$ into C- and H-factors

For $f \in \mathcal{S}(\nu) = LGR$, we write $f^C = (f^G)^C$ and $f^H = (f^G)^H$. Consequently, the mapping $z \mapsto (z^L, z^C, z^H, z^R)$ is the inverse of the product mapping $L \times C \times H \times R \ni (x, c, h, y) \mapsto xchy \in \mathcal{S}(\nu)$. For $\boldsymbol{x} \in LGW$, we write $\boldsymbol{x}^C = (\boldsymbol{x}^G)^C$ and $\boldsymbol{x}^H = (\boldsymbol{x}^G)^H$. Consequently, the mapping $\boldsymbol{x} \mapsto (\boldsymbol{x}^L, \boldsymbol{x}^C, \boldsymbol{x}^H, \boldsymbol{x}^W)$ is the inverse of the product mapping $L \times C \times H \times W \ni (x, c, h, \boldsymbol{w}) \mapsto xch\boldsymbol{w} \in LGW$.

Since H is a normal subgroup of G, we have

$$(g_1 g_2)^C H = (g_1 g_2) H = (g_1 H)(g_2 H) = (g_1^C H)(g_2^C H) = (g_1^C g_2^C) H, \tag{4.3}$$

so that $(g_1 g_2)^C = (g_1^C g_2^C)^C$.

We proceed to prove part of Theorem 1.13.

Proposition 4.2. *Claim* (1.25) *of Theorem 1.13 holds and it holds that*

$$\mathbb{X}_k^C H = \gamma^k Y_C H \text{ a.s. for } k \in \mathbb{Z} \text{ for some } C\text{-valued random variable } Y_C. \tag{4.4}$$

Consequently, $\mathbb{X}_k^C = (\gamma^k Y_C)^C$ *a.s. for* $k \in \mathbb{Z}$.

Proof. Set

$$N_{k,l} := N_k N_{k-1} \cdots N_{l+1}, \quad k > l. \tag{4.5}$$

Since $e \in S = \bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^n$, we can find $f_1, f_2, \ldots, f_{n_0} \in \mathcal{S}(\mu)$ such that $f_{n_0} f_{n_0 - 1} \cdots f_1 = e$, and hence we have

$$T_k^e := \sup\{l < k - n_0 : N_{l+n_0, l} = e\} > -\infty \quad \text{a.s.} \tag{4.6}$$

We now see that $N_{k,T^e_k} = N_{k,T^e_k+n_0} N_{T^e_k+n_0,T^e_k} = N_{k,T^e_k+n_0} e \in Se = LG$ by Proposition 1.6.

Let us prove Claim (1.25). Since $\mathbb{X}_k = N_{k,T^e_k} \mathbb{X}_{T^e_k}$, we have

$$\mathbb{X}^L_k = (N_{k,T^e_k})^L, \quad \mathbb{X}^G_k = (N_{k,T^e_k})^G \, \mathbb{X}^G_{T^e_k} \quad \text{a.s.} \tag{4.7}$$

Hence we obtain $\mathbb{X}^L_k \in \mathcal{F}^N_k$ a.s. Since $\mathbb{X}_k = N_k \mathbb{X}^L_{k-1} \mathbb{X}^G_{k-1} \mathbb{X}^W_{k-1}$ and $N_k \mathbb{X}^L_{k-1} \in SL = SLe \subset Se = LG$ by Proposition 1.6, we have

$$\mathbb{X}^L_k = (N_k \mathbb{X}^L_{k-1})^L, \quad \mathbb{X}^G_k = (N_k \mathbb{X}^L_{k-1})^G \, \mathbb{X}^G_{k-1} \quad \text{a.s.} \tag{4.8}$$

Hence we obtain $M^G_k = \mathbb{X}^G_k (\mathbb{X}^G_{k-1})^{-1} = (N_k \mathbb{X}^L_{k-1})^G \in \mathcal{F}^N_k$ a.s. We thus obtain Claim (1.25).

Let ξ be a random variable such that $\xi \stackrel{\mathrm{d}}{=} \omega_H$ and ξ is independent of $(\mathbb{X}, N)$. Let $k \in \mathbb{Z}$. By $N_k \mathbb{X}^L_{k-1} \in LG$, we have

$$M^G_k \xi = (N_k \mathbb{X}^L_{k-1})^G \xi = (N_k \mathbb{X}^L_{k-1} \xi)^G. \tag{4.9}$$

Since $\omega_H * \eta^R * \delta_e = \omega_H$ by $Re = \{e\}$ and $e \in H$, we have $\eta * \delta_e = \eta_L * \omega_H$, so

$$N_k \mathbb{X}^L_{k-1} \xi \stackrel{\mathrm{d}}{=} \mu * \eta^L * \omega_H = \mu * \eta * \delta_e = \eta^L * \delta_\gamma * \omega_H * \eta^R * \delta_e = \eta^L * \delta_\gamma * \omega_H, \tag{4.10}$$

we have $M^G_k \xi \stackrel{\mathrm{d}}{=} \delta_\gamma * \omega_H \stackrel{\mathrm{d}}{=} \gamma \xi$, which shows $(M^G_k)^C = \gamma$ a.s. for $k \in \mathbb{Z}$. We now see that

$$\mathbb{X}^C_k H = \mathbb{X}^G_k H = M^G_k \mathbb{X}^G_{k-1} H = (M^G_k)^C (\mathbb{X}^G_{k-1})^C H = \gamma \mathbb{X}^C_{k-1} H \quad \text{a.s. for } k \in \mathbb{Z}. \tag{4.11}$$

Hence $\gamma^{-k} \mathbb{X}^C_k H = \gamma^{-(k-1)} \mathbb{X}^C_{k-1} H$ a.s., which shows that there exists a C-valued random variable Y_C such that $\gamma^{-k} \mathbb{X}^C_k H = Y_C H$ a.s. for all $k \in \mathbb{Z}$, which yields (4.4). The proof is now complete. □

Remark 4.3. Let $(\mathbb{X}, N)$ be a stationary m_μ-particle μ-evolution. Let us pick up [1, Proposition 10 and Theorem 11], whose conclusion applied to our μ-evolutions is as follows: *If $\mathbb{X}$ is an irreducible aperiodic recurrent Markov chain on W_μ, then the random set*

$$R_0 := \bigcap_{j<0} N_{0,j} W_\mu \tag{4.12}$$

has exactly m_μ elements, the remote past noise $\mathcal{F}^{\mathbb{X}}_{-\infty}$ is trivial, and the conditional law of $\mathbb{X}_0$ given $\sigma(N)$ is uniform on R_0. Let us derive this result from our results. Since $f \in eS$ implies $fV = eV$ by (v) of Proposition 1.11, and since eW_μ is a subset of the set of all permutations of eV by (3.2), we see that $f \in eS$ implies $fW_\mu = eW_\mu$. For $j < T^e_0$, we have

$$N_{0,j} W_\mu = N_{0,T^e_0} N_{T^e_0,j} W_\mu = (N_{0,T^e_0})^L (N_{0,T^e_0})^G N_{T^e_0,j} W_\mu = \mathbb{X}^L_0 e W_\mu, \tag{4.13}$$

since $(N_{0,T^e_0})^G N_{T^e_0,j}$ takes values in $GS = eGS \subset eS$. Hence we obtain

$$R_0 = \mathbb{X}^L_0 e W_\mu. \tag{4.14}$$

(We did not need aperiodicity so far.) Let us now consider the special case where $\mathbb{X}$ is an irreducible aperiodic Markov chain on W_μ. For every $\boldsymbol{w} \in W$, the law $\mu^n * \delta_{\boldsymbol{w}}$ converges to the unique μ-invariant probability measure $\Lambda \in \mathcal{P}(W_\mu)$. This shows that $\Lambda = \eta * \delta_{\boldsymbol{w}} = \eta^L * \omega_H * \delta_{\boldsymbol{w}}$. By (iii) of Proposition 1.12, we obtain $H = G$ and $W = \{\boldsymbol{w}\}$ is a singleton, and consequently $\mathcal{F}_{-\infty}^{\mathbb{X}}$ is trivial, and $eW_\mu = GW = G\{\boldsymbol{w}\}$. As we have seen it in Section 1.5, the conditional law $\mathbb{P}(\mathbb{X}_0 \in \cdot \mid \sigma(N))$ is uniform on the random finite set $\{\mathbb{X}_0^L g\boldsymbol{w} : g \in G\} = \mathbb{X}_0^L eW_\mu = R_0$.

4.3 Finding the third noise

The following lemma plays a key role.

Lemma 4.4. *Let $\{a_n\}$ and $\{b_n\}$ be two deterministic sequences of $\mathcal{S}(\nu)$. Then*

$$(\delta_{a_n} * \mu^n * \delta_{b_n})^H \xrightarrow{\mathrm{d}} \omega_H. \tag{4.15}$$

Proof. Since $\mathcal{P}(H)$ is compact, it suffices to show that ω_H is the only one cluster point of the sequence $\{(\delta_{a_n} * \mu^n * \delta_{b_n})^H\}$. Let $\{n(m)\}$ be a subsequence of $\mathbb{N}$ such that $(\delta_{a_n} * \mu^n * \delta_{b_n})^H$ converges as $n \to \infty$. Taking a further subsequence if necessary, we can and do assume that $a_{n(m)} \to a_0$ and $b_{n(m)} \to b_0$ for some $a_0, b_0 \in \mathcal{S}(\nu)$ and

$$\mu^{n(m)} \to \mu^r * \eta = \eta^L * \delta_{\gamma^r} * \omega_H * \eta^R \tag{4.16}$$

for some $r = 0, 1, \ldots, p-1$. Hence we have

$$(\delta_{a_{n(m)}} * \mu^{n(m)} * \delta_{b_{n(m)}})^H \xrightarrow{\mathrm{d}} (\delta_{a_0} * \eta^L * \delta_{\gamma^r} * \omega_H * \eta^R * \delta_{b_0})^H. \tag{4.17}$$

Since $RL \subset H$ and $g^{-1}Hg = H$ for all $g \in G$, we have

$$(\delta_{a_0} * \eta^L * \delta_{\gamma^r} * \omega_H * \eta^R * \delta_{b_0})^G \tag{4.18}$$

$$=\delta_{a_0^G} * \delta_{a_0^R} * \eta^L * \delta_{\gamma^r} * \omega_H * \eta^R * \delta_{b_0^L} * \delta_{b_0^G} \tag{4.19}$$

$$=\delta_{a_0^C \gamma^r} * \delta_{\gamma^{-r} a_0^H \gamma^r} * (\delta_{\gamma^{-r}} * \delta_{a_0^R} * \eta^L * \delta_{\gamma^r}) * \omega_H * (\eta^R * \delta_{b_0^L}) * \delta_{(b_0^C b_0^H (b_0^C)^{-1})} * \delta_{b_0^C} \tag{4.20}$$

$$=\delta_{a_0^C \gamma^r} * \omega_H * \delta_{b_0^C} = \delta_{a_0^C \gamma^r b_0^C} * \omega_H = \delta_{(a_0^C \gamma^r b_0^C)^C} * \omega_H, \tag{4.21}$$

which yields $(\delta_{a_0} * \eta^L * \delta_{\gamma^r} * \omega_H * \eta^R * \delta_{b_0})^H = \omega_H$. We thus obtain (4.15). □

We proceed to prove part of Theorem 1.13.

Proposition 4.5. *Let $k \in \mathbb{Z}$ be fixed and set $U_k := \mathbb{X}_k^H = Y_k^H$. Then $U_k \overset{\mathrm{d}}{=} \omega_H$ and the three σ-fields $\sigma(N)$, $\mathcal{F}_{-\infty}^{\mathbb{X}}$ and $\sigma(U_k)$ are independent. (Consequently Claims (ii) and* (1.26) *of Theorem 1.13 hold.)*

Proof. Set

$$\mathcal{F}^N_{k,l} = \sigma(N_k, N_{k-1}, \dots, N_{l+1}), \quad k > l. \tag{4.22}$$

Let $k \in \mathbb{Z}$ be fixed and let $\varphi : H \to \mathbb{R}$ be a test function. Let $l < k$, $k_0 > k$, $n \in \mathbb{N}$, $A \in \mathcal{F}^N_{k_0,l}$ and $B \in \mathcal{F}^{\mathbb{X}}_{-\infty}$. Recall that the symbols T^e_l and $N_{k,l}$ have been introduced in the proof of Proposition 4.2. Since $T^e_l - n < T^e_l < l$, we see that the three σ-fields $\sigma(N_{k,T^e_l}, 1_A)$, $\sigma(N_{T^e_l, T^e_l - n})$ and $\sigma(Y_{T^e_l - n}, 1_B)$ are independent. Since $N_{T^e_l, T^e_l - n} \overset{\mathrm{d}}{=} \mu^n$, we have

$$\begin{aligned}
\mathbb{E}[\varphi(U_k) 1_A 1_B] =& \mathbb{E}\big[\varphi(Y^H_k) 1_A 1_B\big] && (4.23)\\
=& \mathbb{E}\big[\varphi\big((N_{k,T^e_l} N_{T^e_l, T^e_l - n} Y_{T^e_l - n})^H\big)\, 1_A 1_B\big] && (4.24)\\
=& \mathbb{E}\left[\int \varphi \,\mathrm{d}(\delta_a * \mu^n * \delta_{b_n})^H \bigg|_{\substack{a = N_{k,T^e_l} \\ b_n = Y_{T^e_l - n}}} 1_A 1_B \right]. && (4.25)
\end{aligned}$$

Noting that $N_{k,T^e_l} \in \mathcal{S}(\nu)$ (see the proof of Proposition 4.2) and $Y_{T^e_l - n} \in \mathcal{S}(\nu)$, we apply Lemma 4.4 to see that

$$(4.25) \underset{n\to\infty}{\longrightarrow} \int \varphi \,\mathrm{d}\omega_H \cdot \mathbb{E}[1_A 1_B] = \int \varphi \,\mathrm{d}\omega_H \cdot \mathbb{P}(A)\mathbb{P}(B). \tag{4.26}$$

Since $l < k$ and $k_0 > k$ are arbitrary, we obtain

$$\mathbb{E}[\varphi(U_k) 1_A 1_B] = \int \varphi \,\mathrm{d}\omega_H \cdot \mathbb{P}(A)\mathbb{P}(B), \quad A \in \sigma(N),\ B \in \mathcal{F}^{\mathbb{X}}_{-\infty}, \tag{4.27}$$

which leads to the desired result. □

4.4 Determining the remote past noise

We need the following lemma.

Lemma 4.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three sub-σ-fields of $\mathcal{F}$. Suppose that $\mathcal{A} \subset \mathcal{B} \vee \mathcal{C}$ a.s. and that $\mathcal{A} \vee \mathcal{B}$ be independent of $\mathcal{C}$. Then $\mathcal{A} \subset \mathcal{B}$ a.s.*

The proof of Lemma 4.6 can be found in [2, Section 2.2], and so we omit it.

We shall now complete the proof of Theorem 1.13.

Proof of Theorem 1.13. What remains unproved are Claims (iv), (v) and (vi).

We have shown that $\mathbb{X}^C_k = (\gamma^k Y_C)^C$, $\mathbb{X}^H_k = U_k$ and $\mathbb{X}^W_k = \mathbb{Z}_W$. Let $j \le k$. Since $\mathbb{X}^G_k = M^G_k \mathbb{X}^G_{k-1}$ by definition of M^G_k, we have $\mathbb{X}^G_k = M^G_{k,j} \mathbb{X}^G_j$. Hence we obtain

$$\mathbb{X}_j = \mathbb{X}^L_j \mathbb{X}^G_j \mathbb{Z}_W = \mathbb{X}^L_j (M^G_{k,j})^{-1} \mathbb{X}^G_k \mathbb{Z}_W = \mathbb{X}^L_j (M^G_{k,j})^{-1} (\gamma^k Y_C)^C U_k \mathbb{Z}_W \quad \text{a.s.}, \tag{4.28}$$

which shows Claim (iv) and leads to

$$\mathcal{F}_k^{\mathbb{X}} = \mathcal{G}_k^N \vee \sigma(Y_C, \mathbb{Z}_W) \vee \sigma(U_k) \quad \text{a.s.} \tag{4.29}$$

Since $\sigma(Y_C, \mathbb{Z}_W) \subset \mathcal{F}_{-\infty}^{\mathbb{X}}$ a.s., which is obvious by definition, and by (1.26), we can apply Lemma 4.6 for $\mathcal{A} = \mathcal{F}_{-\infty}^{\mathbb{X}}$, $\mathcal{B} = \sigma(Y_C, \mathbb{Z}_W)$ and $\mathcal{C} = \mathcal{F}_k^N \vee \sigma(U_k)$, and hence we obtain $\mathcal{F}_{-\infty}^{\mathbb{X}} \subset \sigma(Y_C, \mathbb{Z}_W)$ a.s. We thus obtain (1.27). Combining (4.29) and (1.27), we obtain (1.24), which shows Claim (v).

Since $\mathbb{X}_k = \mathbb{X}_k^L (\gamma^k Y_C)^C U_k \mathbb{Z}_W$ and since $\Lambda = \eta^L * \omega_G * \Lambda_W$, we see that $(\gamma^k Y_C)^C U_k$ and $\mathbb{Z}_W$ are independent and that $(\gamma^k Y_C)^C U_k \overset{\mathrm{d}}{=} \omega_G$ and $\mathbb{Z}_W \overset{\mathrm{d}}{=} \Lambda_W$. Since $\omega_G = \omega_C * \omega_H$, we see that $\gamma^k Y_C H$ and U_k are independent, $\gamma^k Y_C H \overset{\mathrm{d}}{=} \omega_{G/H}$ and $U_k \overset{\mathrm{d}}{=} \omega_H$, which yields that Y_C and U_k are independent and $Y_C \overset{\mathrm{d}}{=} \omega_C$. We now obtain Claim (vi).

The proof of Theorem 1.13 is therefore complete. □

5 The non-stationary case

Throughout this section we adopt the settings of Subsection 1.4.

Proposition 5.1. *Let $(\mathbb{X}, N)$ be an m-particle μ-evolution. Then, for almost every sample path and for any $1 \le i < j \le m$, either [$X^i = X^j$ (i.e., $X_k^i = X_k^j$ holds for all $k \in \mathbb{Z}$)] or [the pair $\{X_k^i, X_k^j\}$ is a deadlock for all $k \in \mathbb{Z}$]. Consequently, the number of distinct states among $\{X_k^1, \dots, X_k^m\}$ does not depend upon $k \in \mathbb{Z}$ a.s.*

Proof. Recall T_k^e, which has been introduced in the proof of Proposition 4.2. We then have, for almost every sample path, $\mathbb{X}_k = N_{k,T_k^e} \mathbb{X}_{T_k^e} \in LGV^m$ for all $k \in \mathbb{Z}$. By Lemma 3.1, we see that [every distinct pair from $\{X_k^1, \dots, X_k^m\}$ is a deadlock] for all $k \in \mathbb{Z}$.

Suppose that $X^i \ne X^j$. Then $X_{k_0}^i \ne X_{k_0}^j$ for some $k_0 \in \mathbb{Z}$. Since the pair $\{X_{k_0}^i, X_{k_0}^j\}$ is a deadlock, we have $X_k^i \ne X_k^j$ for all $k \ge k_0$. We also see that $X_k^i \ne X_k^j$ for all $k \le k_0$, since $(X_{k_0}^i, X_{k_0}^j) = N_{k_0,k}(X_k^i, X_k^j)$. For every $k \in \mathbb{Z}$, we see that $\{X_k^i, X_k^j\}$ is a distinct pair, and hence is a deadlock. The proof is now complete. □

Proposition 5.2. *For a sequence $(\Lambda_k)_{k \in \mathbb{Z}}$ from $\mathcal{P}(V_\times^{m_\mu})$, the following are equivalent:*

(i) $\Lambda_k = \mu * \Lambda_{k-1}$ *for all* $k \in \mathbb{Z}$.

(ii) There exist $\Lambda_W^0, \dots, \Lambda_W^{p-1} \in \mathcal{P}(W)$ and constants $\alpha_0, \dots, \alpha_{p-1} \ge 0$ with $\alpha_0 + \cdots + \alpha_{p-1} = 1$ such that

$$\Lambda_k = \sum_{r=0}^{p-1} \alpha_r \eta^L * \delta_{\gamma^{k+r}} * \omega_H * \Lambda_W^r \quad \textit{for all } k \in \mathbb{Z}. \tag{5.1}$$

(Consequently, $\Lambda_{k+p} = \Lambda_k$ for all $k \in \mathbb{Z}$, since $\gamma^p \in H$.)

Proof. Suppose that Condition (ii) is satisfied. Since

$$\mu * (\eta^L * \delta_{\gamma^{k+r}} * \omega_H) = \eta^L * \delta_{\gamma^{k+1+r}} * \omega_H, \tag{5.2}$$

we obtain $\mu * \Lambda_{k-1} = \Lambda_k$ for all $k \in \mathbb{Z}$, which shows Condition (i). Since $\gamma^p \in H$, we have $\delta_{\gamma^p} * \omega_H = \omega_H$, which yields $\Lambda_{k+p} = \Lambda_k$ for all $k \in \mathbb{Z}$.

Suppose that Condition (i) is satisfied. Iterating the relation $\Lambda_k = \mu * \Lambda_{k-1}$, we have $\Lambda_k = \mu^p * \Lambda_{k-p}$ for all $k \in \mathbb{Z}$. Let $k \in \mathbb{Z}$ be fixed. Since $\mathcal{P}(V_\times^{m_\mu})$ is compact, we have $\Lambda_{-pn(m)} \to \Lambda_*$ for some subsequence $\{n(m)\}$ of $\mathbb{N}$ and some $\Lambda_* \in \mathcal{P}(V_\times^{m_\mu})$. By Remark 1.9, we have

$$\Lambda_0 = \mu^{pn(m)} * \Lambda_{-pn(m)} \to \eta * \Lambda_*, \tag{5.3}$$

which shows

$$\Lambda_0 = \eta * \Lambda_* = \eta^L * \omega_H * \eta^R * \Lambda_*. \tag{5.4}$$

By the same argument as that after (3.7), we obtain $\mathcal{S}(\Lambda_*) \subset W_\mu$.

Since $\mathcal{S}(\eta^R * \Lambda_*) \subset RW_\mu = eW_\mu = GW = CHW$, we have

$$\begin{aligned} \eta^L * \omega_H * \eta^R * \Lambda_* &= \sum_{r=0}^{p-1} \sum_{h \in H} \sum_{\boldsymbol{w} \in W} (\eta^L * \omega_H * \delta_{\gamma^r h \boldsymbol{w}})\, (\eta^R * \Lambda_*)\{\gamma^r h \boldsymbol{w}\} && (5.5)\\ &= \sum_{r=0}^{p-1} \sum_{\boldsymbol{w} \in W} (\eta^L * \delta_{\gamma^r} * \omega_H * \delta_{\boldsymbol{w}})\, (\eta^R * \Lambda_*)(\gamma^r H \boldsymbol{w}) && (5.6)\\ &= \sum_{r=0}^{p-1} \alpha_r \eta^L * \delta_{\gamma^r} * \omega_H * \Lambda_W^r, && (5.7) \end{aligned}$$

where we set

$$\alpha_r = (\eta^R * \Lambda_*)(\gamma^r HW), \quad \Lambda_W^r = \frac{1}{\alpha_r} \sum_{\boldsymbol{w} \in W} (\eta^R * \Lambda_*)(\gamma^r H \boldsymbol{w}) \delta_{\boldsymbol{w}}. \tag{5.8}$$

We then obtain (5.1) for $k = 0$. By (5.2), we obtain (5.1) also for $k \geq 1$.

Let us prove (5.1) for $k \leq -1$ by induction. Suppose (5.1) for a fixed $k \leq 0$ hold true. We want to prove (5.1) for $k - 1$. By the same argument as for Λ_0, we have

$$\Lambda_{k-1} = \sum_{r=0}^{p-1} \widetilde{\alpha}_r\, \eta^L * \delta_{\gamma^{k-1+r}} * \omega_H * \widetilde{\Lambda}_W^r \tag{5.9}$$

for some $\widetilde{\Lambda}_W^0, \dots, \widetilde{\Lambda}_W^{p-1} \in \mathcal{P}(W)$ and some constants $\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_{p-1} \geq 0$ such that $\widetilde{\alpha}_0 + \cdots + \widetilde{\alpha}_{p-1} = 1$. We then have

$$\Lambda_k = \mu * \Lambda_{k-1} = \sum_{r=0}^{p-1} \widetilde{\alpha}_r\, \eta^L * \delta_{\gamma^{k+r}} * \omega_H * \widetilde{\Lambda}_W^r. \tag{5.10}$$

Comparing this identity with (5.1) and using Proposition 1.12, we obtain $\alpha_r = \widetilde{\alpha}_r$ and $\Lambda_W^r = \widetilde{\Lambda}_W^r$ for $r = 0, 1, \dots, p-1$. We thus obtain (5.1) for $k - 1$. We have proved (5.1) for $k \leq -1$ by induction. □

We now deal with the non-stationary case by reducing it to the stationary case.

Theorem 5.3. *Suppose the same assumptions of Proposition 1.12 be satisfied. Let $(\mathbb{X}, N)$ be an m_μ-particle μ-evolution such that the sequences $X^1, \dots, X^{m_\mu}$ are distinct a.s., i.e., $[X^i \neq X^j$ whenever $i \neq j]$ a.s. Set $\Lambda_k(\cdot) := \mathbb{P}(\mathbb{X}_k \in \cdot)$ for $k \in \mathbb{Z}$. Then the following hold:*

- *For any $k \in \mathbb{Z}$, the states $X_k^1, \dots, X_k^{m_\mu}$ are distinct and form an F-clique, a.s.*
- *$(\Lambda_k)_{k \in \mathbb{Z}}$ satisfies the equivalent conditions of Proposition 5.2.*

(i) For any fixed $k \in \mathbb{Z}$, it holds that $\mathbb{X}_k \in W_\mu = LGW$ a.s. and $\mathbb{X}_k^L \overset{\mathrm{d}}{=} \eta^L$.

(ii) $\mathbb{X}_k^G = (\gamma^k Y_C)^C U_k$ a.s. for $k \in \mathbb{Z}$ for some C-valued random variable Y_C and some H-valued random variables U_k such that U_k is uniform on H.

(iii) $\mathbb{X}_k^W = \mathbb{Z}_W$ a.s. for $k \in \mathbb{Z}$ for some W-valued random variable $\mathbb{Z}_W$.

(iv) If we write $M_j^G := \mathbb{X}_j^G (\mathbb{X}_{j-1}^G)^{-1}$ for $j \in \mathbb{Z}$ and $M_{k,j}^G := \mathbb{X}_k^G (\mathbb{X}_j^G)^{-1} = M_k^G M_{k-1}^G \cdots M_{j+1}^G$ for $j \leq k$, we have the following factorization:

$$\mathbb{X}_j = \mathbb{X}_j^L (M_{k,j}^G)^{-1} (\gamma^k Y_C)^C U_k \mathbb{Z}_W \quad \text{a.s. for } j \leq k. \tag{5.11}$$

(v) A resolution of the observation holds in the sense that

$$\mathcal{F}_k^{\mathbb{X}} = \mathcal{G}_k^N \vee \mathcal{F}_{-\infty}^{\mathbb{X}} \vee \sigma(U_k) \quad \text{a.s.}, \tag{5.12}$$

where

$$\mathcal{G}_k^N = \sigma\big(\mathbb{X}_j^L,\ M_j^G : j \leq k\big) \subset \mathcal{F}_k^N (\subset \sigma(N)) \quad \text{a.s.}, \tag{5.13}$$

$$\text{the three } \sigma\text{-fields } \sigma(N),\ \mathcal{F}_{-\infty}^{\mathbb{X}} \text{ and } \sigma(U_k) \text{ are independent} \tag{5.14}$$

and

$$\mathcal{F}_{-\infty}^{\mathbb{X}} = \sigma(Y_C,\ \mathbb{Z}_W) \quad \text{a.s.} \tag{5.15}$$

(vi) Let $\alpha_0, \dots, \alpha_{p-1}$ and $\Lambda_W^0, \dots, \Lambda_W^{p-1}$ be as in Proposition 5.2. Then the joint distribution of Y_C and $\mathbb{Z}_W$ is given as

$$\mathbb{P}(Y_C = \gamma^r,\ \mathbb{Z}_W = \boldsymbol{w}) = \alpha_r \Lambda_W^r \{\boldsymbol{w}\} \quad \text{for } r = 0, \dots, p-1 \text{ and } \boldsymbol{w} \in W. \tag{5.16}$$

Proof. By Proposition 5.1 and by the assumption that the sequences $X^1, \dots, X^{m_\mu}$ are distinct a.s., we see that, for every $k \in \mathbb{Z}$, the states $X_k^1, \dots, X_k^{m_\mu}$ are distinct and form an F-clique, a.s.

We now have $\mathbb{X}_k \in V_\times^{m_\mu}$ a.s., which shows $\Lambda_k \in \mathcal{P}(V_\times^{m_\mu})$. By definition of μ-evolution, we see that Condition (i) of Proposition 5.2 is satisfied. Hence we have a representation (5.1).

We write ω_W for the uniform probability on W and write $\widetilde{\Lambda} = \eta^L * \omega_G * \omega_W$, which is a μ-invariant probability supported on W_μ. Let $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{P}}$ be a stationary m_μ-particle μ-evolution such that $\widetilde{\mathbb{X}}$ has $\widetilde{\Lambda}$ as its common law. By (vi) of Theorem 1.13, we know that

$$\widetilde{\mathbb{P}}(\widetilde{Y}_C = \gamma^r,\ \widetilde{\mathbb{Z}}_W = \boldsymbol{w}) = \frac{1}{p} \cdot \frac{1}{\#(W)} > 0 \tag{5.17}$$

and so the conditional probability

$$\widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}(\cdot) := \widetilde{\mathbb{P}}(\cdot \mid \widetilde{Y}_C = \gamma^r,\ \widetilde{\mathbb{Z}}_W = \boldsymbol{w}) \tag{5.18}$$

is well-defined. We then see that $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}$ is a (non-stationary) m_μ-particle μ-evolution; Since the random variables $\widetilde{Y}_C$ and $\widetilde{\mathbb{Z}}_W$ are $\mathcal{F}^{\widetilde{\mathbb{X}}}_{-\infty}$-measurable, conditioning by them preserves Markov property (1.3). Note that, for each $k \in \mathbb{Z}$, the law of $\widetilde{\mathbb{X}}_k$ under $\widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}$ equals to $\eta^L * \delta_{(\gamma^{k+r})^C} * \omega_H * \delta_{\boldsymbol{w}}$. Moreover, by (1.23), we obtain the following factorization:

$$\widetilde{\mathbb{X}}_j = \widetilde{\mathbb{X}}^L_j (\widetilde{M^G_{k,j}})^{-1} (\gamma^{k+r})^C \widetilde{U}_k \boldsymbol{w} \quad \widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}\text{-a.s. for } j \le k, \tag{5.19}$$

where $\widetilde{M^G_{k,j}}$ and $\widetilde{U}_k$ are defined in the same way as in Theorem 1.13. We then see that Claims (i)-(iv) are satisfied for $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}$.

Let us check that (v) is satisfied for $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}$. By (1.25) of Theorem 1.13, there exist measurable maps ϕ_k and ψ_k such that $\widetilde{\mathbb{X}}^L_k = \phi_k(\widetilde{N}_j : j \le k)$ and $\widetilde{M^G_k} = \psi_k(\widetilde{N}_j : j \le k)$, $\widetilde{\mathbb{P}}$-a.s., which yields that these identities also hold $\widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}$-a.s., which shows that Claim (5.13) is satisfied. Since the σ-field $\mathcal{F}^{\widetilde{\mathbb{X}}}_{-\infty} = \sigma(\widetilde{Y}_C,\ \widetilde{\mathbb{Z}}_W)$ is trivial $\widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}$-a.s., we can deduce from the factorization (iv) that Claims (5.12) and (5.14) are satisfied.

Define

$$\widetilde{\mathbb{Q}} = \sum_{r=0}^{p-1} \alpha_r \sum_{\boldsymbol{w} \in W} \Lambda^r_W\{\boldsymbol{w}\}\, \widetilde{\mathbb{P}}_{\gamma^r, \boldsymbol{w}}. \tag{5.20}$$

We then see that the joint law of $(\mathbb{X}, N)$ under $\mathbb{P}$ equals to that of $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{Q}}$; in fact, they are μ-evolutions and

$$\widetilde{\mathbb{Q}}(\widetilde{\mathbb{X}}_k \in \cdot) = \sum_{r=0}^{p-1} \alpha_r \sum_{\boldsymbol{w} \in W} \Lambda^r_W\{\boldsymbol{w}\}\, (\eta^L * \delta_{(\gamma^{k+r})^C} * \omega_H * \delta_{\boldsymbol{w}}) \tag{5.21}$$

$$= \sum_{r=0}^{p-1} \alpha_r\, \eta^L * \delta_{(\gamma^{k+r})^C} * \omega_H * \Lambda^r_W = \Lambda_k = \mathbb{P}(\mathbb{X}_k \in \cdot). \tag{5.22}$$

We now obtain Claim (vi) for $(\mathbb{X}, N)$ under $\mathbb{P}$. We thus derive from (5.19) the following factorization:

$$\widetilde{\mathbb{X}}_j = \widetilde{\mathbb{X}}^L_j (\widetilde{M^G_{k,j}})^{-1} (\gamma^k \widetilde{Y}_C)^C \widetilde{U}_k \widetilde{\mathbb{Z}}_W \quad \widetilde{\mathbb{Q}}\text{-a.s. for } j \le k, \tag{5.23}$$

where $\widetilde{Y}_C$ and $\widetilde{\mathbb{Z}}_W$ are defined in the same way as in Theorem 1.13. We therefore obtain Claims (i)-(v) for $(\mathbb{X}, N)$ under $\mathbb{P}$. □

6 Appendix: The semigroup consisting of mappings

Proof of Proposition 1.11. (i) Let us define K by (1.18) and prove that K is a minimal ideal of S. For $f \in K$ and $g, h \in S$, we have $m_S \le \#(gfhV) \le \#(fV) = m_S$, which shows that K is an ideal of S. Let $I \subset K$ be an ideal of S. Let $f \in K$ and $g \in I$. Since $fgf|_{fV}$ is a permutation of fV, there exists an integer $q \ge 1$ such that $(fgf)^q$ is identity on fV, which implies $(fgf)^q f = f$. Hence

$$f = (fgf)^q f = fg(f(fgf)^{q-1} f) \in SIS \subset I, \tag{6.1}$$

which shows $I = K$, and thus K is the kernel of S.

(ii) This is obvious.

(iii) Let e be a primitive idempotent of S. Since K is completely simple by Proposition 1.6, we may take $f \in E(K)$. Then $efe \in SKS \subset K$. Since $efe|_{efeV}$ is a permutation of $efeV$, there exists an integer $q \ge 1$ such that $(efe)^q$ is identity on $efeV$, which yields $(efe)^{q+1} = efe$. If we write $g := (efe)^{2q}$, we obtain $eg = ge = g \in E(K)$, which implies $g = e$ by primitivity. Thus we obtain $e \in E(K)$. The converse is obvious since all idempotents of K are primitive.

(iv) Let $f \in Se = LG$ and take $(x, g) \in L \times G$ such that $f = xg$. Since $g^{-1} f = e$ and $fe = f$, we have $[fv = fw \iff ev = ew]$ for all $v, w \in V$, which shows $\pi(f) = \pi(e)$.

Conversely, let $f \in S$ be such that $\pi(f) = \pi(e)$. Then $\#(fV) = \#(eV) = m_S$, so that $f \in K$. Let $f = xgy$ with $(x, g, y) \in L \times G \times R$. Since $\pi(y) = \pi(gy) = \pi(ef) = \pi(f) = \pi(e)$ and $ye = e$, we obtain $y = e$, so $f \in Se$.

(v) Let $f \in eS = GR$ and take $(g, y) \in G \times R$ such that $f = gy$. Then $fV = efV \subset eV$. Since $\#(fV) = \#(eV) = m_S$, we have $fV = eV$.

Conversely, let $f \in S$ be such that $fV = eV$. Then $\#(fV) = \#(eV) = m_S$, so that $f \in K$. Take $(x, g, y) \in L \times G \times R$ such that $f = xgy$. Note that $fe = xgye = xg$ and $x = feg^{-1} = fg^{-1}$. Since $xV = fg^{-1}V \subset fV = eV$, we have $xV = eV$. On one hand, since e is identity on $xV = eV$, we have $exv = xv$ for $v \in V$. On the other hand, since $ex = e$, we have $exv = ev$ for $v \in V$. We now obtain $x = e$, so $f \in eS$.

(vi) This is immediate from (iv) and (v), since $G = Se \cap eS$. □

7 Appendix: Another example

Let

$$V = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : a_1, a_2, a_3 \in \{-1, 1\} \right\}. \tag{7.1}$$

Let $D = \{(1,0),(-1,0),(0,1),(0,-1)\}$ and set

$$S = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & b \end{pmatrix} : (a_{11},a_{12}) \in D,\ (a_{21},a_{22}) \in D,\ b \in \{-1,1\} \right\}. \tag{7.2}$$

Note that S is a finite semigroup with respect to the usual matrix product. In fact,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \pm 1 & 0 \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \pm 1 \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & \pm 1 \end{pmatrix} \tag{7.3}$$

for $(a_{21},a_{22}) \in D$, etc. We regard an element of S as a map of V into itself with respect to the usual matrix product. Set

$$s_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{7.4}$$

and

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{7.5}$$

so that $s_0, s_1, s_2, g \in S$. Let $\mu = (\delta_{s_0} + \delta_{s_1} + \delta_{s_2} + \delta_g)/4$ be the uniform law on $\mathcal{S}(\mu) = \{s_0, s_1, s_2, g\}$. It is easy to see that $\mathcal{S}(\mu)$ generates S, i.e., $S = \bigcup_{n=1}^{\infty} \{s_0, s_1, s_2, g\}^n$. Let us apply Propositions 1.8, 1.11 and 1.12.

If we write

$$A = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in V : a_1 = a_2 \right\}, \quad B = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in V : a_1 = -a_2 \right\}, \tag{7.6}$$

then

$$\begin{cases} s_0 A = A \\ s_0 B = B \end{cases} \qquad \begin{cases} s_1 A = B \\ s_1 B = A \end{cases} \qquad \begin{cases} s_2 A = A \\ s_2 B = B \end{cases} \tag{7.7}$$

and $gV = gA = gB = A$, and hence we see that $m_\mu = m_S = \#(A) = \#(B) = 4$.

Set

$$S_+ = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & b \end{pmatrix} \in S : b = 1 \right\}, \quad S_- = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & b \end{pmatrix} \in S : b = -1 \right\}. \tag{7.8}$$

Since $\mathcal{S}(\mu) \subset S_-$, we have $\mathcal{S}(\mu^2) = \mathcal{S}(\mu)\mathcal{S}(\mu) \subset S_- S_- = S_+$. Since $\mathcal{S}(\mu^2)$ generates S_+ and contains the identity map, we see that the left or right random walk on S_+ whose steps have law μ^2 is aperiodic, whereas the random walk on S whose steps have law μ is not. Hence we obtain $p = 2$, and consequently the sequence $\{\mu^{2n}\}_{n=1}^{\infty}$ converges to η.

Let $K = \mathcal{S}(\nu)$ and $K_+ = \mathcal{S}(\eta)$ denote the kernels of S and S_+, respectively. Then

$$K = \{s \in S : \#(sV) = 4\} = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a_{11} & 0 \\ 0 & a_{21} & 0 \\ 0 & 0 & b \end{pmatrix} \in S \right\}, \tag{7.9}$$

$$K_+ = \{s \in S_+ : \#(sV) = 4\} = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a_{11} & 0 \\ 0 & a_{21} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in S \right\}. \tag{7.10}$$

Set

$$e := g^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in S_+, \tag{7.11}$$

which is an idempotent of S_+. Since $\#(eV) = 4$, we have $e \in K_+$. Let us determine the Rees decompositions $K = LGR$ and $K_+ = LHR$ at $e \in E(K_+)$. Set

$$f = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.12}$$

Then we have

$$L = \{s \in E(K_+) : \pi(s) = \pi(e)\} = \{e, f\}, \tag{7.13}$$

$$R = \{s \in E(K_+) : sV = eV\} = \{e, k\}, \tag{7.14}$$

$$H = \{s \in K_+ : \pi(s) = \pi(e),\ sV = eV\} = \{e, -g\}, \tag{7.15}$$

$$G = \{s \in K : \pi(s) = \pi(e),\ sV = eV\} = \{e, -e, g, -g\}. \tag{7.16}$$

We now see that we may choose $\gamma = g$, so that $C = \{e, g\}$ and $G = CH$. We have the following multiplication tables (the table of ab for a and b):

$a\backslash b$	e	f	g	k
e	e	e	g	k
f	f	f	fg	fk
g	g	g	e	gk
k	e	$-g$	g	k

$a\backslash b$	e	f
s_0	g	fg
s_1	fg	g
s_2	g	$-f$
g	g	g

$a\backslash b$	s_0	s_1	s_2	g
e	g	g	gk	g
k	gk	$-k$	g	g

(7.17)

Note that $-f = fg(-g)$ and $-k = g(-g)k$.

Let us compute η^L. Let $\eta^L = \alpha\delta_e + \beta\delta_f$ for some $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Note that

$$\mu * \eta^L * \omega_H = \mu * \eta^L * \omega_H * \eta^R * \omega_H = \mu * \eta * \omega_H \tag{7.18}$$

$$= \eta^L * \delta_g * \omega_H * \eta^R * \omega_H = \eta^L * \delta_g * \omega_H. \tag{7.19}$$

On the one hand, we have

$$\eta^L * \delta_g * \omega_H = (\alpha\delta_e + \beta\delta_f) * \delta_g * \omega_H. \tag{7.20}$$

On the other hand, by (7.17), we have

$$\mu * \eta^L = \frac{1}{4}(\delta_{s_0} + \delta_{s_1} + \delta_{s_2} + \delta_g) * (\alpha\delta_e + \beta\delta_f) = \frac{3\alpha + 2\beta}{4}\delta_g + \frac{\alpha+\beta}{4}\delta_{fg} + \frac{\beta}{4}\delta_{(-f)}. \quad (7.21)$$

Since $-f = fg(-g)$ and $-g \in H$, we have $\delta_{-f} * \omega_H = \delta_{fg} * \omega_H$, so

$$\mu * \eta * \omega_H = \mu * \eta^L * \omega_H = \left(\frac{3\alpha + 2\beta}{4}\delta_e + \frac{\alpha + 2\beta}{4}\delta_f\right) * \delta_g * \omega_H. \quad (7.22)$$

Hence we obtain $\alpha = 2/3$ and $\beta = 1/3$, so that

$$\eta^L = \frac{2}{3}\delta_e + \frac{1}{3}\delta_f. \quad (7.23)$$

By a similar argument, using the identities $-k = g(-g)k$, $\omega_H * \delta_g = \delta_g * \omega_H$ and

$$\omega_H * \eta^R * \mu = \delta_g * \omega_H * \eta^R, \quad (7.24)$$

we obtain

$$\eta^R = \frac{2}{3}\delta_e + \frac{1}{3}\delta_k. \quad (7.25)$$

Note that $eV = \{v_1, v_2, v_3, v_4\}$ and $fV = \{fv_1, fv_2, fv_3, fv_4\}$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}. \quad (7.26)$$

By (3.1) and (3.2), we see that

$$W_\mu = \{(x^1, x^2, x^3, x^4) : \text{a permutation of } (v_1, v_2, v_3, v_4) \text{ or } (fv_1, fv_2, fv_3, fv_4)\} \quad (7.27)$$

and

$$eW_\mu = \{(x^1, x^2, x^3, x^4) : \text{a permutation of } (v_1, v_2, v_3, v_4)\}. \quad (7.28)$$

We have the following multiplication table (the table of sv for s and v):

$s\backslash v$	v_1	v_2	v_3	v_4
e	v_1	v_2	v_3	v_4
$-e$	v_4	v_3	v_2	v_1
g	v_3	v_4	v_1	v_2
$-g$	v_2	v_1	v_4	v_3

(7.29)

From this table, we see that we may take a set W as

$$W = \{(x^1, x^2, x^3, x^4) : x^4 = v_4 \text{ and } (x^1, x^2, x^3) \text{ is a permutation of } (v_1, v_2, v_3)\ \}, \quad (7.30)$$

which is a minimal subset of W_μ such that $eW_\mu = GW$.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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